1. Goldstein, 13-4

\[ \mathcal{L} = \frac{\hbar^2}{8\pi^2 m} \vec{\nabla}_\psi \cdot \vec{\nabla}_\psi^* + V \psi^* \psi + \frac{\hbar}{4\pi i} (\psi^* \psi_t - \psi \psi_t^*) \].  

(1)

We treat \( \psi \) and \( \psi^* \) as two independent fields. The Euler’s equation for \( \psi^* \)

\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \psi_t^*} + \frac{d}{dx^i} \frac{\partial \mathcal{L}}{\partial \psi_{x^i}^*} = \frac{\partial \mathcal{L}}{\partial \psi^*} \]

where \( \psi_{x^i}^* \equiv d\psi^*/dx^i \) with \( i = 1 \ldots 3 \) and summation over \( i \) is implied. Re-writing the first term in Eq.(1) as \( \frac{\hbar^2}{8\pi^2 m} \frac{d}{dx^i} \frac{d}{dx^i} \) and calculating the derivatives, we obtain

\[ -\frac{\hbar}{4\pi i} \psi_t + \frac{\hbar^2}{8\pi^2 m} \frac{d}{dx^i} \frac{d}{dx^i} = V \psi + \frac{\hbar}{4\pi i} \psi_t \]

or

\[ \frac{\hbar i}{2\pi} \psi_t = -\frac{\hbar^2}{8\pi^2 m} \nabla^2 \psi + V \psi. \]

Same for \( \psi^* \) except for the sign of the left-hand side.

Momenta

\[ \Pi_\psi = \frac{\partial \mathcal{L}}{\partial \psi_t} = \frac{\hbar}{4\pi i} \psi^* \]

\[ \Pi_{\psi^*} = \frac{\partial \mathcal{L}}{\partial \psi_t^*} = -\frac{\hbar}{4\pi i} \psi \]

Hamiltonian density

\[ \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \psi_t} \psi_t + \frac{\partial \mathcal{L}}{\partial \psi_t^*} \psi_t^* - \mathcal{L} = \frac{\hbar}{4\pi i} \psi^* \psi_t - \frac{\hbar}{4\pi i} \psi \psi_t^* - \frac{\hbar^2}{8\pi^2 m} \vec{\nabla}_\psi \cdot \vec{\nabla}_\psi^* - V \psi^* \psi - \frac{\hbar}{4\pi i} (\psi^* \psi_t - \psi \psi_t^*) \]

\[ = -\frac{\hbar^2}{8\pi^2 m} \vec{\nabla}_\psi \cdot \vec{\nabla}_\psi^* - V \psi^* \psi \]

2. Goldstein, 13-10.

The Lagrangian density depends on \( \psi_t, \psi_x, \) and \( \psi_{xx} \). We need first to generalize the least action principle for this case. Assuming, as in the class, that the trial field is \( \psi(x,t) = \psi_0(x,t) + \beta \xi(x,t) \) and forming the action as

\[ I = \int dt \int dx \mathcal{L}(\psi_t, \psi_x, \psi_{xx}), \]

we impose the condition \( dI/d\beta = \int dt \int dx d\mathcal{L}(\psi_t, \psi_x, \psi_{xx})/d\beta = 0 \). The derivative of \( \mathcal{L} \) is

\[ \frac{d\mathcal{L}}{d\alpha} = \frac{\partial \mathcal{L}}{\partial \psi_t} \xi_t + \frac{\partial \mathcal{L}}{\partial \psi_x} \xi_x + \frac{\partial \mathcal{L}}{\partial \psi_{xx}} \xi_{xx}. \]
Substitute the $\frac{d}{dt}$ and integrate by parts. In the last term, integrate by parts (with respect to $x$) twice. The Euler equation is

$$\frac{d}{dt} \frac{\partial L}{\partial \psi_t} - \frac{d}{dx} \frac{\partial L}{\partial \psi_x} + \frac{d^2}{dx^2} \frac{\partial L}{\partial \psi_{xx}} = 0.$$ 

Now,

$$\frac{\partial L}{\partial \psi_t} = \frac{1}{2} \psi_x,$$

$$\frac{\partial L}{\partial \psi_x} = \frac{1}{2} \psi_t + \frac{1}{2} \alpha \psi_x^2,$$

$$\frac{\partial L}{\partial \psi_{xx}} = -\nu \psi_{xx},$$

and the Euler equation reduces to

$$-\frac{1}{2} \psi_{xt} - \frac{1}{2} \psi_{xt} - \alpha \psi_x \psi_{xx} - \psi_{xxxx} = 0$$

which is the KdV equation for $\phi = \psi_x$.

3. In addition to a single-soliton solution discussed in the class, the sine-Gordon equation has other solutions. In particular,

$$\phi_1 (x, t) = 4 \tan^{-1} \left[ \frac{\sin (ut)}{u \gamma \sinh (\gamma^{-1} x)} \right]$$

$$\phi_2 (x, t) = 4 \tan^{-1} \left[ \frac{\sinh (u \gamma t)}{u \sinh (\gamma x)} \right]$$

$$\phi_3 (x, t) = 4 \tan^{-1} \left[ \frac{\sin (\gamma x)}{\sinh (u \gamma t)} \right]$$

are all supposed to be solutions of the sine-Gordon equation.

a) Verify that $\phi_{1,2,3}$ indeed satisfy the sine-Gordon equation.

