1. [50 points] Following the discussion in the class and notes posted at http://www.phys.ufl.edu/~maslov/classmech/pendulum.pdf, find the period of oscillations for a pendulum which starts its motion at the angle $\phi_0$ to the vertical and with the initial kinetic energy $T_0$. Obtain an asymptotic result for the period in the limit $\phi_0 \ll 1$ and $T_0 \ll mgl$.

Solution

Energy conservation

$$\frac{m}{2} \ell^2 \left(\dot{\phi}\right)^2 - mgl \cos \phi = T_0 - mgl \cos \phi_0$$

$$\dot{\phi} = -\sqrt{\frac{2g}{\ell}} \left(\cos \phi - \cos \phi_0 + t_0\right),$$

where $t_0 = T_0/mgl$, where the choice of the sign corresponds to $\phi_0 > 0$ and subsequent downward motion. The period is then

$$T = 4 \sqrt{\frac{\ell}{2g}} \int_{\phi_0}^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0 + t_0}},$$

where $\phi_{\text{max}}$ is the maximum angle determined from the condition

$$\cos \phi_{\text{max}} = \cos \phi_0 - t_0.$$ 

The condition $-1 \leq \cos \phi_{\text{max}} \leq 1$ implies that $t_0 \leq \cos \phi_0 + 1$. For larger kinetic energies, the motion becomes unbounded. Equation for the period can be re-written as

$$T = 4 \sqrt{\frac{\ell}{2g}} \int_{\phi_0}^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_{\text{max}}}},$$

upon which the problem reduces to the one considered in the class with the only change $\phi_0 \to \phi_{\text{max}}$. Following the same steps as before:

$$\cos \phi - \cos \phi_{\text{max}} = 2 \left(\sin^2 \frac{\phi_{\text{max}}}{2} - \sin^2 \frac{\phi}{2}\right)$$

$$T = 2 \sqrt{\frac{\ell}{g}} \int_{\phi_0}^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{\sin^2 \left(\frac{\phi_{\text{max}}}{2}\right) - \sin^2 \left(\frac{\phi}{2}\right)}}$$

$$= 2 \sqrt{\frac{\ell}{g}} \int_{\phi_0}^{\phi_{\text{max}}} \frac{d\phi}{\sin \left(\frac{\phi_{\text{max}}}{2}\right) \sqrt{1 - \sin^2 \left(\frac{\phi}{2}\right)}}$$

New variable

$$\sin \alpha = \frac{\sin \left(\frac{\phi}{2}\right)}{\sin \left(\frac{\phi_{\text{max}}}{2}\right)}, \quad 0 \leq \alpha \leq \pi/2$$

$$\cos \alpha = \frac{1}{2} \frac{\cos \left(\frac{\phi}{2}\right)}{\sin \left(\frac{\phi_{\text{max}}}{2}\right)}$$

$$\frac{d\phi}{\sin \left(\frac{\phi_{\text{max}}}{2}\right)} = \frac{2 \cos \alpha}{\cos \left(\frac{\phi}{2}\right)} = \frac{2 \cos \alpha}{\sqrt{1 - \sin^2 \left(\frac{\phi}{2}\right)}} = \frac{\cos \alpha}{\sqrt{1 - \sin^2 \alpha \sin^2 \left(\frac{\phi_{\text{max}}}{2}\right)}}$$
Period

\[ T = 4 \sqrt{\frac{\ell}{g}} \int_0^{\pi/2} d\alpha \frac{1}{\sqrt{1 - \sin^2 \alpha \sin^2 (\phi_{max}/2)}} \]

(1)

\[ = 4 \sqrt{\frac{\ell}{g}} K (\sin (\phi_{max}/2)), \]

where \( K (k) \) is the elliptic integral of the 1st kind.

When \( \phi_0 \ll 1 \) and \( t_0 \ll 1 \), \( \phi_{max} \) is small too. Expanding both cosines in the relation

\[ \cos \phi_{max} = \cos \phi_0 - t_0, \]

we obtain

\[ 1 - \frac{\phi_{max}^2}{2} = 1 - \frac{\phi_0^2}{2} - t_0 \]

\[ \phi_{max}^2 = \phi_0^2 + 2t_0. \]

The square root in Eq.(1) can be expanded for for small \( \sin^2 (\phi_{max}/2) \approx \phi_{max}^2/4 \), which leads to

\[ T = 4 \sqrt{\frac{\ell}{g}} \int_0^{\pi/2} d\alpha \frac{1}{\sqrt{1 - \sin^2 \alpha \sin^2 (\phi_{max}/2)}} \]

\[ \approx 4 \sqrt{\frac{\ell}{g}} \int_0^{\pi/2} d\alpha \left( 1 + \frac{1}{2} \sin^2 \alpha \sin^2 (\phi_{max}/2) \ldots \right) \]

\[ \approx 4 \sqrt{\frac{\ell}{g}} \int_0^{\pi/2} d\alpha \left( 1 + \frac{1}{8} \sin^2 \alpha \phi_{max}^2 \ldots \right) \]

\[ = 4 \sqrt{\frac{\ell}{g}} \left( \frac{\pi}{2} + \frac{1}{8} \phi_{max}^2 \ldots \right) \]

\[ = 2\pi \sqrt{\frac{\ell}{g}} \left( 1 + \frac{1}{16} \phi_{max}^2 \ldots \right) \]

\[ = 2\pi \sqrt{\frac{\ell}{g}} \left( 1 + \frac{1}{16} \phi_0^2 + \frac{1}{8} t_0 \ldots \right). \]

The final result

\[ T = 2\pi \sqrt{\frac{\ell}{g}} \left( 1 + \frac{1}{16} \phi_0^2 + \frac{1}{8} t_0 \ldots \right) \]

2. [50 points] A disk rolls without slipping along a horizontal plane. The disk is constrained to remain vertical. Let be \( \psi \) be the angle between the plane of the disk and the \( x \) axis of a fixed frame and \( \theta \) be the angle measuring spinning of the disk about its center (see Fig. 1). Following the variational formulation, presented in the class, and neglecting the gravitational force, show that the center of mass of the disk moves in a circle, if \( \psi (t = 0) \neq 0 \), and along a straight line, if \( \psi (t = 0) = 0 \).

