
**Solution:** $\eta_{1,2}$ are displacement of the masses from their equilibrium positions. Lagrangian

$$L = \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2 - \frac{1}{2}\left(k\eta_1^2 + k\eta_2^2 + 3k(\eta_1 - \eta_2)^2\right).$$

Equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_{1,2}} = \frac{\partial L}{\partial \eta_{1,2}}$$

or

$$m\ddot{\eta}_1 = -k\eta_1 - 3k(\eta_1 - \eta_2)$$
$$m\ddot{\eta}_2 = -k\eta_2 + 3k(\eta_1 - \eta_2)$$

Looking for solutions in a form of $\eta_{1,2} = A_{1,2}e^{i\omega t}$ gives

$$-\omega^2 A_1 = -\frac{k}{m}A_1 - 3\frac{k}{m}(A_1 - A_2)$$
$$-\omega^2 A_2 = -\frac{k}{m}A_2 + 3\frac{k}{m}(A_1 - A_2)$$

which is equivalent to finding the roots of the equation

$$\begin{vmatrix}
\omega^2 - 4k/m & 3k/m \\
3k/m & \omega^2 - 4k/m
\end{vmatrix} = 0.$$

$$(\omega^2 - 4k/m)^2 - (3k/m)^2 = 0$$
$$\omega_1^2 = 7k/m; \omega_2 = \sqrt{k/m}.$$

2. Find the period of oscillations for a quartic one-dimensional oscillator

$$U(x) = \frac{1}{4}ax^4.$$

**Solution:** energy conservation

$$\frac{1}{2}m\dot{x}^2 + U(x) = E.$$

Turning points, where $\dot{x} = 0$, are at $x = \pm A$. At these points, $U(x) = E$, so that $E = \frac{1}{4}aA^4$. Period

$$T = 4 \int_0^A \frac{dx}{\sqrt{(2/m)(E - U(x))}}$$
$$= 2\sqrt{2m} \int_0^A \frac{dx}{\sqrt{\frac{1}{2}aA^4 - \frac{1}{4}ax^4}}$$
$$= 4\sqrt{\frac{2m}{aA^2}} \int_0^A dx \frac{1}{\sqrt{1 - (x/A)^4}}$$
$$= 4C \sqrt{\frac{2m}{aA}}.$$
\[ C = \int_0^1 dy \frac{1}{\sqrt{1 - y^4}} = \frac{1}{4} B \left( \frac{1}{4}, \frac{3}{2} \right) \approx 1.31 \ldots \]

with \( B(x, y) = \Gamma(x) \Gamma(y) / \Gamma(x + y) \) being a beta-function. Finally,

\[ T = 7.42 \sqrt{\frac{m}{a}}. \]

Note that the period decreases with increasing amplitude.

3. Using the perturbation theory, find the eigenfrequencies and eigenmodes of a one-dimensional oscillator with potential energy

\[ U(x) = \frac{1}{2} kx^2 + \frac{1}{4} ax^4 \]

in the limit \( a \to 0 \). Which dimensionless parameter controls the convergence of the perturbation theory?

**Solution:** Equation of motion

\[ m \ddot{x} = -kx - ax^3 \]

or

\[ \ddot{x} + \omega_0^2 x = -ax^3 \rightarrow \]

\[ \frac{\omega_0^2}{\omega^2} \ddot{x} + \omega_0^2 x = -\alpha x^3 - \left( 1 - \frac{\omega_0^2}{\omega^2} \right) \ddot{x} \]

where \( \omega_0^2 = k/m \) and \( \alpha = a/m \). Seek a solution in a form

\[ x = A \cos \omega t + x_1 \ldots \]

\[ \omega = \omega_0 + \omega_1 + \ldots \]

To leading order in \( \alpha \), this gives

\[ \ddot{x}_1 + \omega_0^2 x_1 = -\alpha A^3 \cos^3 \omega t + 2\omega_0 \omega_1 A \cos \omega t. \]

Transform \( \cos^3 \omega t \) as

\[ \cos^3 \omega t = \cos \omega t \cos^2 \omega t = \cos \omega t \frac{1}{2} (1 + \cos 2\omega t) \]

\[ = \frac{1}{2} \cos \omega t + \frac{1}{2} \cos \omega t \cos 2\omega t = \frac{1}{2} \cos \omega t + \frac{1}{4} (\cos \omega t + \cos 3\omega t) \]

\[ = \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t. \]

The eqn of motion now take the form

\[ \ddot{x}_1 + \omega_0^2 x_1 = -\frac{3}{4} \alpha A^3 \cos \omega t - \frac{1}{4} \alpha A^3 \cos \omega t + 2\omega_0 \omega_1 A \cos \omega t. \]

The term oscillating with frequency \( \omega \) in the RHS looks like a force almost at resonance with the original frequency. This resonance is unphysical as our anharmonic oscillator is an isolated system. To avoid the resonance, we require that the coefficient of the \( \cos \omega t \) term vanishes. This gives

\[ \omega_1 = \frac{3}{8} \frac{A^2}{\omega_0}. \]

Finally, the equation for \( x_1 \) reduces to

\[ \ddot{x}_1 + \omega_0^2 x_1 = \frac{\alpha A^3}{4} \cos 3\omega t. \]
The solution is
\[ x_1 = -\frac{\alpha A^3}{32\omega_0} \cos 3\omega t. \]

Collecting everything together, the lowest order solution is
\[
x = A \cos \omega t + \frac{\alpha A^3}{32\omega_0} \cos 3\omega t
\]
\[
\omega = \omega_0 + \frac{3}{8} \frac{A^2}{\omega_0} = \omega_0 \left( 1 + \frac{3}{8} \frac{A^2}{\omega_0} \right).
\]

The dimensionless parameter controlling the convergence of the perturbative expansion is \( \frac{3}{8} \frac{A^2}{\omega_0} \).