Reduction of a Dyadic to a Sum of Spherical Tensor Operators

• As an example of a reducible rank-2 tensor, consider the dyadic

$$T_{jk} = U_j V_k,$$

where $U$ and $V$ are vector operators satisfying

$$[U_j, J_k] = i\hbar \sum_l \epsilon_{jkl} U_l, \quad [V_j, J_k] = i\hbar \sum_l \epsilon_{jkl} V_l.$$

It is an identity that

$$U_j V_k = \left[ \frac{1}{3} U \cdot V \delta_{jk} \right] + \left[ \frac{1}{2} (U_j V_k - U_k V_j) \right] + \left[ \frac{1}{2} (U_j V_k + U_k V_j) - \frac{1}{3} U \cdot V \delta_{jk} \right].$$

The three parts in square brackets have very different properties:

$$\frac{1}{3} U \cdot V \delta_{jk} = \text{a multiple of the identity matrix},$$

$$\frac{1}{2} (U_j V_k - U_k V_j) = \frac{1}{2} \sum_l \epsilon_{jkl} (U \times V)_l, \text{ an asymmetric (traceless) tensor},$$

$$\frac{1}{2} (U_j V_k + U_k V_j) - \frac{1}{3} U \cdot V \delta_{jk} = \text{a symmetric, traceless tensor}.$$

We will show below that each of these parts is associated with a spherical tensor operator, each of which transforms as a different irreducible representation of the rotation group SO(3).

• Associated with the vector operators $U$ and $V$ are a pair of rank-1 spherical tensor operators $U^{(1)}_q$ and $V^{(1)}_q$, where

$$U^{(1)}_1 = \frac{1}{\sqrt{2}} (-U_x - iU_y), \quad U^{(1)}_0 = U_z, \quad U^{(1)}_{-1} = \frac{1}{\sqrt{2}} (U_x - iU_y),$$

and similarly for $V$.

These two rank-1 spherical tensor operators can be combined to form three compound spherical tensor operators:

1. A rank-0 operator,

$$T^{(0)}_0 = \frac{1}{3} \left( U^{(1)}_1 V^{(1)}_{-1} - U^{(1)}_0 V^{(1)}_0 + U^{(1)}_{-1} V^{(1)}_1 \right) = -\frac{1}{3} U \cdot V,$$

which is associated with the scalar $S = U \cdot V$.

2. A rank-1 operator $T^{(1)}_q$, with

$$T^{(1)}_1 = \frac{1}{\sqrt{2}} \left( U^{(1)}_1 V^{(1)}_0 - U^{(1)}_0 V^{(1)}_1 \right) = \frac{i}{2} (U \times V)_x + \frac{1}{2} (U \times V)_y = -\frac{i}{\sqrt{2}} (U \times V)^{(1)}_1,$$

and

$$T^{(1)}_{-1} = \frac{i}{\sqrt{2}} (U \times V)^{(1)}_{-1},$$

which is associated with the vector $W = U \times V$. 
3. A rank-2 operator $T_q^{(2)}$, where

$$T_{\pm 2}^{(2)} = U_{\pm 1}^{(1)} V_{\pm 1}^{(1)},$$
$$T_{\pm 1}^{(2)} = \frac{1}{\sqrt{2}} \left( U_{\pm 1}^{(1)} V_0^{(1)} + U_0^{(1)} V_{\pm 1}^{(1)} \right),$$
$$T_0^{(2)} = \frac{1}{\sqrt{6}} \left( U_1^{(1)} V_{-1}^{(1)} + 2 U_0^{(1)} V_1^{(1)} + U_{-1}^{(1)} V_1^{(1)} \right),$$

which is associated with the quadrupole tensor

$$Q_{jk} = \frac{1}{2} (U_j V_k + U_k V_j) - \frac{1}{3} U \cdot V \delta_{jk}.$$

• It is obvious that

$$\frac{1}{3} U \cdot V \delta_{jk} = \begin{pmatrix}
-T_0^{(0)} & 0 & 0 \\
0 & -T_0^{(0)} & 0 \\
0 & 0 & -T_0^{(0)}
\end{pmatrix}.$$

After some tedious algebra, one can also show that

$$\frac{1}{2} (U_j V_k - U_k V_j) = \frac{1}{2} \begin{pmatrix}
0 & i \sqrt{2} T_0^{(1)} & T_1^{(1)} + T_{-1}^{(1)} \\
-i \sqrt{2} T_0^{(1)} & 0 & -i \left( T_1^{(1)} - T_{-1}^{(1)} \right) \\
-T_1^{(1)} + T_{-1}^{(1)} & i \left( T_1^{(1)} - T_{-1}^{(1)} \right) & 0
\end{pmatrix},$$

and

$$\frac{1}{2} (U_j V_k + U_k V_j) - \frac{1}{3} U \cdot V \delta_{jk} =$$

$$\frac{1}{2} \begin{pmatrix}
T_2^{(2)} + T_{-2}^{(2)} - \sqrt{\frac{2}{3}} T_0^{(2)} & -i \left( T_2^{(2)} - T_{-2}^{(2)} \right) & -T_1^{(2)} + T_{-1}^{(2)} \\
-i \left( T_2^{(2)} - T_{-2}^{(2)} \right) & -T_2^{(2)} - T_{-2}^{(2)} - \sqrt{\frac{2}{3}} T_0^{(2)} & i \left( T_1^{(2)} + T_{-1}^{(2)} \right) \\
-T_1^{(2)} + T_{-1}^{(2)} & i \left( T_1^{(2)} + T_{-1}^{(2)} \right) & \sqrt{\frac{2}{3}} T_0^{(2)}
\end{pmatrix}.$$

Each of these parts is irreducible, i.e., the nonzero elements of each matrix form an invariant subspace under the action of $U[R]$. 