PHZ 7427 SOLID STATE II:
Electron-electron interaction and the Fermi-liquid theory

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I. NOTATIONS

- $k_B = 1$ (replace $T$ by $k_B T$ in the final results)
- $\hbar = 1$ (momenta and wave numbers have the same units, so do frequency and energy)
- $\nu(\varepsilon) \equiv$ density of states

II. ELECTROSTATIC SCREENING

A. Thomas-Fermi model

For Thomas-Fermi model, see AM, Ch. 17.

B. Effective strength of the electron-electron interaction. Parameter $r_s$.

The ratio of the Coulomb energy at a typical inter-electron distance to the Fermi energy is

$$\frac{U_C}{E_F} = \frac{e^2/\langle r \rangle}{E_F}.$$  

$\langle r \rangle$ is found from

$$\frac{4}{3} \pi \langle r \rangle^3 n = 1 \rightarrow \langle r \rangle = \left( \frac{3}{4\pi} \right)^{1/3} n^{-1/3}$$

$$\frac{U_C}{E_F} = \left( \frac{4\pi}{3} \right)^{1/3} \frac{1}{\left( \frac{3\pi^2}{2} \right)^{2/3} 2/3} \frac{e^2 n^{1/3} m}{\pi \, n^{2/3}} = \frac{2}{3} \frac{(2)^{2/3} e^2 m}{\pi n^{1/3}} = 0.34 \frac{e^2 m}{n^{1/3}}.$$  

Lower densities correspond to stronger effective interactions and vice versa.
Parameter $r_s$ is introduced as the average distance between electrons measured in units of the Bohr radius

$$\langle r \rangle = r_s a_B = r_s / me^2.$$

Expressing $r_s$ in terms of $n$ and relating density to $k_F$, we find

$$r_s = \left( \frac{3}{4\pi} \right)^{1/3} \frac{1}{n^{1/3} a_B} = \left( \frac{9\pi}{4} \right)^{1/3} \frac{me^2}{k_F}.$$

In terms of $r_s$,

$$U_C = \frac{e^2}{r_s a_B}$$

and

$$E_F = \frac{1}{2} \left( \frac{9\pi}{4} \right)^{2/3} \frac{1}{ma_B^2 r_s^2},$$

$$\frac{U_C}{E_F} = 2 \left( \frac{4}{9\pi} \right)^{2/3} r_s \approx 0.54 r_s.$$

C. Full solution (Lindhard function)

In the Thomas-Fermi model, one makes two assumptions: a) the effective potential acting on electrons is weak and b) the effective potential (and corresponding density) varies slowly on the scale of the electron’s wavelength. Assumption a) allows one to use the perturbation theory whereas assumption b) casts this theory into a quasi-classical form. In a full theory, one discards assumption b) but still keeps assumption a). So now we want to do a complete quantum-mechanical (no quasi-classical assumptions) form.

Let the total electrostatic potential acting on an electron be

$$\phi = \phi_{\text{ext}} + \phi_{\text{ind}},$$

where $\phi_{\text{ext}}$ is the potential of external charges and $\phi_{\text{ind}}$ is that of induced charges. Correspondingly, the potential energy

$$v = -e\phi = -e\phi_{\text{ext}} - e\phi_{\text{ind}}.$$

Because we are doing the linear-response theory, the form of the external perturbation does not matter. Let’s choose it as a plane-wave

$$v(\vec{r}, t) = \frac{1}{2} v_q e^{i(\vec{q} \cdot \vec{r} - \omega t)} + c.c. \tag{1}$$
Before the perturbation was applied, the wavefunction was
\[ \Psi_0 = \frac{1}{L^{3/2}} e^{i(k \cdot \vec{r} - \varepsilon_k t)}. \]

The wavefunction in the presence of the perturbation is given by standard expression from the first-order perturbation theory
\[ \Psi = \Psi_0 \left[ 1 + \frac{v_q}{2} \frac{e^{i(q \cdot \vec{r} - \omega t)}}{\varepsilon_k - \varepsilon_{k+q} + \omega} + \frac{v_q}{2} \frac{e^{-i(q \cdot \vec{r} - \omega t)}}{\varepsilon_k - \varepsilon_{k-q} - \omega} \right], \]
where the last term is a response to a c.c. term in Eq. (1). The Fourier component of the wavefunction
\[ \Psi_k = \Psi_{0k} \left[ 1 + \frac{1}{2} \frac{v_q}{\varepsilon_k - \varepsilon_{k+q} + \omega} + \frac{v_q}{2} \frac{e^{-i(q \cdot \vec{r} - \omega t)}}{\varepsilon_k - \varepsilon_{k-q} - \omega} \right]. \]

The induced charge density is related to the wavefunction
\[ \rho_{\text{ind}} = -2e \sum_k f_k \left( |\Psi_k|^2 - |\Psi_{0k}|^2 \right), \quad (2) \]

where \( f_k \) is the Fermi function, factor of 2 is from the spin summation and the homogeneous (unperturbed) charge density was subtracted off. Keeping only the first-order terms in \( v_q \), Eq. (2) gives
\[ \rho_{\text{ind}} = -2e \frac{1}{L^3} \sum_k f_k v_q \left[ \frac{1}{\varepsilon_k - \varepsilon_{k+q} + \omega} + \frac{1}{\varepsilon_k - \varepsilon_{k-q} - \omega} \right], \quad (3) \]
\[ = -2e v_q \int \frac{d^3k}{(2\pi)^3} \frac{f_k - f_{k+q}}{\varepsilon_k - \varepsilon_{k+q} + \omega}, \quad (4) \]

where we shifted the variables as \( k - \vec{q} \to k \) and \( k \to k + \vec{q} \) in the last term.

