1. A sphere rolls without slipping along a horizontal surface. Using Eulerian angles and nonholonomic rolling constraint, show that the center of the sphere is moving along a straight line while the angular velocity remains constant in time.

*(Hint: use the projections of the angular velocity on the axes of the fixed system; see Goldstein, Derivation 14, p. 182.)*

**Solution:** $\Omega_{1,2,3}$ are the projections of the angular velocity on the axes of the fixed system.

a) Let’s first solve the problem in a simple way, without using the non-holonomic constraints.

Frictional force $f$ acts in the plane. Equations of motion

\[
\begin{align*}
    m\ddot{x}_1 &= f_1 \\
    m\ddot{x}_2 &= f_2.
\end{align*}
\]

This force also produces a torque about the c.o.m. of the sphere. The corresponding equations describing the dynamics of the angular momentum are

\[
\begin{align*}
    N_1 &= f_2 R = I \dot{\Omega}_1 \\
    N_2 &= -f_1 R = I \dot{\Omega}_2, \\
    N_3 &= 0.
\end{align*}
\]

where $I$ is the moment of inertia.

Rolling constraint

\[
\begin{align*}
    \dot{x}_1 &= R\Omega_2 \rightarrow \ddot{x}_1 = R \dot{\Omega}_2 \\
    \dot{x}_2 &= -R\Omega_1 \rightarrow \ddot{x}_2 = -R \dot{\Omega}_1.
\end{align*}
\]

Substituting expressions for $\ddot{x}_1$ into the equation of motion and using the equations for torque, we find that we have two results for $f_1$

\[
\begin{align*}
    f_1 &= m\ddot{x}_1 = mR \dot{\Omega}_2 \\
    f_1 &= -I \dot{\Omega}_2 \\
    \left( mR + \frac{I}{R} \right) \dot{\Omega}_2 &= 0.
\end{align*}
\]

As the coefficient in front of $\dot{\Omega}_2$ is strictly positive, the last equations can be satisfied only if

\[
\dot{\Omega}_2 = 0.
\]

Similarly, one shows that

\[
\dot{\Omega}_1 = 0.
\]

Then, it follows that $\ddot{x}_1 = \ddot{x}_2 = 0$. The ratio of velocity components is obtained by dividing the first constraint equation by the second one

\[
\frac{\dot{x}_1}{\dot{x}_2} = \frac{\Omega_2}{\Omega_1}
\]
Thus, we see that the sphere rolls along a straight line whose tangent is determined by the ratio of the angular velocities
\[ \tan \theta = \frac{\dot{x}_2}{\dot{x}_1} = \frac{\Omega_1}{\Omega_2}. \]

2) In order to apply the technique of non-holonomic constraints, we write the constraint equation as
\[ F_\alpha = A_\alpha^\alpha q_\alpha^\alpha = 0 \]
and add \( A_\alpha^\alpha = \partial F_\alpha / \partial q_\alpha \) multiplied by the Lagrange factors \( \mu_\alpha \) to the right-hand side of the Euler equation
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j^\alpha} - \frac{\partial L}{\partial q_j^\alpha} = \mu_\alpha A_j^\alpha. \]

The Lagrangian is
\[ L = \frac{1}{2} m (\dot{r})^2 + \frac{1}{2} I \sum_{i=1}^{3} \Omega_i^2, \]
where \( r \) is the position of the c.o.m. and \( \Omega_i \) are defined with respect to a stationary system (XYZ). Projecting the angular velocities on the axes of XYZ, we find
\[ \Omega_1 = \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi \\
\Omega_2 = \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi \\
\Omega_3 = \dot{\phi} + \dot{\psi} \cos \theta. \]

Substituting these expressions into \( \Omega_1^2 + \Omega_2^2 + \Omega_3^2 \), we find
\[ \Omega_1^2 + \Omega_2^2 + \Omega_3^2 = \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2 \dot{\phi} \dot{\psi} \cos \theta. \]