Equation of motion

$$\phi_{xx} = \phi_{tt} + \sin \phi \quad (2)$$

For any solutions $\phi = 4 \tan^{-1} [(F(x) g(t))] \equiv 4 \alpha$. A little bit of trigonometry

$$\sin 4\alpha = 2 \sin (2\alpha) \cos (2\alpha) = 4 \sin \alpha \cos \alpha \left( \cos^2 \alpha - \sin^2 \alpha \right) = 4 \sin \alpha \cos^3 \alpha \left( 1 - \tan^2 \alpha \right) = 4 \tan \alpha \cos^4 \alpha \left( 1 - \tan^2 \alpha \right)$$

$$= 4 \tan \alpha \frac{1 - \tan^2 \alpha}{(1 + \tan^2 \alpha)^2} = 4 F G \frac{1 - (FG)^2}{(1 + (FG)^2)^2}$$

$$\frac{d\phi_3}{dx} = 4 \frac{F_x G}{1 + (FG)^2}$$

$$\frac{d^2\phi_3}{dx^2} = 4 \left( \frac{F_{xx} G}{1 + (FG)^2} - \frac{2 F_x^2 F G^3}{(1 + (FG)^2)^2} \right) = 4 \left( \frac{1 + (FG)^2}{(1 + (FG)^2)^2} \right)^2 \left( F_{xx} G \left( 1 + (FG)^2 \right) - 2 F_x^2 F G^3 \right)$$

$$\frac{d\phi_3}{dt} = 4 \frac{F G_t}{1 + (FG)^2}$$

$$\frac{d^2\phi_3}{dt^2} = 4 \left( \frac{F G_{tt}}{1 + (FG)^2} - \frac{2 G_t^2 G F^3}{(1 + (FG)^2)^2} \right) = \frac{4 F}{(1 + (FG)^2)^2} \left( G_{tt} F \left( 1 + (FG)^2 \right) - 2 G_t^2 G F^3 \right)$$
Substituting these results into Eq.(2), we obtain
\[
\left(F_{xx}G \left(1 + (FG)^2\right) - 2F_x^2FG^3\right) - \left(G_{tt}F \left(1 + (FG)^2\right) - 2G_t^2GF^3\right) = FG \left(1 - (FG)^2\right)
\]
or, dividing by \(FG \left(1 + (FG)^2\right)\)
\[
\frac{F_{xx}}{F} - \frac{G_{tt}}{G} - \frac{2F_x^2G^2}{1 + (FG)^2} + \frac{2G_t^2F^2}{1 + (FG)^2} = \frac{1 - (FG)^2}{1 + (FG)^2} \tag{3}
\]
For the solution \(\phi_3\) it is convenient to introduce a function
\[
G = \frac{1}{g}
\]
\[
G_t = -\frac{g_t}{g^2}
\]
\[
G_{tt} = -\frac{g_{tt}}{g^2} + 2\frac{g_t^2}{g^3}
\]
\[
G_{tt}/G = -\frac{g_{tt}}{g} + 2\frac{g_t^2}{g^2}.
\]
Eq. (3b) is re-written as
\[
\frac{F_{xx}}{F} + \frac{g_t}{g} - 2\frac{g_t^2}{g^2} - 2\frac{F_x^2}{F^2 + g^2} + 2\frac{g_t^2}{g^2} \frac{F^2}{F^2 + g^2} = \frac{g^2 - F^2}{g^2 + F^2}
\]
or
\[
\frac{F_{xx}}{F} + \frac{g_t}{g} - 2\frac{g_t^2}{g^2} - 2\frac{F_x^2}{F^2 + g^2} = \frac{g^2 - F^2}{g^2 + F^2}
\]
For the solution \(\phi_3\), \(f = u \sinh(\gamma x)\) and \(G = \sinh(\omega \gamma t)\)
\[
f_x = u\omega \cosh(\gamma x)
\]
\[
f_{xx} = u\omega^2 \sinh(\gamma x)
\]
\[
f_{xx}/f = \gamma^2
\]
\[
g_t = u\omega \cosh(\omega \gamma t)
\]
\[
g_{tt} = u^2\omega \sinh(\omega \gamma t)
\]
\[
g_{tt}/g = u^2\omega^2
\]
\[
(\gamma^2 + u^2 \omega^2) (\sinh^2 u \gamma t + u^2 \sinh^2 \gamma x) - 2u^2 \omega^2 \cosh^2 u \gamma t - 2u^2 \omega \cosh^2 u \gamma x = \sinh^2 (u \gamma t) - u^2 \sinh^2 u \gamma x
\]
\[
(\gamma^2 + u^2 \omega^2) (\sinh^2 u \gamma t + u^2 \sinh^2 \gamma x) - 2u^2 \omega^2 [2 + \sinh^2 u \gamma t + \sinh^2 u \gamma x] = \sinh^2 (u \gamma t) - u^2 \sinh^2 u \gamma x
\]
Equating the coefficients in front of \(\sinh^2 u \gamma t\) and \(\sinh^2 \gamma x\), we see that the equation is satisfied if
\[
\gamma^2 = \frac{1}{1 - u^2},
\]
Same for other solutions.

b) One of these solutions describes a soliton-soliton pair. Another one describes a solition-antisoliton pair. The remaining one describes the localized perturbation (so-called "breather"). Identify which one is which by plotting the solutions.
\(\phi_3\) : soliton-soliton
\(\phi_2\) : soliton-antisoliton
\(\phi_1\) : breather