The charge susceptibility, \( \chi \), is defined as
\[ \rho_{\text{ind}} = e \frac{q^2}{4\pi} \chi v_q = -e^2 \frac{q^2}{4\pi} \chi \phi_q, \quad (5) \]

where \( \phi_q \) is the Fourier component of the net electrostatic potential. Comparing Eqs. (4) and (5), we see that
\[ \chi = \frac{4\pi}{q^2} (-2) \int \frac{d^3k}{(2\pi)^3} \frac{f_k - f_{k+q}}{\varepsilon_k - \varepsilon_{k+q} + \omega}. \]

The meaning of \( \chi \) becomes more clear, if we write down the Poisson equation (in a Fourier-transformed form)
\[ q^2 \phi_q = 4\pi (\rho_{\text{ext}} + \rho_{\text{ind}}). \]
External charges and potentials satisfy a Poisson equation on their own

\[ q^2 \phi_{\text{ext}} = 4\pi \rho_{\text{ext}}. \]

Now,

\[ q^2 \phi_q = q^2 \phi_{\text{ext}} + 4\pi \rho_{\text{ind}} = q^2 \phi_{\text{ext}} - 4\pi e^2 q^2 \frac{\chi}{4\pi} \phi_q \rightarrow \]

\[ \phi_q = \frac{\phi_{\text{ext}}}{1 + \chi e^2}. \]

Using a definition of the dielectric function

\[ \phi_q = \frac{\phi_{\text{ext}}}{\epsilon(q, \omega)}, \quad (6) \]

we see that

\[ \epsilon(q, \omega) = 1 + \chi e^2 = 1 + \frac{4\pi e^2}{q^2} (-2) \int \frac{d^3k}{(2\pi)^3} \frac{f_k - f_{k+\vec{q}}}{\varepsilon_k - \varepsilon_{k+\vec{q}} + \omega}. \quad (7) \]

This is the Lindhard’s expression for the dielectric function.

a. Check: Let’s make sure that the general form of \( \epsilon(q, \omega) \) [Eq.(7)] does reduce to the Thomas-Fermi one in the limit \( \omega = 0 \) and \( q \ll k_F \).

\[ \epsilon(q, 0) = 1 + \frac{4\pi e^2}{q^2} (-2) \int \frac{d^3k}{(2\pi)^3} \frac{f_k - f_{k+\vec{q}}}{\varepsilon_k - \varepsilon_{k+\vec{q}}} \]

For small \( q \),

\[ f_{k+\vec{q}} = f(\varepsilon_{k+\vec{q}}) = f(\varepsilon_{k+\vec{q}} - \varepsilon_k + \varepsilon_k) = f(\varepsilon_k) + \frac{\partial f}{\partial \varepsilon_k}(\varepsilon_{k+\vec{q}} - \varepsilon_k) + \ldots \]

and

\[ \epsilon(q, 0) = 1 + \frac{4\pi e^2}{q^2} 2 \int \frac{d^3k}{(2\pi)^3} \left( -\frac{\partial f}{\partial \varepsilon_k} \right) . \]

At \( T = 0, -\frac{\partial f}{\partial \varepsilon_k} = \delta(\varepsilon_k - E_F) \). The density of states at the Fermi energy

\[ \nu_F = 2 \int \frac{d^3k}{(2\pi)^3} \delta(\varepsilon_k - E_F) . \]

Now,

\[ \epsilon(q, 0) = 1 + \frac{4\pi e^2}{q^2} \nu_F = 1 + \frac{\kappa^2}{q^2} , \]

where \( \kappa^2 \equiv 4\pi e^2 \nu_F \), which is just the Thomas-Fermi form.
D. Lindhard function

As shown in AM, the static form of the Lindhard’s dielectric function is given by

\[ \epsilon (q, 0) = 1 + \frac{4 \pi e^2}{q^2} \left[ \frac{1}{2} + \frac{1 - x^2}{4x} \ln \frac{1 + x}{1 - x} \right] , \]

where \( x \equiv q/2k_F \).

Notice that the derivative of \( \epsilon (q, 0) \) is singular for \( q = 2k_F \), i.e., \( x = 1 \). This singularity gives rise to a very interesting phenomenon—Friedel oscillations in the induced charge density (and corresponding potentials). Mathematically, it arises because of the property of the Fourier transform. To find the net electrostatic potential in the real space we need to Fourier transform back to real space Eq.(6). Let’s say that the external perturbation is a single point charge \( Q \). Then (in the \( q \)-space) , \( \phi_{\text{ext}} = 4\pi Q/q^2 \) and

\[ \phi (r) = \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q} \cdot \vec{r}} \frac{4\pi Q}{q^2 \epsilon (q, 0)} . \] (8)

1. A discourse: properties of Fourier transforms

Fourier transforms have the following property. Suppose we want to find the large \( t \) limit of

\[ F (t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} F (\omega) . \] (9)

If function \( F (\omega) \) is analytic, the integral in Eq.(6) can be done by closing the contour in the complex plane. \( F (t) \) for \( t \to \infty \) will be then given by an exponentially decaying function \( \exp(-\omega''_{\text{min}} t) \), where \( \omega''_{\text{min}} \) is the imaginary part of that pole of \( F (\omega) \) which is closest to the real axis. For example, if \( F (\omega) = (\omega^2 + a^2)^{-1} \), \( F (t) \propto \exp(-at) \) . Thus, the large-\( t \) asymptotes of analytic functions decay exponentially in time. On the other hand, if \( F (\omega) \) is non-analytic, \( F (t) \) decays much slower—only as a power-law. For example, for \( F (\omega) = \exp(-a |\omega|) \), we have

\[ F (t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \exp(-a |\omega|) = \int_0^\infty \frac{d\omega}{2\pi} \left( e^{-i\omega t} + e^{i\omega t} \right) e^{-a\omega} \]

\[ = 2 \text{Re} \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega t} e^{-a\omega} = 2 \text{Re} \frac{1}{2\pi a + it} \frac{1}{\pi a^2 + t^2} \propto \frac{1}{t^2} \text{ for } t \to \infty . \]

In addition, if \( F (\omega) \) has a divergent derivative of order \( n \) at finite \( \omega \), e.g., for \( \omega = \omega_0 \), that \( F (t) \) oscillates in \( t \). This can be seen by doing the partial integration in Eq.(6) \( n + 1 \) times

\[ 2\pi F (t) = \int d\omega e^{-i\omega t} F (\omega) = \int_{-\infty}^{\omega_0} d\omega e^{-i\omega t} F (\omega) + \int_{\omega_0}^{\infty} d\omega e^{-i\omega t} F (\omega) = \]

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= \frac{1}{-i\omega} e^{-i\omega t} F(\omega) \big|_{\omega_0}^{\infty} + \frac{1}{-i\omega} e^{-i\omega t} F(\omega) \big|_{\omega_0}^{\infty} - \left( \frac{1}{-i\omega} \right) \int d\omega e^{-i\omega t} \frac{d}{d\omega} F(\omega) + \ldots ,

i.e., until the boundary terms gives the divergent expression \( \frac{d}{d\omega} n \) \( F(\omega_0) e^{-i\omega_0 t} \) which oscillates as \( e^{-i\omega_0 t} \).

2. End of discourse

Coming back to Eq.(9), we can now understand why the induced density around the point charge oscillates as \( \cos 2k_F r \) and falls off only as a power law of the distance

\[ \rho_{\text{ind}} \propto \frac{\cos 2k_F r}{r^3} . \]

Both of these effects are the consequences of the singularity of \( \epsilon(q,0) \) at \( q = 2k_F \).

E. Friedel oscillations

The physics of Friedel oscillations is very simple: they arise due to standing waves formed as a result of interference between incoming and backscattered electron waves. For the sake of simplicity, let’s analyze a 1D case. Suppose that at \( x = 0 \), we have an infinitely high barrier (wall). For each \( k \), the wavefunction is a superposition of the incoming plane wave \( L^{-1/2} e^{i k x} \) and a reflected wave \( L^{-1/2} e^{-i k x} \):

\[ \Psi = L^{-1/2} e^{i k x} - L^{-1/2} e^{-i k x} = \frac{2i}{L^{1/2}} \sin k x . \]

The probability density \( |\Psi|^2 = (4/L) \sin^2 k x \) oscillates in space. If the probability that the state with momentum \( k \) is occupied is a smooth function of \( k \) (as it is the case for the Maxwell-Boltzmann or Bose-Einstein statistics), then summation over \( k \) would smear out the oscillations. However, for the Fermi statistics, \( f_k \) has a sharp (at \( T = 0 \)) boundary between the occupied and empty states. As a result, oscillations survive even after the summation of \( k \). The profile of the density is described by

\[ n(x) = 2 \sum_k \frac{dk}{2\pi} f_k |\Psi|^2 = 8 \int_0^{k_F} \frac{dk}{2\pi} \sin^2 k x = 4 \int_0^{k_F} \frac{dk}{2\pi} (1 - \cos 2k x) \]

\[ = n_0 - \frac{\sin 2k_F x}{\pi x} , \]

where \( n_0 = 2k_F / \pi \) is the density of the homogeneous electron gas. Away from the barrier, oscillations die off as \( x^{-1} \). At the barrier, \( n(0) = 0 \).
A 3D case, is different in that the $x^{-1}$ decay changes to a $r^{-3}$ one. (In general $D$-dimensional case, the Friedel oscillations fall off as $r^{-D}$.) Friedel oscillations were observed in STM experiment (see attached figures).

1. Enhancement of the backscattering probability due to Friedel oscillations

As it was discussed in the previous Section, Friedel oscillations arise already in the single-particle picture. However, they influence scattering of interacting electrons at impurities and other imperfections. Once a Friedel oscillation is formed, the effective potential barrier seen by other electrons is the sum of the bare potential plus the potential produced by the Friedel oscillation. Consider a simple example when 1D electrons interact via a contact potential $U(x) = u\delta(x)$. The potential produced by the Friedel oscillation is

$$V_F(x) = \int dx' (n(x') - n_0) V(x - x') = u(n(x) - n_0) = -u\frac{\sin 2k_F x}{\pi x}.$$ 

Backscattering at an oscillatory potential is enhanced due to resonance. In the Born approximation, the backscattering amplitude for an electron with momentum $k$ is

$$A = \int dx \left( e^{-ikx} \ast V_F(x) e^{ikx} \right) = -\frac{u}{\pi} \int_0^\infty \frac{dx}{x} \sin 2k_F x e^{i2kx}$$

The first term gives a convergent integral (we remind that our $k > 0$), so forget about it. (It’s only role is to guarantee the convergence at $x \to 0$, but we will take this into account by cutting the integral at $x \approx k_F^{-1}$.) The second term becomes log-divergent at large distances if $k = k_F$. To estimate the integral, notice that it diverges for $k = k_F$ and converges for $k \neq k_F$. Thus

$$A = -\frac{u}{\pi} \int_{k_F^{-1}}^{[k-k_F]^{-1}} \frac{dx}{x} = -\frac{u}{\pi} \ln \frac{k_F}{|k - k_F|}.$$ 

Precisely at the Fermi surface ($k = k_F$), the backscattering amplitude (and thus the probability) blows up which means that impurity becomes impenetrable. This effect in 1D is usually described as arising due to the non-Fermi-liquid nature of the ground state. However, as we just saw that this effect can be simply understood in terms of backscattering from the Friedel oscillations. Higher orders in the e-e interaction can be summed up (see
Ref.\(^1\)). It turns out that each next order in \(u\) brings in an additional log of \(|k - k_F|\). The transmission amplitude of the barrier becomes

\[
t = t_0 \left( \frac{1}{2} g \ln \frac{k_F}{|k - k_F|} + \frac{1}{2} g^2 \ln^2 \frac{k_F}{|k - k_F|} - \frac{1}{6} g^3 \ln^3 \frac{k_F}{|k - k_F|} + \ldots \right)
\]

\[
= t_0 \exp \left( -g \ln \frac{k_F}{|k - k_F|} \right) = t_0 \left( \frac{|k - k_F|}{k_F} \right)^g,
\]

where \(g = u/\pi v_F\) is the dimensionless coupling constant and \(t_0\) is the transmission amplitude in the absence of \(e-e\) interaction. At \(k = k_F\), the transmission amplitude vanishes. Suppose one measures a tunneling conductance of the barrier inserted into a 1D system. Then typical \(|k - k_F| \simeq \max \{T/v_F, eV/v_F\}\), where \(V\) is the applied voltage. With the help of the Landauer formula \(G = (2e^2/h) |t_0|^2\) we conclude that the tunneling conductance exhibits a power-law scaling in the voltage or temperature

\[
G \propto (\max \{T, eV\})^{2g}.
\]

Such a power-law scaling was indeed observed in carbon nanotubes (which are essentially quantum wires with two channels)

\[\text{F. Hamiltonian of the jellium model}\]

The model system we study are electrons in the presence of a positively charged ions. The ionic charge is assumed to be spread uniformly over the system volume (“jellium model”). The net electron charge is equal in magnitude and opposite in sign to that of ions so that overall the system is electro-neutral. This part of the model is common for all tractable models of e-e interactions in solids, including but not limited to the Hartree-Fock approximation. The classical energy of the system electrons + ions is

\[
E_{\text{int}} = \frac{1}{2} \int d^3r_1 \int d^3r_2 n(\vec{r}_1) V_{ee}(\vec{r}_1 - \vec{r}_2) n(\vec{r}_2) + \int d^3r_1 \int d^3r_2 n(\vec{r}_1) V_{ei}(\vec{r}_1 - \vec{r}_2) n_i(10)
\]

\[
+ \frac{1}{2} \int d^3r_1 \int d^3r_2 n_i V_{ii}(\vec{r}_1 - \vec{r}_2) n_i,
\]

where \(n(\vec{r})\) is the (non-uniform) electron density, \(n_i\) is the (uniform) density of ions, \(V_{ee,ei,ii}\) are the potentials of electron-electron, electron-ion, and ion-ion interactions. In the Hartree-Fock approximation,

\[
V_{ee}(\vec{r}_1 - \vec{r}_2) = \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} = -V_{ei}(\vec{r}_1 - \vec{r}_2) = V_{ii}(\vec{r}_1 - \vec{r}_2).
\]

\[(11)\]
Using (11), Eq.(12) can be re-arranged as

\[ E_{\text{int}} = \frac{1}{2} \int d^3r_1 \int d^3r_2 \left[ n(r_1) - n_i \right] V_{ee} (r_1 - r_2) \left[ n(r_2) - n_i \right]. \]

In this form, the energy has a very simple meaning. At point \( r_1 \) the electron density deviates from the ionic density, so that the net charge density is \( n(r_1) - n_i \). Similarly, at point \( r_2 \) the net charge density is \( n(r_2) - n_i \). These local fluctuations in density interact via the Coulomb potential. In the absence of any external perturbations, boundaries, and impurities, the average electron density is uniform and equal to the density of ions: \( \langle n(r_1) \rangle = n_i \). However, the ground state energy involves the product of densities at different points \( \langle n(r_1) n(r_2) \rangle \) (a correlation function) which is not uniform. In classical systems, local fluctuations in the charge density are due to thermal motion of electrons. In a quantum-mechanical system at \( T = 0 \) they are due to the zero-point motion of electrons (in a Fermi system, the kinetic energy is finite at \( T = 0 \)). To treat the QM system, we need to pass from the classical energy to a Hamiltonian

\[ E_{\text{int}} \rightarrow \hat{H}_{\text{int}} = \frac{1}{2} \int d^3r_1 \int d^3r_2 \left[ \hat{n}(r_1) - n_i \right] V_{ee} (r_1 - r_2) \left[ \hat{n}(r_2) - n_i \right], \tag{12} \]

where now \( \hat{n}(r) \) is a number-density operator. Densities of ions do not fluctuate so we can leave them as classical variables (c-numbers). Also, the potential is the same as in the classical system. It is convenient to re-write Eq. (12) in a Fourier-transformed form

\[ \hat{H}_{\text{int}} = \frac{1}{2L^3} \sum_q \left[ \hat{n}_q - n_i \right] V_{ee} (q) \left[ \hat{n}_{-q} - n_i \right], \]

where

\[ \hat{n}(r) - n_i = \frac{1}{L^3} \sum_q \left[ \hat{n}_q (r) - n_i \right] e^{-i\vec{q}\cdot\vec{r}} \]

