Problem 1 A pendulum consists of a uniform rigid rod of length $L$, mass $M$, and of a snail of mass $M/3$ which can crawl along the rod (Fig.1—turn the page over). The rod is pivoted at one end and swings in a vertical plane. Initially, the snail is at the pivot-end of the rod and then it crawls slowly with constant speed $v$ along the rod towards the bottom end of the rod. Assume that the snail can be treated as a point mass.

a) Construct the Lagrangian for the rod-snail system and derive the equations of motion. [20 points]

When the snail has crawled distance $\ell$, the moment of inertia is

$$I = \frac{1}{3}ML^2 + \frac{1}{3}M\ell^2.$$

Lagrangian

$$L = \frac{I\dot{\theta}^2}{2} + \frac{Mv^2}{6} + \frac{1}{2}MgL\cos\theta + \frac{1}{3}MgL\cos\theta.$$

E.o.m.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(I\dot{\theta}) = -\frac{1}{2}MgL\sin\theta - \frac{1}{3}MgL\sin\theta.$$

$$I\ddot{\theta} + \frac{dI}{dt}\dot{\theta} = -Mg\left(\frac{L}{2} + \frac{\ell}{3}\right).$$

$$\frac{dI}{dt} = \frac{2}{3}M\ell\dot{\ell}$$

$$\ddot{\theta} + \frac{\ell\dot{\theta}}{L^2 + \ell^2} = -g\sin\theta \frac{\frac{2}{3}L + \ell}{L^2 + \ell^2}.$$

b) Find the frequency of small oscillations of the pendulum when the bug has crawled a distance $\ell$ along the rod. Assume that the snail crawls so slowly that the change in $\ell$ over the period of oscillations can be neglected (that is, discard the $\dot{\ell}$ term in the equation and treat $\ell$ as a constant). [20 points]

Neglecting the term containing $\dot{\ell}$ and expanding $\sin\theta = \theta$, we obtain

$$\ddot{\theta} = -g\theta \frac{\frac{2}{3}L + \ell}{L^2 + \ell^2}$$

and

$$\omega = \sqrt{g\frac{\frac{2}{3}L + \ell}{L^2 + \ell^2}}.$$

Problem 2 Consider a damped linear oscillator described by the following equation of motion

$$\ddot{q} + 2\gamma\dot{q} + \omega^2q = 0. \quad (1)$$
a) Assuming the Lagrangian of this system can be written as

\[ L = f(t) \left( \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right), \]  

find such a function \( f(t) \) that the Euler equation reproduces the equation of motion (1). To define \( f(t) \) uniquely, assume that \( f(0) = 1 \). [10 points]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}
\]

\[
\frac{d}{dt} (f(t) m \dot{q}) = -f(t) m \omega^2 q
\]

\[
f m \ddot{q} + f m \dot{q} = -f(t) m \omega^2 q
\]

\[
\ddot{q} + \frac{f}{f} \dot{q} = -\omega^2 q.
\]

The e.o.m. is reproduced, if

\[
\frac{f}{f} = 2 \gamma.
\]

Solving this equation with the initial condition \( f(0) = 1 \), we obtain

\[
f(t) = \exp(2 \gamma t).
\]

b) Construct the Hamiltonian \( H(q, p) \) using the Lagrangian from Eq. (2) with \( f(t) \) found in part a). [15 points]

\[
p = \frac{\partial L}{\partial \dot{q}} = f(t) m \dot{q}
\]

\[
\dot{q} = \frac{p}{mf(t)}.
\]

The Hamiltonian

\[
H = \dot{q} p - L = \frac{p^2}{2m f(t)} + \frac{1}{2} f(t) m \omega^2 q^2
\]

\[
= e^{-2 \gamma t} \frac{p^2}{2m} + e^{2 \gamma t} \frac{1}{2} m \omega^2 q^2.
\]

c) For the generating function

\[
F_2(q, P, t) = \exp(\gamma t) qP,
\]

find the transformed Hamiltonian \( K(Q, P, t) \) and show that \( K \) is conserved. [15 points]

\[
p = \frac{\partial F_2}{\partial q} = e^{\gamma t} P
\]

\[
Q = \frac{\partial F_2}{\partial P} = e^{\gamma t} q
\]

\[
q = e^{-\gamma t} Q
\]

\[
K = H + \frac{\partial F_2}{\partial t} = H + \gamma qP e^{\gamma t}.
\]

