1. **Solution:** Lagrangian

\[ L = \frac{1}{2} m \dddot{z}^2 + \frac{1}{2} m \dddot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - m g z. \]

As the wire rotates at a constant angular frequency, \( \dot{\phi} = \omega = \text{const} \) and

\[ L = \frac{1}{2} m \dddot{z}^2 + \frac{1}{2} m \dddot{r}^2 + \frac{1}{2} m r^2 \omega^2 - m g z. \]

Constraint function

\[ f = z - A r^4. \]

Eqs of motions

\[
\frac{d}{dt} \frac{\partial L}{\partial \dddot{z}} = \frac{\partial L}{\partial z} + \lambda \frac{\partial f}{\partial z} \rightarrow m \dddot{z} = -m g + \lambda \\
\frac{d}{dt} \frac{\partial L}{\partial \dddot{r}} = \frac{\partial L}{\partial r} + \lambda \frac{\partial f}{\partial r} \rightarrow m \dddot{r} = m r \omega^2 - 4 \lambda A r^3.
\]

If the bead is rotating in a circle of radius \( R \), \( r = R , \dddot{r} = 0 \), and \( \dddot{z} = 0 \) →

\[ \lambda = m g \]
\[ m R \omega^2 = 4 \lambda A R^3 \]

\[ \omega = 2 \sqrt{\frac{A g R}{E}} \]

2. **Solution:**

a.

\[ E = \frac{p^2}{2m} + m g z \]

Action variable

\[
J = \int pdz = \int_0^{z_0} (-) \sqrt{2m (E - mg z)}dz + \int_0^{z_0} \sqrt{2m (E - mg z)}dz \\
\quad \quad = 2 \int_0^{z_0} \sqrt{2m (E - mg z)}dz = 2 \sqrt{2m g} \int_0^{z_0} (z_0 - z)^{1/2} dz = \frac{4 \sqrt{2}}{3} m \sqrt{g} \left( \frac{E}{mg} \right)^{3/2} = C E^{3/2},
\]

where

\[ z_0 = \frac{E}{mg} ; C \equiv \frac{4 \sqrt{2}}{3} \frac{1}{\sqrt{mg}}. \]
Expressing the Hamiltonian in terms of $J$

\[ H = E = \frac{J^{2/3}}{C^{2/3}} \]

Frequency

\[ \nu = \frac{\partial H}{\partial J} = \frac{2}{3C^{2/3}J^{1/3}} \]

Period

\[ T = \frac{1}{\nu} = \frac{3C^{2/3}J^{1/3}}{2} = \frac{3}{2} C^{2/3} C^{1/3} E^{1/2} = \frac{3}{2} C E^{1/2} = \frac{3}{2} \frac{1}{3 \sqrt{mg}} E^{1/2} \]
\[ = 2\sqrt{\frac{E^{1/2}}{mg}} \]

b. Assuming the naive quantization rule $J = \hbar n$, we obtain for the energy of quantized levels

\[ E_n = \frac{J^{2/3}}{C^{2/3}} = \left( \frac{3\sqrt{2}}{8} \right)^{2/3} \hbar^{2/3} n^{2/3} m^{1/3} g^{2/3}. \]

The exact quantum-mechanical formula (see S. Flugge, Practical Quantum Mechanics I, Problem 40) gives in the limit of large $n$

\[ E_n = \left( \frac{3\pi\sqrt{2}}{8} \right)^{2/3} \hbar^{2/3} n^{2/3} m^{1/3} g^{2/3}. \]

By comparing the two results, we see that the correct quantization rule must have been $J = \pi \hbar n$

3. **Solution:** The Hamiltonian is

\[ H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mg y. \]

The variable $x$ is cyclic and thus $p_x$ is a constant. In addition, the energy is conserved. One of the “new momenta” (constants $\alpha_j$) is just the energy $E$, other one is conjugated to $x$. The principal function can be written as

\[ S(x, \alpha_x, y, \alpha) = \alpha x + W_y(y, E) - Et. \]

