The Bound-State Aharonov-Bohm Effect

There are a number of situations in which the electromagnetic vector potential affects the physical properties of a particle, even though the electric and magnetic potentials are identically zero throughout the region accessible to the particle. The best-known example is the Aharonov-Bohm effect (Shankar pp. 497–499).

However, a variant—the “bound-state Aharonov-Bohm effect” (Ballentine pp. 323–325)—is easier to analyze rigorously in the wave-function formulation of quantum mechanics:

A particle of mass \( M \) and charge \( q \) is confined to the interior of a toroidal box, so that its cylindrical coordinates \((\rho, \phi, z)\) satisfy \( a < \rho < b \), \( 0 < z < l \). The magnetic field is nonzero in the region \( \rho < a \), but vanishes everywhere inside the box.

We take the electromagnetic scalar potential to be zero. The electromagnetic vector potential inside the box must satisfy two conditions: (1) \( B = \nabla \times A = 0 \). (2) For any closed path \( C \) that encircles the “hole” in the box, \( \oint_C A \cdot dl = \int_S \nabla \times A \cdot dS = \int_S B \cdot dS = \Phi \), the flux through the hole. These conditions are met by \( A_\rho = A_z = 0 \), \( A_\phi = \Phi/(2\pi \rho) \), which also obeys the Coulomb gauge condition, \( \nabla \cdot A = 0 \).

Inside the box, the particle’s Hamiltonian is

\[
H = -\frac{\hbar^2}{2M} \nabla^2 + \frac{i\hbar q}{2Mc} [2A \cdot \nabla + (\nabla \cdot A)] + \frac{q^2}{2Mc^2} A \cdot A \\
= -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right) + \frac{i\hbar q\Phi}{2\pi Mc \rho^2} \frac{\partial}{\partial \phi} + \frac{q^2 \Phi^2}{8\pi^2 Mc^2 \rho^2}.
\]

The stationary states (which must vanish on the surface of the box) satisfy

\[
H \psi_{n,m,k_z} = \varepsilon(n, m, k_z) \psi_{n,m,k_z}, \quad \text{where} \quad \psi_{n,m,k_z}(\rho, \phi, z) = R_n(\rho) e^{im\phi} \sin k_z z.
\]

Here \( n, m, \) and \( l k_z/\pi \) are all integers, and \( R_n \) is a solution of

\[
R_n'' + R_n'/\rho + (\alpha^2 - \nu^2/\rho^2) R_n = 0,
\]

with

\[
\alpha = \sqrt{2M\varepsilon/\hbar^2 - k_z^2}, \quad \nu = m - \Phi q/hc.
\]

The transformation \( \rho \rightarrow \tilde{\rho}/\alpha \) converts Eq. (1) into Bessel’s equation. Thus,

\[
R_n(\rho) = A_n J_\nu(\rho/\alpha_n) + B_n N_\nu(\rho/\alpha_n),
\]

where \( \alpha_n \) (and \( \varepsilon \)) is determined, along with \( B_n/A_n \), by the conditions \( R_n(a) = R_n(b) = 0 \).

Key point: The values of \( \varepsilon \) that satisfy the boundary conditions depend on the enclosed flux \( \Phi \) (via \( \nu \)), even though \( B = 0 \) throughout the region in which \( \psi \neq 0 \). Two alternative interpretations are plausible: (1) \( B \) acts non-locally—an idea that is abhorrent to many physicists. (2) The electromagnetic vector potential is physically significant despite being gauge-dependent; this significance is such that all observable effects of \( A \) are gauge-invariant.