1. Consider an isolated point-like particle of mass $m$ and charge $q$, situated in a uniform magnetic field $B$, and described by the Hamiltonian

$$H = \frac{1}{2m} \left| \mathbf{P} - \frac{q}{c} \mathbf{A} \right|^2 + q\phi. \quad (1)$$

Let us consider the family of gauges $\phi(\mathbf{r}) = 0$, $A^{(\Lambda)}(\mathbf{r}) = \frac{1}{2}B \times \mathbf{r} - \nabla \Lambda(\mathbf{r})$, where each gauge is specified by a different function $\Lambda(\mathbf{r})$.

The real-space wave function corresponding to an abstract state $|\psi\rangle$ is gauge-dependent; let us denote it $\psi^{(\Lambda)}(\mathbf{r})$.

The particle’s orbital magnetic moment operator $\mathbf{\mu}_L$ can be defined (by analogy with classical electromagnetism) to be the operator which enters the Hamiltonian in the combination $-B \cdot \mathbf{\mu}_L$. The explicit form of this operator is also gauge-dependent, so we denote it $\mathbf{\mu}^{(\Lambda)}_L$.

(a) Substitute the coordinate-representations of $\mathbf{P}$, $\phi$ and $A^{(\Lambda)}$ into the Hamiltonian (1), expand all parentheses, and then simplify the resulting expression.

(b) Find an explicit expression for $\mathbf{\mu}^{(\Lambda)}_L$. Make it resemble as closely as possible its form in the gauge $\Lambda = 0$, i.e., $\mathbf{\mu}^{(0)}_L = q/(2mc)\mathbf{L}$, where $\mathbf{L}$ is the orbital angular momentum.

(c) Show that the expectation value of $\mathbf{\mu}^{(\Lambda)}_L$ in the state $\psi^{(\Lambda)}(\mathbf{r})$ is gauge-independent.

(d) Show that if the magnetic operator is given the alternative definition $\mathbf{\mu}^{(\Lambda)}_L = -\partial H^{(\Lambda)}/\partial B$, then this operator can be expressed in the gauge-independent form

$$\mathbf{\mu}_L = \frac{q}{2c} \mathbf{R} \times \mathbf{V},$$

where $\mathbf{V}$ is the velocity operator.

2. Based on Ballentine Problem 12.2: The most general state operator for a spin-$\frac{1}{2}$ system has the form $\rho = \frac{1}{2}(I + \mathbf{p} \cdot \mathbf{\sigma})$, where $\sigma_j$ ($j = 1, 2, 3$) is a Pauli operator and $\mathbf{p}$ is the (real) polarization vector, whose length cannot exceed 1. If the system has a magnetic moment $\mathbf{\mu} = \frac{1}{2}\gamma\hbar\mathbf{\sigma}$ and is in a constant, uniform magnetic field $\mathbf{B}$, calculate the time dependence of $\rho(t)$ in the Schrödinger picture. Describe the result geometrically in terms of the variation of the vector $\mathbf{p}$.

3. Shankar Exercise 14.4.3.

Before tackling this question, you may find it helpful to read the section “Paramagnetic Resonance” on pp. 392–394 of Shankar.