Problem 1. 1

$m$ suspended by the oscillations.

SOLUTION. The energy of the pendulum is \( E = \frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos \phi = -mgl \cos \phi_0 \), where \( \phi \) is the angle between the string and the vertical, and \( \phi_0 \) the maximum value of \( \phi \). Calculating the period as the time required to go from \( \phi = 0 \) to \( \phi = \phi_0 \), multiplied by four, we find

\[
T = 4 \sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}
\]

\[
= 2 \sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{1}{2} \phi_0 - \sin^2 \frac{1}{2} \phi}}.
\]

The substitution \( \sin \xi = \sin \frac{1}{2} \phi / \sin \frac{1}{2} \phi_0 \) converts this to \( T = 4 \sqrt{(l/g)K(\sin \frac{1}{2} \phi_0)} \), where

\[
K(k) = \int_0^{\frac{1}{2}\pi} \frac{d\xi}{\sqrt{1 - k^2 \sin^2 \xi}}
\]

is the complete elliptic integral of the first kind. For \( \sin \frac{1}{2} \phi_0 \approx \frac{1}{2} \phi_0 \ll 1 \) (small oscillations), an expansion of the function \( K \) gives

\[
T = 2\pi \sqrt{(l/g)(1 + \frac{1}{16} \phi_0^2 + \ldots)}.
\]