The first term written using the fact that the part of the principal function, corresponding to a cyclic variable ($x$) satisfies

\[ p_x = \alpha_x = \frac{\partial W_x}{\partial x} \rightarrow W_x = \alpha x. \]

The Hamilton-Jacobi equation

\[ \frac{1}{2m} \left[ \left( \frac{\partial S}{\partial x} \right)^2 + \left( \frac{\partial S}{\partial y} \right)^2 \right] + mgy = -\frac{\partial S}{\partial t} \]

reduces to

\[ \frac{\alpha_x^2}{2m} + \frac{1}{2m} \left( \frac{\partial W_y}{\partial y} \right)^2 + mgy = E. \]

Solving for $\frac{\partial W_y}{\partial y}$ gives

\[ \frac{\partial W_y}{\partial y} = \sqrt{2mE - \alpha_x^2 - 2m^2gy}. \]
Integrating, we obtain

\[ W_y = -\frac{1}{3m^2g} (2mE - \alpha_x^2 - 2m^2gy)^{3/2} \]

and thus

\[ S (x, \alpha_x, y, \alpha) = \alpha_x x - \frac{1}{3m^2g} (2mE - \alpha_x^2 - 2m^2gy)^{3/2} - Et. \]

Now we need to find the “new coordinates” (constants \( \beta_j \)):

\[ \beta = \frac{\partial S}{\partial E} = -\frac{1}{mg} (2mE - \alpha_x^2 - 2m^2gy)^{1/2} - t \]
\[ \beta_x = \frac{\partial S}{\partial \alpha_x} = x + \frac{\alpha_x}{m^2g} (2mE - \alpha_x^2 - 2m^2gy)^{1/2}. \]

Solving the equation for \( \beta \) gives

\[ \frac{1}{mg} (2mE - \alpha_x^2 - 2m^2gy)^{1/2} = - (\beta + t). \]

Substituting this result into the equation for \( \beta_x \), we find

\[ x(t) = \beta_x - \frac{\alpha_x}{mg} (2mE - \alpha_x^2 - 2m^2gy)^{1/2} = \beta_x + \frac{\alpha_x}{m} (\beta + t). \]

Constants in this equations are determined from the initial conditions: \( x(0) = 0, \dot{x}(0) = v_0 \cos \theta \). We have

\[ x(0) = \beta_x + \frac{\alpha_x}{m} \beta = 0 \]
\[ \dot{x}(0) = \frac{\alpha_x}{m} = v_0 \cos \theta, \]

hence

\[ x(t) = v_0 t \cos \theta. \]

Coming back to the equation for \( \beta \) and solving it in terms of \( y \) gives

\[ y(t) = \frac{E}{mg} - \frac{\alpha_x^2}{2m^2g} - \frac{g(\beta + t)^2}{2}. \]

Initial conditions \( y(0) = 0 \) and \( \dot{y}(0) = v_0 \sin \theta \) lead to

\[ 0 = \frac{E}{mg} - \frac{\alpha_x^2}{2m^2g} - \frac{g\beta^2}{2} \]
\[ v_0 \sin \theta = -g\beta \]

Solving the second eqn for \( \beta = - (1/g) v_0 \sin \theta \) and substituting this result into the first equation gives

\[ \frac{\alpha_x^2}{2m^2g} = \frac{E}{mg} - \frac{v_0^2 \sin^2 \theta}{2g}. \]

Recalling that \( E = mv_0^2/2 \), we obtain

\[ \frac{\alpha_x^2}{2m^2g} = \frac{v_0^2 \cos^2 \theta}{2g} \]
\[ \alpha_x = mv_0 \cos \theta. \]

With these constants, the equation for \( y(t) \) assumes the familiar form

\[ y(t) = v_0 t \sin \theta - \frac{gt^2}{2}. \]