Solution

A disk rolls without slipping along a horizontal plane. The disk is constrained to remain vertical. Let be \( \psi \) be the angle between the plane of the disk and the \( x \) axis of a fixed frame and \( \theta \) be the angle measuring spinning of the disk about its center (see Fig. 1). Following the variational formulation, presented in the class, and neglecting the gravitational force, show that the center of mass of the disk moves in a circle, if \( \psi (t = 0) \neq 0 \), and along a straight line, if \( \psi (t = 0) = 0 \).
Solution

Lagrangian

\[ L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I_0\dot{\theta}^2 + \frac{1}{2}I_1\dot{\psi}^2 \]

Components of the angular velocity

\[ \omega_x = -\dot{\theta}\sin\psi \]
\[ \omega_y = \dot{\theta}\cos\psi \]

Rolling constraint

\[ \vec{v} = \vec{\omega} \times \vec{a} = \omega_y\vec{R}\dot{x} - \omega_x\vec{R}\dot{y}, \]

where \( \vec{a} \) is the vector from the contact point to the center of rotation. In our case, \( \vec{a} = (0, 0, R) \), where \( R \) is the radius of the disk. Constraint functions

\[ f_1 = R\dot{\theta}\cos\psi - \dot{x} \]
\[ f_2 = R\dot{\theta}\sin\psi - \dot{y} \]

Lagrange’s equations:

\[ m\ddot{x} = \mu_1 \frac{\partial f_1}{\partial \dot{x}} + \mu_2 \frac{\partial f_2}{\partial \dot{x}} = -\mu_1 \]
\[ m\ddot{y} = \ldots = -\mu_2 \]
\[ I_0\ddot{\theta} = \mu_1 \frac{\partial f_1}{\partial \theta} + \mu_2 \frac{\partial f_2}{\partial \theta} = \mu_1 R\cos\psi + \mu_2 R\sin\psi \]
\[ I_1\ddot{\psi} = 0. \]

From the last equation, it follows that \( \dot{\psi} = \text{const} \) or \( \psi = \psi(0) + \dot{\psi}(0) t \). From the constraint equations \( f_1 = f_2 = 0 \),

\[ R\cos\psi = \frac{\dot{x}}{\dot{\theta}} \]
\[ R\sin\theta = \frac{\dot{y}}{\dot{\theta}} \]

Substituting these two equations into the equation for \( \dot{\theta} \) gives

\[ I_0\ddot{\theta} = \mu_1 \dot{x} + \mu_2 \dot{y} \]

or, substituting the \( \mu_{1,2} \) from the equations for \( \dot{x} \) and \( \dot{y} \)

\[ I_0\ddot{\theta} = -m(\dot{x}\dot{x} + \dot{y}\dot{y}). \]

Integrating,

\[ I_0 \left( \dot{\theta} \right)^2 + m \left( (\dot{x})^2 + (\dot{y})^2 \right) = C = \text{const} \]

Recalling that \( \dot{x}^2 = \dot{\theta}^2 R^2 \cos^2\psi \) and \( \dot{y}^2 = \dot{\theta}^2 R^2 \sin^2\psi \), we see that

\[ I_0 \left( \dot{\theta} \right)^2 + mR^2\dot{\theta}^2 = \text{const}, \]

from which it follows that

\[ \dot{\theta} = \text{const} \]
\[ \theta = \theta (0) + \dot{\theta} (0) t. \]

This means that
\[
\begin{align*}
\dot{x} &= \dot{\theta} R \cos \psi (t) = \dot{\theta} (0) R \cos \left( \psi (0) + \dot{\psi} (0) t \right) \\
\dot{y} &= \dot{\theta} R \sin \psi (t) = \dot{\theta} (0) R \sin \left( \psi (0) + \dot{\psi} (0) t \right)
\end{align*}
\]
or
\[
\begin{align*}
x(t) &= x (0) + \frac{\dot{\theta} (0) R}{\psi (0)} \sin \left( \psi (0) + \dot{\psi} (0) t \right) \\
y(t) &= y (0) - \frac{\dot{\theta} (0) R}{\psi (0)} \cos \left( \psi (0) + \dot{\psi} (0) t \right)
\end{align*}
\]

\[
[x (t) - x (0)]^2 + [y (t) - y (0)]^2 = R^2 \frac{\dot{\psi}^2 (0)}{\psi^2 (0)}.
\]

\( \dot{\psi} (0) = 0 \) requires a special treatment. Coming back to Eqs. for \( \dot{x} \) and \( \dot{y} \) and setting \( \dot{\psi} (0) \) in there gives
\[
\begin{align*}
\dot{x} &= \dot{\theta} R \cos \psi (t) = \dot{\theta} (0) R \cos (\psi (0)) \\
\dot{y} &= \dot{\theta} R \sin \psi (t) = \dot{\theta} (0) R \sin (\psi (0))
\end{align*}
\]
or
\[
\begin{align*}
x (t) &= x (0) + \dot{\theta} (0) R \cos (\psi (0)) t \\
y (t) &= y (0) + \dot{\theta} (0) R \sin (\psi (0)) t,
\end{align*}
\]

which describes a motion along a straight line.
FIG. 1:

FIG. 2: