non-holonomic constraint

Goldstein:

\[ f(q, \dot{q}) \]

\[ S = \int_{t_1}^{t_2} \left[ L + \lambda(t) f(q, \dot{q}) \right] dt \]

\[ \delta S = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \delta \lambda f \right] dt \]

by path

\[ = \int_{t_1}^{t_2} \left[ \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \lambda \frac{\partial f}{\partial q} - \frac{d}{dt} \left( \lambda \frac{\partial f}{\partial \dot{q}} \right) \right) \delta q \right. \]

\[ + \delta \lambda f \left. \right] dt = 0 \]

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \lambda \frac{\partial f}{\partial q} - \frac{d}{dt} \left( \lambda \frac{\partial f}{\partial \dot{q}} \right) = 0 \]

\[ f = 0 \]

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \lambda \frac{\partial f}{\partial q} - \lambda \frac{\partial f}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial f}{\partial q} = 0 \]

1st order differential equation for \( \lambda \) \( \Rightarrow \) need \( \dot{q}(0) \)

\( \lambda \) is a force \( \Rightarrow \) need to know acceleration at \( t=0 \)

Contradiction with the general principle of CM:

a trajectory is uniquely determined by specifying \( q(0) \) and \( \dot{q}(0) \). No initial acceleration is needed.
Variational principle works only for a special class of non-holonomic constraints
\[ f(q) = A(q) \dot{q} \] (Pfaffian)

\[ \lambda \frac{\partial L}{\partial \dot{q}} - \dot{\lambda} \frac{\partial L}{\partial q} = \lambda \frac{\partial A(q)}{\partial \dot{q}} - \dot{\lambda} \frac{\partial A(q)}{\partial q} \]

\[ = \lambda \frac{\partial A(q)}{\partial \dot{q}} - \dot{\lambda} A(q) - \dot{\lambda} \frac{\partial A(q)}{\partial q} + \frac{\partial A(q)}{\partial q} \dot{q} \]

\[ = \dot{\lambda} A(q) = \mu A(q) \]

\[ \dot{\lambda} = \text{total time derivative} = \mu \]

\[ \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = -\mu A(q) \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \mu A(q) = \mu \frac{\partial A(q)}{\partial q} \]

Only for \[ f(q) = A(q) \dot{q} \]

Generalization for an arbitrary set of Pfaffian constraints is trivial

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \sum_{i} \mu_{a} A_{a} = \sum_{a=1}^{k} \frac{\partial L}{\partial q_{a}} \]