\[ V_{ee}(q) = \frac{1}{L^3} \sum_q V_{ee} (q) e^{-i\vec{q}\cdot\vec{r}}, \]

with \( V_{ee}(q) = 4\pi e^2/q^2 \) and where we took into account that \( V_{ee}(q) \) depends only on the magnitude of \( q \). The Fourier transform of a uniform density of ions is \( n_i L^3 \delta_{\vec{q},0} \) so

\[ \hat{H}_{\text{int}} = \frac{1}{2L^3} \sum_q \left[ \hat{n}_q - n_i L^3 \delta_{\vec{q},0} \right] V_{ee} (q) \left[ \hat{n}_{-q} - n_i \right] \]

\[ = \frac{1}{2L^3} \sum_q \left[ \hat{n}_q - N \delta_{\vec{q},0} \right] V_{ee} (q) \left[ \hat{n}_{-q} - N \delta_{\vec{q},0} \right], \]

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where $N$ is the total number of ions (equal to that of electrons). On the other hand,

$$\hat{n}_{\vec{q}} = \int d^3 r \hat{n} (\vec{r}) = \hat{N},$$

where $\hat{N}$ is an operator of the total number of particles. Because the total number of particles is fixed this operator is simply equal to its expectation value $N$. Thus the $\vec{q} = 0$ term gives no contribution to the sum. Physically, it means that the fluctuation of infinite size in real space (corresponding to $\vec{q} = 0$) do not interact as they are compensated by uniform charge of ions. With that, we re-write $\hat{H}_{\text{int}}$ as

$$\hat{H}_{\text{int}} = \frac{1}{2} \frac{1}{L^3} \sum_{\vec{q} \neq 0} \hat{n}_{\vec{q}} V_{ee} (q) \hat{n}_{-\vec{q}}. \quad (13)$$

At $T = 0$, the ground-state energy is simply an expectation value of $\hat{H}_{\text{int}}$:

$$E_{\text{int}} = \langle 0 | \hat{H}_{\text{int}} | 0 \rangle = \frac{1}{2} \frac{1}{L^3} \sum_{\vec{q} \neq 0} V_{ee} (q) \langle 0 | \hat{n}_{\vec{q}} \hat{n}_{-\vec{q}} | 0 \rangle.$$

As we simply do the perturbation theory with respect to $V_{ee}$, the expectation value is calculated using the wave-functions of the free system (vacuum average).

Now we need the second quantized form of $\hat{n}_{\vec{q}}$. To this end, we recall that in real space

$$\hat{n} (\vec{r}) = \sum_{\alpha} \Psi_{\alpha}^\dagger (\vec{r}) \Psi_{\alpha} (\vec{r}).$$

Operator $\Psi_{\alpha}^\dagger (\vec{r})$ ($\Psi_{\alpha} (\vec{r})$) creates (annihilates) a particle with spin projection $\alpha$ at point $\vec{r}$. Using the plane waves as our basis set,

$$\Psi_{\alpha} (\vec{r}) = \frac{1}{L^{3/2}} \sum_{\vec{p}} c_{p\alpha} e^{-i\vec{p} \cdot \vec{r}},$$
$$\Psi_{\alpha}^\dagger (\vec{r}) = \frac{1}{L^{3/2}} \sum_{\vec{p}} c_{p\alpha}^\dagger e^{i\vec{p} \cdot \vec{r}}.$$

$$\hat{n}_{\vec{q}} = \int d^3 r e^{i\vec{q} \cdot \vec{r}} \hat{n} (\vec{r}) = \int d^3 r e^{i\vec{q} \cdot \vec{r}} \sum_{\alpha} \Psi_{\alpha}^\dagger (\vec{r}) \Psi_{\alpha} (\vec{r})$$
$$= \frac{1}{L^3} \int d^3 r e^{i\vec{q} \cdot \vec{r}} \sum_{\alpha} \sum_{\vec{p}, \vec{p}'} c_{p\alpha}^\dagger e^{i\vec{p} \cdot \vec{r}} c_{p\alpha} e^{-i\vec{p}' \cdot \vec{r}}$$
$$= \sum_{\alpha} \sum_{\vec{p}} c_{p - \vec{q}, \alpha}^\dagger c_{p\alpha}.$$
Substituting this last result into Eq.(13), we obtain

\[ \hat{H}_{\text{int}} = \frac{1}{2} \frac{1}{L^3} \sum_{\vec{q} \neq 0} \sum_{p,k} \sum_{\alpha,\beta} V_{ee}(q) c_{p-\vec{q},\alpha}^\dagger c_p^\dagger c_{k+\vec{q},\beta}^\dagger c_k^\beta. \]

It is convenient to re-write the second-quantized Hamiltonians in the *normal-ordered form*, when all creation operators are positioned to the right of the annihilation ones. Interchanging the positions of two fermionic operators twice (no sign change!) and discarding the terms which result from the \( \delta \)-function part of the anti-commutation relations (those will give only a shift of the chemical potential), we arrive at

\[ \hat{H}_{\text{int}} = \frac{1}{2} \frac{1}{L^3} \sum_{\vec{q} \neq 0} \sum_{p,k} \sum_{\alpha,\beta} V_{ee}(q) c_{p-\vec{q},\alpha}^\dagger c_{k+\vec{q},\beta}^\dagger c_k^\beta c_p^\alpha. \]

Thus,

\[ E_{\text{int}} = \langle 0 \vert \hat{H}_{\text{int}} \vert 0 \rangle = \frac{1}{2} \frac{1}{L^3} \sum_{\vec{q} \neq 0} \sum_{p,k} \sum_{\alpha,\beta} V_{ee}(q) \langle 0 \vert c_{p-\vec{q},\alpha}^\dagger c_{k+\vec{q},\beta}^\dagger c_k^\beta c_p^\alpha \vert 0 \rangle. \]