As before, the rolling constraint reads
\[ \dot{x}_1 = \Omega_2 R \\
\dot{x}_2 = -\Omega_1 R \]
or, using equations for \( \Omega_1 \) and \( \Omega_2 \),
\[ f_1 = \dot{\theta} \sin \phi - \dot{\psi} \sin \theta \cos \phi - \frac{\dot{x}_1}{R} = 0 \\
f_2 = \dot{\theta} \cos \phi + \dot{\psi} \sin \theta \sin \phi + \frac{\dot{x}_2}{R} = 0. \]

Equations of motion
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j^\alpha} - \frac{\partial L}{\partial q_j^\alpha} = \mu_1 \frac{\partial F_1}{\partial q_j^\alpha} + \mu_2 \frac{\partial F_2}{\partial q_j^\alpha}, \]
where \( j \) labels \( x_1, x_2, \phi, \psi, \) and \( \theta \). Performing differentiations, we find
\[ m\dot{x}_1 = -\mu_1 / R \\
m\dot{x}_2 = \mu_2 / R \\
I \frac{d}{dt} \left( \dot{\phi} + \dot{\psi} \cos \theta \right) = I \frac{d}{dt} \dot{\Omega}_3 = 0 \\
I \frac{d}{dt} \left( \dot{\psi} + \dot{\phi} \cos \theta \right) = \sin \theta (-\mu_1 \cos \phi + \mu_2 \sin \phi) \\
I \dot{\theta} + I \dot{\phi} \dot{\psi} \sin \theta = \mu_1 \sin \phi + \mu_2 \cos \phi \]
Note that \( \Omega_3 = \text{const.} \) Multiply the last equation by \( \sin \phi / \sin \theta \) and the one before the last one by \( \cos \phi \), and add them up. This gives a relation for \( \mu_2 \)

\[
\mu_2 = I \left( \ddot{\theta} \cos \phi + \dot{\phi} \dot{\psi} \sin \theta \cos \phi + \dot{\psi} \frac{\sin \phi}{\sin \theta} + \frac{\dot{\phi} \sin \phi \cos \theta}{\sin \theta} - \dddot{\phi} \sin \phi \right).
\]

Noticing that
\[
\dddot{\phi} + \dot{\psi} \cos \theta - \dot{\psi} \dot{\phi} \sin \theta = 0,
\]
the expression for \( \mu_2 \) is reduced to

\[
\mu_2 = \frac{I}{\Omega_1} - \frac{I}{R} \ddot{x}_2.
\]

Thus,

\[
\ddot{x}_2 = -R \ddot{\Omega}_1.
\]

On the other hand,

\[
m \ddot{x}_2 = \mu_2 = \frac{I}{\Omega_1} - \frac{I}{R} \ddot{x}_2.
\]

It follows then that

\[
\ddot{x}_2 = 0
\]

\[
\ddot{\Omega}_1 = 0.
\]

Similarly,

\[
\ddot{x}_1 = 0
\]

\[
\ddot{\Omega}_2 = 0.
\]

The conclusions are made in the same way as in the “simple” solution.

2. Goldstein, Problem 5.7

Choose the the principal axes so that \( I_1 < I_2 < I_3 \). Focus on the axis \( x_1 \) first. The angular velocity is almost aligned with this axis, so that we can write

\[
\omega = \omega_1 \dot{x}_1 + \varepsilon_2 \dot{x}_2 + \varepsilon_3 \dot{x}_3,
\]

where projections \( \varepsilon_2 \) and \( \varepsilon_3 \) are small.

The Euler equations for a torque-free motion

\[
\begin{align*}
I_1 \ddot{\omega}_1 &= \omega_2 \omega_3 (I_2 - I_3) \\
I_2 \ddot{\omega}_2 &= \omega_1 \omega_3 (I_3 - I_1) \\
I_3 \ddot{\omega}_3 &= \omega_1 \omega_2 (I_1 - I_2)
\end{align*}
\]

become

\[
\begin{align*}
I_1 \ddot{\omega}_1 &= \varepsilon_2 \varepsilon_3 (I_2 - I_3) \\
I_2 \ddot{\omega}_2 &= \omega_1 \varepsilon_3 (I_3 - I_1) \\
I_3 \ddot{\omega}_3 &= \omega_1 \varepsilon_2 (I_1 - I_2)
\end{align*}
\]

