1. Problem 1 [40 points]
A bead of mass $m$ slides without friction in a uniform gravitational field on a vertical hoop of radius $R$. The hoop rotates at a fixed angular velocity $\omega$ about its vertical diameter (see Fig. 1).

a) Write down the Lagrangian for the bead and derive the equations of motion. [10 points]

Solution
Placing the origin of the Cartesian reference frame at the center of the hoop, we get

- $x = R \sin \theta \cos \phi$
- $y = R \sin \theta \sin \phi$
- $z = -R \cos \theta$,

where $\phi = \omega t$ is the polar angle. Differentiating,

- $\dot{x} = \dot{\theta} \cos \theta \cos \phi - \omega \sin \theta \sin \phi$
- $\dot{y} = R \left( \dot{\theta} \cos \theta \sin \phi + \omega \sin \theta \cos \phi \right)$.

The Lagrangian

$$L = T - U = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + mg R \cos \theta$$

$$= \frac{1}{2} m R^2 \left( \dot{\theta}^2 + \sin^2 \theta \omega^2 \right) + mg R \cos \theta$$

E.O.M.:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \rightarrow$$

$$\ddot{\theta} = \sin \theta \cos \theta \omega^2 - \frac{g}{R} \sin \theta$$

b) Determine all stable equilibrium positions of the bead. [10 points]

Solution
Equilibrium positions correspond to $\ddot{\theta} = 0$ or

$$\sin \theta \left( \omega^2 \cos \theta - \frac{g}{R} \right) = 0.$$  

In the interval of physically distinguishable values of $\theta$ ($0 \leq \theta \leq \pi$), there are three roots:

$$\theta_1 = 0, \theta_2 = \pi, \cos \theta_3 = \frac{g}{R \omega^2}.$$  

The third root is real for $\omega > \sqrt{g/R}$.

Now we need to analyze the stability of the equilibrium positions. Expanding the RHS of Eq.(1) to the 1st order in $\theta$:

$$\ddot{\theta} = \theta \times 1 \times \omega^2 - \frac{g}{R} \theta = \theta \left( \omega^2 - \frac{g}{R} \right).$$

(2)
The position $\theta = 0$ is stable for $\omega < \sqrt{g/R}$ and unstable otherwise.

Expanding Eq. (1) in $\alpha = \pi - \theta$, we get

$$-\ddot{\alpha} = -\alpha \left( \omega^2 + \frac{g}{R} \right).$$

Since $\omega^2 + g/R > 0$, $\alpha$ increases with time. Hence, $\theta = \pi$ is an unstable position. Finally, expanding Eq. (1) in $\alpha = \theta - \theta_3$ to first order, we obtain

$$\ddot{\alpha} = \left[ \sin \theta_3 + \alpha \cos \theta_3 \left( \omega^2 (\cos \theta_3 - \alpha \sin \theta_3) - \frac{g}{R} \right) \right]
= \alpha \left[ \omega^2 (\cos^2 \theta_3 - \sin^2 \theta_3) - \frac{g}{R} \cos \theta_3 \right]
= \alpha \left[ \frac{g^2}{R^2 \omega^2} - \omega^2 \right].$$

(3)

The equilibrium is stable, if $\omega > \sqrt{g/R}$, which is when $\theta_3$ corresponds to a real solution.

(c) Find the frequency of small oscillations about these positions. [10 points]

**Solution**

The frequencies of small oscillations can be read off the RHS of Eqs. (2,3)

$$\Omega_0 = \sqrt{\frac{g}{R} - \omega^2},
\Omega_3 = \sqrt{\omega^2 - \frac{g^2}{R^2 \omega^2}}.$$

d) Derive the conservation law following from the fact that the Lagrangian is independent of time. [10 points]

**Solution**

Since the Lagrangian does not depend on time explicitly, the energy function

$$h = \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L$$

must be conserved. Simple algebra gives

$$h = \frac{1}{2} m R^2 \dot{\theta}^2 - \frac{1}{2} m R^2 \omega^2 \sin^2 \theta - m g R \cos \theta = \text{const.}$$

Notice that $h$ is not equal to the total energy because and external force is doing work on spinning the hoop.

2. **Problem 2 [30 points]**

Find the condition for the rotation of a symmetrical top about a vertical axis in a uniform gravitational field to be stable.

**Solution**

Following Landau, p. 112, Problem 1, the effective potential energy for a symmetrical top in a gravitational field is

$$U_{\text{eff}} = \frac{(M_z - M_3 \cos \theta)^2}{2 I_3 \sin^2 \theta} - m g l (1 - \cos \theta),$$

where

$$M_3 = I_3 \left( \dot{\psi} + \dot{\phi} \cos \theta \right),
M_z = (I_1 \sin^2 \theta + I_3 \cos^2 \theta) \dot{\phi} + I_3 \dot{\psi} \cos \theta.$$
For the vertical top, the $x_3$ and $Z$ axes coincide, hence $M_z = M_3$. For small deviations from the vertical ($\theta \ll 1$),

$$U_{\text{eff}} \approx \left( \frac{M_3^2}{8I_1} - \frac{1}{2}mg\theta \right) \theta^2.$$

The motion is stable if

$$\frac{M_3^2}{4I_1} > mg\ell$$

3. **Problem 3 [30 points]**

Find the cross-section for a particle of mass $m$, incoming from “infinity” with speed $v_0$, to fall on the center of a potential $U(r) = -A/r^3$, where $A > 0$.

**Solution**

The effective potential energy for radial motion

$$U_{\text{eff}}(r) = -\frac{A}{r^3} + T_0 \frac{s^2}{r^2},$$

where $T_0 = mv_0^2/2$. The curve $U_{\text{eff}}$ has a maximum determined by

$$\frac{dU_{\text{eff}}(r)}{dr} = 0$$

which gives

$$r_{\text{max}} = \frac{3A}{2T_0s^2}.$$

The value at the maximum,

$$U_{\text{eff}}^{\text{max}} = \frac{4}{27} \frac{T_0^3s^6}{A^2}.$$

If $T_0 > U_{\text{eff}}^{\text{max}}$, the point $r = 0$ is accessible and the particle falls on the center. For $T_0 < U_{\text{eff}}^{\text{max}}$, there are two regions accessible to motion: to the left and to the right of the maximum. However, the region to the left cannot be reached if the particle is coming from far away because it will be bounced back. Therefore, the range of impact parameters for falling on the center is determined by the condition

$$T_0 \geq U_{\text{eff}}^{\text{max}} = \frac{4}{27} \frac{T_0^3s^6}{A^2} \Rightarrow$$

$$s^2 \leq s_{\text{max}}^2 = \frac{3}{4^{1/3}} \left( \frac{A}{T_0} \right)^{2/3},$$

from which

$$\sigma = 2\pi \int_0^{s_{\text{max}}} ds = \pi s_{\text{max}}^2 = 3\pi \left( \frac{A}{2T_0} \right)^{2/3} = 3\pi \left( \frac{A}{mv_0^2} \right)^{2/3}.$$
FIG. 1: ±3 ±2 ±1 0 1 2 0.5 1 1.5 2 2.5 3

FIG. 2: Problem 3
FIG. 3: Problem 1