1. Consider a particle of mass $m$ which is constrained to move on the surface of a sphere of radius $R$. There are no external forces of any kind on the particle.

a) Derive the Hamiltonian of the particle. Is it conserved? [15 points]

b) Using the Hamiltonian equations of motion, prove that the motion of the particle is along a great circle of the sphere. [15 points]

NB: A great circle on a sphere is a circle on the sphere’s surface whose center is the same as the center of the sphere

Solution:

\[
\begin{align*}
    x &= R \sin \theta \cos \phi \\
    y &= R \sin \theta \sin \phi \\
    z &= R \cos \theta \\
    \dot{x} &= R \left( \dot{\theta} \cos \theta \cos \phi - \sin \theta \dot{\phi} \sin \phi \right) \\
    \dot{y} &= R \left( \dot{\theta} \cos \theta \sin \phi + \sin \theta \dot{\phi} \cos \phi \right) \\
    \dot{z} &= -R \dot{\theta} \sin \theta 
\end{align*}
\]

The Lagrangian

\[
L = T = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) = \frac{mR^2}{2} \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right).
\]

Moments

\[
\begin{align*}
    p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta} \\
    p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mR^2 \dot{\phi} \sin^2 \theta
\end{align*}
\]

The Hamiltonian

\[
H = \dot{\theta} p_\theta + \dot{\phi} p_\phi - L = \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{mR^2 \sin^2 \theta}
\]

$H$ does not depend on $\phi \rightarrow p_\phi$ is conserved

\[
\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0.
\]

By a suitable choice of the initial conditions, $p_\phi$ can always be made equal to zero. e.g., starting the motion at $\theta = 0$. Velocity

\[
\begin{align*}
    \dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2 \sin^2 \theta} = 0 \\
    mR^2 \dot{\phi} \sin^2 \theta &= 0.
\end{align*}
\]

As $\theta$ cannot be equal to zero for any instant of time, we conclude that $\dot{\phi} = 0$ or $\phi = \text{const}$. This equation defines a great circle.
2. Goldstein, Problem 10.5. [30 points]

\[ S = \frac{m\omega}{2} \left( q^2 + \alpha^2 \right) \cot \omega t - m\omega q \alpha \csc \omega t \]
\[ = \frac{m\omega}{2} \left( q^2 + \alpha^2 \right) \frac{\cos \omega t}{\sin \omega t} - m\omega q \alpha \frac{1}{\sin \omega t} \]

Hamilton-Jacobi equation

\[ H \left( p = \frac{\partial S}{\partial \dot{q}}, q \right) = \frac{1}{2m} \left[ \left( \frac{\partial S}{\partial q} \right)^2 + m^2 \omega^2 q^2 \right] = -\frac{\partial S}{\partial t} \]

\[ -\frac{\partial S}{\partial t} = \frac{m\omega}{2} \left( q^2 + \alpha^2 \right) \csc^2 \omega t - m\omega q \alpha \cot \omega t \csc \omega t \]
\[ \frac{\partial S}{\partial q} = m\omega q \cot \omega t - m\omega \alpha \csc \omega t \]
\[ \left( \frac{\partial S}{\partial \dot{q}} \right)^2 = (m\omega q \cot \omega t)^2 + (m\omega \alpha \csc \omega t)^2 - 2m^2 \omega^2 q \alpha \cot \omega t \csc \omega t \]
\[ = \frac{1}{2m} \left[ m^2 \omega^2 q^2 \left( \cot^2 \omega t + 1 \right) + (m\omega \alpha \csc \omega t)^2 - 2m^2 \omega^2 q \alpha \csc \omega t \right] \]
\[ = \frac{1}{2m} \left[ m^2 \omega^2 \left( q^2 + \alpha^2 \right) \csc^2 \omega t - 2m^2 \omega^2 q \alpha \csc \omega t \right] \]

Comparing the first and last lines, we see that they are the same.