To lowest order in \( \varepsilon_2 \) and \( \varepsilon_3 \), the RHS of the first equation is equal to zero, which means that \( \omega_1 = \text{const.} \)

Differentiating the second equation and using the third one gives

\[
\ddot{\varepsilon}_2 = -\Omega_1^2 \varepsilon_2,
\]
where
\[ \Omega_1^2 = \omega_1^2 \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3} \]

Likewise,
\[ \varepsilon_3 = -\Omega_1^2 \varepsilon_3 \]

The motion is stable if
\[ \Omega_1^2 > 0 \]

which is the case because of the ordering of \( I \)'s. Thus rotation about the \( x_1 \) axis is stable. Components of \( \omega \),
transverse to this axis, rotate with the frequency
\[ \Omega_1 = \omega_1 \sqrt{\frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}}. \]

Results for other axes can be simply obtained by a cyclic permutation of indices \( 1, 2, 3 \). This gives
\[ \Omega_2^2 = \omega_2^2 \frac{(I_1 - I_2)(I_3 - I_2)}{I_1 I_3} \]

\( \Omega_2^2 < 0 \) because \( I_1 < I_2 \), and thus rotation about the \( x_2 \) axis is unstable. Finally,
\[ \Omega_3^2 = \omega_3^2 \frac{(I_2 - I_3)(I_1 - I_3)}{I_1 I_3} > 0 \]

and rotation about \( x_3 \) is stable.

3. Goldstein, Problem 4.23

Choose the frame as shown in the figure: the \( xy \) plane is tangent to the Earth surface, the \( z \) axis is along the radius. The pivoting point of the pendulum is on the \( z \) axis. Consider first the motion of the pendulum for stationary Earth. The pendulum swings in the plane normal to the \( xy \) plane. The motion in this plane is described by
\[ \begin{align*}
\dot{x} + \omega_0^2 x &= 0 \\
\dot{y} + \omega_0 y &= 0,
\end{align*} \]

where \( \omega_0 = \sqrt{\frac{g}{l}} \) and \( l \) is the length of the pendulum. Earth's rotation add extra terms to the RHS of these equations. For slow rotation, one can neglect the centrifugal force (proportional to \( \Omega^2 \)) and keep only the Coriolis force. Then the equations change to
\[ \begin{align*}
\dot{x} + \omega_0^2 x &= 2 (\mathbf{v} \times \mathbf{\Omega})_x = 2\dot{y}\Omega_z \\
\dot{y} + \omega_0 y &= 2 (\mathbf{v} \times \mathbf{\Omega})_y = -2\dot{x}\Omega_z,
\end{align*} \]

where \( \Omega_z = \Omega \cos \theta \). Multiplying the second equation by \( i \) adding up the two equations, we obtain
\[ \dot{\zeta} + 2i\Omega_z \dot{\zeta} + \omega_0^2 \zeta = 0, \]

where \( \zeta = x + iy \). A solution of the form \( \zeta \propto e^{i\omega t} \) gives an equation for \( \omega \)
\[ \omega^2 + 2\Omega_z \omega - \omega_0^2 = 0 \]

or
\[ \omega_{1,2} = -\Omega_z \pm \sqrt{\omega_0^2 + \Omega_z^2}. \]

As \( \Omega_z \ll \omega_0 \), one can expand the last result as
\[ \omega_{1,2} = -\Omega_z \pm \omega_0. \]
Hence the general solution for $\zeta$ is

$$\zeta = e^{-i\Omega_z t} \left( A e^{i\omega_0 t} + B e^{-i\omega_0 t} \right).$$

For $\Omega_z = 0$, $\zeta = x_0(t) + iy_0(t)$, where $x_0(t)$ and $y_0(t)$ are trajectories for $\Omega_z = 0$. Therefore,

$$\zeta = x + iy = e^{-i\Omega_z t} (x_0 + iy_0).$$

Separating real and imaginary parts, we find

$$x = x_0 \cos \Omega_z t + y_0 \sin \Omega_z t$$
$$y = -x_0 \sin \Omega_z t + y_0 \cos \Omega_z t.$$

The plane of the pendulum rotates about the $z$ axis with angular frequency $\Omega_z$. 