The expectation value of the product of \( c^- \)-operators can be calculated directly. However, it is much more convenient to use the result known as the *Wick’s theorem*\(^2,^3\). The Wick theorem states that an expectation value of a product of any number of \( c^- \)-operators splits into products of expectation values of pairwise averages (called *contractions*):

\[ \langle 0 \vert \prod_{i=1}^{M_1} c_{p_i,\alpha_i}^\dagger \prod_{j=1}^{M_2} c_{p_j,\alpha_j} \vert 0 \rangle = \sum_P \prod_{i=1}^{M_1} F_{ij} \langle 0 \vert c_{p_i,\alpha_i}^\dagger c_{p_j,\alpha_j} \vert 0 \rangle \]

for \( M_1 = M_2 \) and is equal to zero otherwise. The sum goes over permutations. The factor \( F_{ij} \) is equal to one if it takes an even number of permutations to bring \( c_{p_i,\alpha_i}^\dagger c_{p_j,\alpha_j} \) together (no permutation is an even permutation of order zero) and it is equal to minus one if takes an odd number of permutations. The The pairwise averages are nothing more than occupation numbers

\[ \langle 0 \vert c_{p_i,\alpha_i}^\dagger c_{p_j,\alpha_j} \vert 0 \rangle = \delta_{\alpha_i\alpha_j} \delta_{p_i,p_j} f_{p_i}. \]

where we suppress the spin index of \( f_{p_i} \) assuming that the ground state is paramagnetic. For our case, we can pair operators in two ways

\[ \langle 0 \vert c_{p-\vec{q},\alpha}^\dagger c_{k+\vec{q},\beta}^\dagger c_k^\beta c_p^\alpha \vert 0 \rangle = \langle 0 \vert c_{p-\vec{q},\alpha}^\dagger c_p^\alpha \vert 0 \rangle \langle 0 \vert c_{k+\vec{q},\beta}^\dagger c_k^\beta \vert 0 \rangle - \langle 0 \vert c_{p-\vec{q},\alpha}^\dagger c_k^\beta \vert 0 \rangle \langle 0 \vert c_{k+\vec{q},\beta}^\dagger c_p^\alpha \vert 0 \rangle \]

\[ = \delta_{\vec{q},0} f_p f_k - \delta_{\alpha\beta} \delta_{\vec{q}_p-k} f_p f_k. \]
The first term drops out from the sum, whereas the second one gives

\[ E_{\text{int}} = -\frac{1}{2} \int \frac{1}{L^3} \sum_{p \neq k} \sum_\alpha V_{ee} (|p - k|) f_p f_k. \]

The effective single-particle energy can be found as a variational derivative of the ground state energy

\[ \varepsilon_p = \frac{\delta}{\delta f_p} (E_0 + E_{\text{int}}) = \frac{p^2}{2m} - \frac{1}{L^3} \sum_k V(|k - k|) f_k = \frac{p^2}{2m} - \int \frac{d^3 k}{(2\pi)^3} V(|k - k|) f_k \]

\[ \equiv \frac{p^2}{2m} + \delta \varepsilon_p. \]  (14)

Calculate the integral over \( k \) at \( T = 0 \)

\[ \delta \varepsilon_p (p) = -\frac{1}{\pi} e^2 \int_0^{k_F} dk k^2 \int_{-1}^1 d \cos \theta \frac{1}{p^2 + k^2 - 2pk \cos \theta} \]

\[ = -\frac{1}{\pi} e^2 \int_0^{k_F} dk \ln \frac{p + k}{|p - k|} \]

\[ = -\frac{e^2 k_F}{\pi} \left( 1 + \frac{1 - y^2}{2y} \ln \left| \frac{1 + y}{1 - y} \right| \right), \]

where \( y \equiv p/k_F \).

Near the bottom of the band when \( p \ll k_F \), i.e., \( y \ll 1 \), we have

\[ \Sigma (p) = -\frac{e^2 k_F}{\pi} \left( 2 - \frac{2}{3} y^2 \right) = -\frac{2e^2 k_F}{\pi} + \frac{2e^2 p^2}{3\pi k_F}. \]

Adding this up with the free spectrum, we obtain

\[ \varepsilon_p^* = -\frac{2e^2 k_F}{\pi} + \frac{p^2}{2m} + \frac{2e^2 p^2}{3\pi k_F} = -\frac{2e^2 k_F}{\pi} + \frac{p^2}{2m^*}. \]

A constant term means that the chemical potential is shifted by the interaction. The effect we are after is the change of the coefficient in front of \( p^2 \):

\[ \frac{1}{m^*} = \frac{1}{m} \left( 1 + \frac{4e^2 m}{3\pi k_F^2} \right) \]

or

\[ m^* = \frac{m}{1 + \frac{e^2 m}{3\pi k_F}} = \frac{m}{1 + 0.22 r_s}. \]

Near the bottom of the band, the effective mass of electrons is smaller than the band mass.