Show that \( S \) generates a correct solution to the equations of motion:

\[ p = \frac{\partial S}{\partial q} = m\omega (q \cot \omega t - \alpha \csc \omega t) \]
\[ \beta = \frac{\partial S}{\partial \alpha} = m\omega (\alpha \cot \omega t - q \csc \omega t) \]

Solve the last equation for \( q = q(\alpha, \beta, t) \)

\[ q = \alpha \cos \omega t - \frac{\beta}{m\omega} \sin \omega t \]
\[ q_0 = q(t = 0) = \alpha \]
\[ \dot{q}_0 = \dot{q}(t = 0) = \frac{\beta}{m} \]

\[ p = m\omega^2 \left[ \alpha \cos \omega t - \frac{\beta}{m\omega} \sin \omega t \right] \cot \omega t - \alpha \csc \omega t \]
\[ = -m\omega \left[ \alpha \sin \omega t + \frac{\beta}{m\omega} \cos \omega t \right] = m\dot{q} \]

3. A tennis ball of mass \( m \) is bouncing off the floor. The total energy of the ball is \( E \). The ball is moving strictly along the vertical. The collision between the ball and the floor is perfectly elastic.

a. Derive the action-angle variables for the tennis ball and determine the period of motion. [30 points]

\[ E = \frac{p^2}{2m} + mgz \]
\[ p = \pm \sqrt{2m(E - mgz)} \]
The action variable

\[ J = \oint pdz = \int_{z_0}^{0} (-) \sqrt{2m(E - mgz)}dz + \int_{0}^{z_0} \sqrt{2m(E - mgz)}dz \]

\[ = 2 \int_{0}^{z_0} \sqrt{2m(E - mgz)} = 2\sqrt{2m^2g} \int_{0}^{z_0} (z_0 - z)^{1/2} dz = \frac{4\sqrt{2}}{3}m\sqrt{g} \left( \frac{E}{mg} \right)^{3/2} \equiv CE^{3/2}, \]

where

\[ z_0 = E/mg; \quad C = \frac{4\sqrt{2}}{3} \frac{1}{\sqrt{mg}}. \]

Expressing the Hamiltonian in terms of \( J \)

\[ H = E = \frac{J^{2/3}}{C^{2/3}} \]

Frequency

\[ \nu = \frac{\partial H}{\partial J} = \frac{2}{3C^{2/3}J^{1/3}} \]

Period

\[ T = \frac{1}{\nu} = \frac{3C^{2/3}J^{1/3}}{2} = \frac{3}{2}C^{2/3}C^{1/3}E^{1/2} = \frac{3}{2}CE^{1/2} = \frac{3}{2} \frac{4\sqrt{2}}{3} \frac{1}{\sqrt{mg}}E^{1/2} \]

\[ = 2\sqrt{2}E^{1/2} \frac{1}{\sqrt{mg}} \]

b. Suppose the tennis ball is a quantum-mechanical object. Using the results of part [a], surmise the dependence of the quantized energy levels on the level number. [10 points]

Assuming the naive quantization rule \( J = \hbar n \), we obtain for the energy of quantized levels

\[ E_n = \frac{J^{2/3}}{C^{2/3}} = \left( \frac{3\sqrt{2}}{8} \right)^{2/3} \frac{\hbar^{2/3}n^{2/3}}{m^{1/3}}g^{2/3}. \]

The exact quantum-mechanical formula (see, e.g., S. Flugge, Practical Quantum Mechanics I, Problem 40) gives in the limit of large \( n \)

\[ E_n = \left( \frac{3\pi\sqrt{2}}{8} \right)^{2/3} \frac{\hbar^{2/3}n^{2/3}}{m^{1/3}}g^{2/3}. \]

By comparing the two results, we see that the correct quantization rule must have been \( J = \pi \hbar n \). The wave-function of the problem is known as the Airy function.