1. A quantum mechanical system is described by a two-dimensional vector space spanned by orthonormal basis vectors \(|1\rangle\) and \(|2\rangle\). The Hamiltonian for this system is

\[ H = \epsilon \left( -4|1\rangle\langle 1| + 4|2\rangle\langle 2| + 3|1\rangle\langle 2| + 3|2\rangle\langle 1| \right), \]

where \(\epsilon > 0\). We will also consider the operator

\[ \Lambda = \lambda_0 (|1\rangle\langle 2| + |2\rangle\langle 1|). \]

(a) [4 points] Provide the matrix representations of \(H\) and \(\Lambda\) in the basis \(|1\rangle, |2\rangle\).

(b) [8 points] Find the eigenvalues \((E_1 \text{ and } E_2, \text{ with } E_1 < E_2)\) and normalized eigenvectors \((|E_1\rangle \text{ and } |E_2\rangle)\) of \(H\) in the basis \(|1\rangle, |2\rangle\).

(c) [13 points] Find the matrix representation of the propagator \(U(t)\) in the basis \(|1\rangle, |2\rangle\).

(d) [20 points] Suppose that the state vector at time 0 is \(|\psi(0)\rangle = |2\rangle\). The value of the dynamical variable \(\lambda\) corresponding to the operator \(\Lambda\) is measured at time \(t > 0\). What are the possible measured values of \(\lambda\) and their respective probabilities \(P_\lambda(t)\)?
2. Consider a particle of mass $m$ moving in one dimension under the influence of the potential
\[ V(x) = \frac{\hbar^2 Q}{2m} [\delta(x + a) - \delta(x - a)], \]
where $\delta(x)$ is the Dirac delta function, and $Q$ and $a$ are positive, real quantities. Let $\psi_E(x)$ be the wave function describing a stationary state of energy $E$.

Let us first consider unbound solutions ($E > 0$) and write
\[ \psi_E(x) = A_j e^{ikx} + B_j e^{-ikx}, \quad k = |\sqrt{2mE}|/\hbar, \]
where $j = 1$ describes $x \leq -a$, $j = 2$ describes $|x| \leq a$, and $j = 3$ describes $x \geq a$.

Define the $2 \times 2$ transfer matrix $P_E(n, m)$ by the relation
\[ \begin{pmatrix} A_n \\ B_n \end{pmatrix} = P_E(n, m) \begin{pmatrix} A_m \\ B_m \end{pmatrix}. \]

(a) [13 points] State the boundary conditions on the wave function at $x = -a$. By applying these boundary conditions, find the transfer matrix $P(2, 1)$ in terms of variables introduced above.

(b) [8 points] Find the transfer matrix $P(3, 2)$ in terms of variables introduced above. Hint: You can do this by applying boundary conditions at $x = a$, but you can obtain $P(3, 2)$ more efficiently by suitably modifying $P(2, 1)$.

(c) [4 points] Show that the bottom-right element of the overall transfer matrix $P \equiv P(3, 1)$ is
\[ P_{22} = 1 + \frac{Q^2}{4k^2} \left( 1 - e^{i4ka} \right). \]

(d) [8 points] Argue that the transmission coefficient for this potential is $T = |P_{22}|^{-2}$. (Hint: Start with a general expression for $T$, valid for any real, piecewise-constant potential, and specialize to the problem at hand.) Hence, find $T$ as a function of $k$, $Q$, and $a$.

(e) [6 points] Find the value(s) of $k$ at which $T$ takes its maximum value.

Let us now seek bound-state solutions $\psi_E(x)$ having $E < 0$.

(f) [8 points] Recalling that bound states occur when $P_{22} = 0$, obtain a condition relating the bound-state value(s) of $\kappa = |\sqrt{-2mE}|/\hbar$ to $Q$ and $a$. Hint: You should be able to obtain $P_{22}$ for $E < 0$ by suitably modifying the result of part (c).

(g) [8 points] Show that bound state solutions can exist only for $0 < \kappa < Q/2$. 