However, this is not the part of the spectrum we are interested in thermodynamic and transport phenomena. In this phenomena, the vicinity of \( k_F \) plays the major role. Away from the bottom of the band the self-energy is not a quadratic function of \( p \), so the resulting spectrum is not parabolic (see Fig. ??). We need to understand what is the meaning of the effective mass in this situation.
G. Effective mass near the Fermi level

In a Fermi gas, the Fermi momentum is obtained by requiring that the number of states within the Fermi sphere is equal to the number of electrons:

\[ \frac{2\frac{4}{3}\pi k_F^3}{(2\pi)^3} = n. \]

This condition does not change in the presence of the interaction. Therefore, \( k_F \) of the interacting system is the same as in a free one. (This statement can be proven rigorously and is known as the Luttinger theorem.\(^3\)) The energy of a topmost state is the chemical potential. In a Fermi gas,

\[ \mu = \varepsilon_{p_F} = \frac{p_F^2}{2m}. \]

When the spectrum is renormalized by the interaction,

\[ \mu = \varepsilon_{p_F}^*. \]

The energy of a quasi-particle is defined as

\[ \varepsilon_p^* = \varepsilon_p^* - \mu. \]

From the definition of the self-energy,

\[ \varepsilon_p^* = \varepsilon_p^* - \mu = \varepsilon_p + \Sigma(p) - \mu. \]

At the Fermi surface, this equality reduces to

\[ 0 = \frac{p_F^2}{2m} + \Sigma(p_F) - \mu \rightarrow \mu = \frac{p_F^2}{2m} + \Sigma(p_F). \]

The chemical potential is changed due to the interaction. Near the Fermi surface, i.e., for \( |p - p_F| \ll p_F \),

\[ \varepsilon_p^* = \frac{p^2}{2m} + \Sigma(p - p_F + p_F) - \frac{p_F^2}{2m} - \Sigma(p_F). \]

Near the Fermi surface, the spectrum can always be linearized

\[ \frac{p^2}{2m} - \frac{p_F^2}{2m} = \frac{(p - p_F)(p + p_F)}{2m} \approx v_F(p - p_F). \]
Expand the self-energy in Taylor series

\[ \Sigma (p - p_F + p_F) - \Sigma (p_F) \approx \Sigma' (p - p_F), \]

where \( \Sigma' \equiv \partial \Sigma / \partial p |_{p=p_F} \). The renormalized spectrum can be also linearized near the Fermi surface

\[ \epsilon_p^* \approx v_F^* (p - p_F), \]

where \( v_F^* \) is the renormalized Fermi velocity. Now we have

\[ v_F^* (p - p_F) = v_F (p - p_F) + \Sigma' (p - p_F) \to \]

\[ v_F^* = v_F + \Sigma' \]

The effective mass near the Fermi surface is defined as

\[ \frac{1}{m^*} \equiv \frac{v_F^*}{p_F} \]

so that

\[ \frac{1}{m^*} = \frac{1}{m} + \frac{1}{p_F \Sigma'} \]

or

\[ m^* = \frac{m}{1 + \Sigma'/v_F}. \]  

(16)

\( m^* \) is determined by the derivative of the self-energy at the Fermi surface.

1. **Effective mass in the Hartree-Fock approximation**

   Calculating the derivative of the self-energy in the Hartree-Fock approximation [Eq.(15)], we arrive at an unpleasant surprise: \( \Sigma' = \infty \). According to Eq.(16), this means that \( m^* = 0 \). This unphysical result is the penalty we pay for using an oversimplified model in which electrons interact via the unscreened Coulomb potential.

2. **Beyond the Hartree-Fock approximation**

   The deficiency of the Hartree-Fock approximation is cured by using the screened Coulomb potential. Because electrons exchange energies, the polarization clouds of induced charges around them are dynamic, i.e., the two-body potential depends not only on the distance but
also on time. In the Fourier space, it means the interaction is a function not only of \( q \) but also of frequency \( \omega \)

\[
V_{ee}(q, \omega) = \frac{4\pi e^2}{q^2 \epsilon(q, \omega)},
\]

where \( \epsilon(q, \omega) \) is the full Lindhard function. A calculation of the self-energy in this case is a rather arduous task, so I will give here only the result for the effective mass near the Fermi level\(^3\)

\[
\frac{m^*}{m} = 1 - \frac{e^2}{\pi v_F} \ln \frac{2p_F}{e\kappa},
\]

where \( e = 2.718... \) where \( \kappa \) is the screening wavevector

\[
\kappa^2 = 4\pi e^2 \nu_F.
\]

In terms of \( r_s \).

\[
\frac{m^*}{m} = 1 - \frac{1}{2\pi} \left( \frac{4}{9\pi} \right)^{1/3} r_s \ln \left( \frac{\pi}{(2.718)^2} \left( \frac{4}{9\pi} \right)^{1/3} \frac{1}{r_s} \right)
\]

\[
= 1 - 0.8 r_s \ln \frac{0.22}{r_s}.
\]

This expression is valid for a weak interaction, i.e., \( r_s \ll 1 \). For very small \( r_s \), \( m^* \) decreases with \( r_s \). At \( r_s^{\text{min}} \approx 0.08 \), \( m^* \) has a minimum and it becomes equal to \( m \) at \( r_s \approx 0.22 \).