\( H \) is new variables

\[
H = e^{-2 \gamma t} \frac{(e^{\gamma t} P)^2}{2m} + e^{2 \gamma t} \frac{1}{2} m \omega^2 (e^{-\gamma t} Q)^2 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 Q^2
\]
\[ K = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 Q^2 + \gamma (e^{-\gamma t}Q) Pe^{\gamma t} = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 Q^2 + \gamma Q P. \]

\( K \) does not depend on time and is, therefore, conserved.

**Problem 3** A non-linear oscillator has a potential given by

\[ V(x) = \frac{1}{2} k x^2 - \frac{\lambda}{3} m x^3, \]

where \( \lambda \) is a small parameter. Find the solution of the equation of motion to first order in \( \lambda \), assuming that \( x(0) = 0 \) and \( \dot{x}(0) = v_0 \). [40 points] Hint: apply the initial conditions at each step of the perturbation theory.

E.o.m.

\[ \ddot{x} + \omega_0^2 x = \lambda x^2, \]

where \( \omega_0^2 = k/m \). Perturbation theory

\[ x = x_0 + x_1, \]

where \( x_0 = A \cos \omega t + B \sin \omega t \). To zeroth order in \( \lambda \), substitute \( x = x_0 \) into the e.o.m. and neglect the non-linear term:

\[-A \omega^2 \cos \omega t + A \omega_0^2 \cos \omega t - B \omega^2 \sin \omega t + B \omega_0^2 \sin \omega t = 0 \rightarrow \omega = \omega_0.\]

Taking into account the initial conditions,

\[ x_0 = \frac{v_0}{\omega_0} \sin \omega_0 t. \]

To first order in \( \lambda \), keep only the \( \lambda x_0^2 \) term in the RHS of the e.o.m.

\[ \ddot{x}_1 + \omega_0^2 x_1 = \lambda x_0^2 = \lambda \left( \frac{v_0}{\omega_0} \right)^2 \sin^2 \omega_0 t = \frac{1}{2} \lambda \left( \frac{v_0}{\omega_0} \right)^2 - \frac{1}{2} \lambda \left( \frac{v_0}{\omega_0} \right)^2 \cos 2 \omega_0 t. \]

A general solution of the inhomogeneous linear differential equation is a sum of a fundamental solution of the homogeneous equation and a particular solution of the inhomogeneous equation: \( x_1 = x_f + x_p \). A fundamental solution

\[ x_f = D \cos \omega_0 t + E \sin \omega_0 t. \]

We can look for a particular solution of the following form

\[ x_p = F + G \cos 2 \omega_0 t. \]

Substituting back into the equation, we find

\[ F = \frac{\lambda}{2 \omega_0^2} \left( \frac{v_0}{\omega_0} \right)^2 \]

\[-4 G \omega_0^2 \cos 2 \omega_0 t + G \omega_0^2 \cos 2 \omega_0 t = - \frac{1}{2} \lambda \left( \frac{v_0}{\omega_0} \right)^2 \cos 2 \omega_0 t, \]

\[-3 G \omega_0^2 \cos 2 \omega_0 t = - \frac{1}{2} \lambda \left( \frac{v_0}{\omega_0} \right)^2 \cos 2 \omega_0 t, \]

\[ G = \frac{\lambda}{6 \omega_0^2} \left( \frac{v_0}{\omega_0} \right)^2. \]
The zeroth order solution already satisfies the initial condition $\dot{x}_0(0) = v_0$. Therefore, the first-order correction must satisfy $\dot{x}_1(0) = 0$, which gives $E = 0$. Finally,

$$x_1(t) = -\frac{2\lambda}{3\omega_0^2} \left( \frac{v_0}{\omega_0} \right)^2 \cos \omega_0 t + \frac{\lambda}{2\omega_0^2} \left( \frac{v_0}{\omega_0} \right)^2 + \frac{\lambda}{6\omega_0^2} \left( \frac{v_0}{\omega_0} \right)^2 \cos 2\omega_0 t$$