The Bound-State Aharonov-Bohm Effect

There are a number of situations in which the electromagnetic vector potential affects the physical properties of a particle, even though the electric and magnetic potentials are identically zero throughout the region accessible to the particle. The best-known example is the Aharonov-Bohm effect (Shankar pp. 497–499).

However, a variant—the “bound-state Aharonov-Bohm effect” (Ballentine pp. 323–325)—is easier to analyze rigorously in the wave-function formulation of quantum mechanics:

A particle of mass $M$ and charge $q$ is confined to the interior of a toroidal box, so that its cylindrical coordinates $(\rho, \phi, z)$ satisfy $a < \rho < b$, $0 < z < l$. The magnetic field is nonzero in the region $\rho < a$, but vanishes everywhere inside the box.

We take the electromagnetic scalar potential to be zero.

The electromagnetic vector potential inside the box must satisfy two conditions: (1) $B = \nabla \times A = 0$. (2) For any closed path $C$ that encircles the “hole” in the box, $\int_C A \cdot dl = \int_S \nabla \times A \cdot dS = \int_S B \cdot dS = \Phi$, the flux through the hole. These conditions are met by $A_{\rho} = A_z = 0$, $A_\phi = \Phi/(2\pi\rho)$, which also obeys the Coulomb gauge condition, $\nabla \cdot A = 0$. Inside the box, the particle’s Hamiltonian is

$$H = -\frac{\hbar^2}{2M} \nabla^2 + \frac{i\hbar q}{2Mc^2} [2A \cdot \nabla + (\nabla \cdot A)] + \frac{q^2}{2Mc^2} A \cdot A$$

$$= -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right) + \frac{i\hbar q\Phi}{2\pi Mc^2} \frac{\partial}{\partial \phi} + \frac{q^2\Phi^2}{8\pi^2 Mc^2 \rho^2}.$$ 

The stationary states (which must vanish on the surface of the box) satisfy

$$H \psi_{n,m,k_z} = \varepsilon(n, m, k_z) \psi_{n,m,k_z}, \quad \text{where} \quad \psi_{n,m,k_z}(\rho, \phi, z) = R_n(\rho)e^{im\phi}\sin k_z z.$$

Here $n$, $m$, and $lk_z/\pi$ are all integers, and $R_n$ is a solution of

$$R''_n + R'_n/\rho + (\alpha^2 - \nu^2/\rho^2)R_n = 0,$$

with

$$\alpha = \sqrt{2M\varepsilon/\hbar^2 - k_z^2}, \quad \nu = m - \Phi q/\hbar c.$$

The transformation $\rho \rightarrow \tilde{\rho}/\alpha$ converts Eq. (1) into Bessel’s equation. Thus,

$$R_n(\rho) = A_n J_\nu(\rho/\alpha_n) + B_n N_\nu(\rho/\alpha_n),$$

where $\alpha_n$ (and $\varepsilon$) is determined, along with $B_n/A_n$, by the conditions $R_n(a) = R_n(b) = 0$.

The key point is that the values of $\varepsilon$ that satisfy the boundary conditions depend on the enclosed flux $\Phi$ (via $\nu$), even though $B = 0$ throughout the region in which the wave function is non-vanishing. Possible interpretations are that the electromagnetic vector potential is physically significant (even though its observable effects must be gauge-invariant), or that the magnetic field acts non-locally.