Rotation Through $2\pi$: SO(3) vs SU(2), Superselection, and Time Reversal

• It is a general feature of all states of half-integer angular momentum that any rotation through an angle of $2\pi$,

$$U[R(2\pi\hat{\omega})] \equiv U[R(2\pi)] = -1.$$ 

- We saw this explicitly for $j = \frac{1}{2}$ (actually, $s = \frac{1}{2}$).
- The proof is also straightforward for arbitrary $j$ in the case $\hat{\omega} = \hat{z}$:

$$U[R(2\pi\hat{z})]\ket{j, m} = \exp(-i2\pi J_z/\hbar)\ket{j, m} = \exp(-i2\pi m)\ket{j, m} = (-1)^{2j}\ket{j, m},$$

since $m = j - k$, where $k$ is an integer.

- For arbitrary $j$ and $\hat{\omega}$, it is necessary to rotate the $\ket{j, m}$'s into eigenstates of $\hat{\omega} \cdot \vec{J}$, apply $U[R(2\pi\hat{\omega})]$, then rotate back. The conclusion is again

$$U[R(2\pi\hat{z})]\ket{j, m} = (-1)^{2j}\ket{j, m}.$$ 

- Important: $U[R(2\pi)] = -1$ means that the state vector is multiplied by $-1$, not that the spin (or its expectation value $\langle S \rangle$) changes sign.

• This peculiar feature can be traced to the properties of the matrices that represent the symmetry operators:

- Spatial rotations have the SO(3) group properties of $3 \times 3$ special orthogonal matrices, for which a $2\pi$ rotation equals the identity.

- The rotation operators for a spin-$\frac{1}{2}$ system, $U[R(\omega)] = \cos(\omega/2)I - i\sin(\omega/2)\hat{\omega} \cdot \vec{\sigma}$, span the set of $2 \times 2$ unitary unimodular\(^1\) matrices

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$

where $|a|^2 + |b|^2 = 1$. This set forms the group SU(2) under matrix multiplication.

- There is a 2:1 mapping of the elements of SU(2) onto those of SO(3). Formally, $U(a, b)$ and $U(-a, -b)$ correspond to the same $3 \times 3$ matrix of SO(3).

- Strictly, the $D^{(j)}$ matrices introduced previously are irreducible representations of SU(2). The odd-dimensional (integer $j$) representations do not preserve the distinction between $\omega$ and $\omega + 2\pi$ rotations; these matrices also serve as representations of SO(3).

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\(^1\)A general $2 \times 2$ unitary matrix can be written $U(a, b) = \begin{pmatrix} a & b \\ -b^* e^{i\theta} & a^* e^{i\theta} \end{pmatrix}$, where $|a|^2 + |b|^2 = 1$ and $\theta$ is real; then $\det U = e^{i\theta}$. “Unimodular” means $\det U = 1$, or $\theta = 2\pi$ times an integer.
We have seen that rotations through $\omega$ and $\omega + 2\pi$ about the same axis are identical for spatial rotations, but inequivalent for spin rotations. However, we will take it as axiomatic that all physical observables $\Omega$ are invariant under $2\pi$ rotations, i.e.,

$$[\Omega, U[R(2\pi)]] = 0.$$  

This assumption leads to a superselection rule: no operator corresponding to a physical observable can have nonvanishing matrix elements between a state $|+\rangle$ of integer angular momentum and a state $|-\rangle$ of half-integer angular momentum.

Proof:

$$\langle +| U[R(2\pi)] \Omega |-\rangle = \langle +| \Omega U[R(2\pi)] |-\rangle,$$

$$+\langle +|\Omega |-\rangle = -\langle +|\Omega |-\rangle,$$

$$\langle +|\Omega |-\rangle = 0.$$  

Finally, $2\pi$ rotations are connected with time-reversal symmetry:

Recall that in the passive picture, we defined the time-reversal operator $T$ by

$$R' = T^{-1} R T = R,$$

and

$$P' = T^{-1} P T = -P,$$

$$\Rightarrow L' = T^{-1} L T = -L.$$  

In order to be consistent, we must require

$$S' = T^{-1} S T = -S.$$  

As shown by Ballentine (p. 382), the form of the time-reversal operator appropriate for the representation $\psi_o(r,t)\chi(t) = \langle r| \otimes \langle s,m|\psi(t)\rangle$ is

$$T = \exp(-i\pi S_y/\hbar) C,$$

where $C$ is the complex conjugation operator (acting on both $\psi_o$ and $\chi$).

Recalling that $iS_y = \frac{1}{2}(S_+ + S_-)$, where $S_{\pm}|s,m\rangle = \sqrt{(s \pm m)(s \pm m + 1)}\hbar|s,m \pm 1\rangle$, we see that $\exp(-i\pi S_y/\hbar)$ has a real matrix representation in the basis of common eigenkets of $S^2$ and $S_z$. Then

$$T^2 = \exp(-i\pi S_y/\hbar) C \exp(-i\pi S_y/\hbar) C$$

$$= \exp(-i\pi S_y/\hbar) \exp(-i\pi S_y/\hbar) C C$$

$$= \exp(-i2\pi S_y/\hbar).$$

Since $\exp(-i2\pi L_y/\hbar) = 1$ always, and $[S_y, L_y] = 0$, we can write

$$T^2 = \exp(-i2\pi J_y/\hbar) \equiv U[R(2\pi)] = (-1)^{2j}.$$  

This justifies the claim made previously that $T^2 = +1 [-1]$ for particles having integer [half-integer] angular momentum.