### III. STONER MODEL OF FERROMAGNETISM IN ITINERANT SYSTEMS

Consider a model of fermions interacting via a delta-function potential \( V(\vec{r}_1 - \vec{r}_2) = g\delta(\vec{r}_1 - \vec{r}_2) \) (a good model for He\(^3\) atoms). The interaction part of Hamiltonian reads (in real space)

\[
H_{\text{int}} = \frac{1}{2} \int d^3r_1 \int d^3r_2 V(\vec{r}_1 - \vec{r}_2) \Psi_\alpha^\dagger(\vec{r}_1) \Psi_\beta(\vec{r}_2) \Psi_\beta(\vec{r}_2) \Psi_\alpha(\vec{r}_1).
\]

For a delta-function interaction, we need to take into account that \( \Psi_\alpha(\vec{r}) \Psi_\beta(\vec{r}) \neq 0 \) only if \( \alpha \neq \beta \). \( H_{\text{int}} \) then reduces to

\[
H_{\text{int}} = g \int d^3r \Psi_\uparrow^\dagger(\vec{r}) \Psi_\downarrow(\vec{r}) \Psi_\downarrow^\dagger(\vec{r}) \Psi_\uparrow(\vec{r})
\]

An expectation value of the interaction energy per unit volume is given by

\[
E_{\text{int}} = L^{-3} \langle 0 | H_{\text{int}} | 0 \rangle = L^{-3} g \int d^3r \langle 0 | \Psi_\uparrow^\dagger(\vec{r}) \Psi_\downarrow^\dagger(\vec{r}) \Psi_\downarrow(\vec{r}) \Psi_\uparrow(\vec{r}) | 0 \rangle \tag{17}
\]

(\text{using Wick’s theorem})

\[
= L^{-3} g \int d^3r \langle 0 | \Psi_\uparrow^\dagger(\vec{r}) \Psi_\uparrow^\dagger(\vec{r}) | 0 \rangle \langle 0 | \Psi_\downarrow^\dagger(\vec{r}) \Psi_\downarrow^\dagger(\vec{r}) | 0 \rangle \tag{18}
\]

\[
= L^{-3} g \int d^3r n_\uparrow n_\downarrow = gn_\uparrow n_\downarrow \tag{19}
\]
where \( n_{\uparrow,\downarrow} \) is the (expectation value of) number density of spin-up (down) fermions, which is position independent. In a paramagnetic state, \( n_{\uparrow} = n_{\downarrow} = n/2 \), where \( n \) is the total number density. Let’s analyze the possibility of a transition into a state with a finite spin polarization (ferromagnetic state), in which \( n_{\uparrow} \neq n_{\downarrow} \).

The kinetic energy of a partially spin-polarized system is:

\[
E_0 = \int_{0<k<k_F^+} \frac{d^3k}{(2\pi)^3} \varepsilon_k + \int_{0<k<k_F^+} \frac{d^3k}{(2\pi)^3} \varepsilon_k,
\]

where \( k_F^{+,-} \) are the Fermi momenta of the spin-up (down) fermions related to \( n_{\uparrow,\downarrow} \) via

\[
\frac{4\pi}{3} \left( k_F^{\uparrow,\downarrow} \right)^3 \frac{1}{(2\pi)^3} = n_{\uparrow,\downarrow}
\]

\[
\Rightarrow k_F^{\uparrow,\downarrow} = \left( 6\pi^2 \right)^{1/3} n_{\uparrow,\downarrow}^{1/3}
\]

\[
\int_{0<k<k_F^+} \frac{d^3k}{(2\pi)^3} \varepsilon_k = \frac{4\pi}{(2\pi)^3} \int_0^{k_F^{\uparrow,\downarrow}} dk k^2 = \frac{k_F^{\uparrow,\downarrow}^5}{20\pi^2m}
\]

Introducing the full density \( n = n_{\uparrow} + n_{\downarrow} \) and the difference in densities \( \delta n = n_{\uparrow} - n_{\downarrow} \), so that \( n_{\uparrow,\downarrow} = (1/2) (n \pm \delta n) \), the equation for \( E_0 \) takes the form

\[
E_0 = \frac{(6\pi^2)^{5/3}}{20\pi^2m} \left[ n_{\uparrow}^{5/3} + n_{\downarrow}^{5/3} \right]
\]

Obviously, \( E_0 \) has a minimum at \( \delta n = 0 \), i.e., in a paramagnetic state. The interaction term changes the balance:

\[
E_{\text{int}} = gn_{\uparrow}n_{\downarrow} = \left( g/4 \right) \left( n^2 - (\delta n)^2 \right).
\]

For a repulsive interaction \( (g > 0) \), the interaction part of the energy is lowered by spin polarization. The total energy is the sum of two contributions

\[
E = E_0 + E_{\text{int}} = \frac{(6\pi^2)^{5/3}}{20\pi^2m} \left[ (n + \delta n)^{5/3} + (n - \delta n)^{5/3} \right] + \left( g/4 \right) \left( n^2 - (\delta n)^2 \right).
\]

Introducing dimensionless quantity

\[
\zeta = \delta n/n
\]
and using the relation between the Fermi energy and the density, the expression for energy is simplified to

\[ E = \frac{3}{10} nE_F \left[ (1 + \zeta)^{5/3} + (1 - \zeta)^{5/3} \right] + (n^2 g/4) \left( 1 - \zeta^2 \right). \]

Suppose that \( \zeta \) is small, then the first term can be expanded to the second order (the first order contribution vanishes), using \((1 + \zeta)^{5/3} + (1 - \zeta)^{5/3} = 2 + \frac{10}{9} \zeta^2 + \frac{10}{243} \zeta^4 + O(x^5) : \)

\[
E (\zeta) = \frac{3}{5} nE_F + \frac{1}{3} nE_F \zeta^2 + (n^2 g/4) \left( 1 - \zeta^2 \right) + \frac{1}{81} gnE_F \zeta^4 + \ldots
= E (0) + a\zeta^2 + \frac{1}{81} gnE_F \zeta^4 + \ldots,
\]

where \( E (0) = \frac{3}{5} nE_F + n^2 g/4 \) and

\[
a = \frac{1}{3} nE_F \left( 1 - \frac{3g}{4E_F} n \right) = \frac{1}{3} nE_F \left( 1 - g\nu_F^{(1)} \right),
\]

with \( \nu_F^{(1)} = 3g/4E_F \) being the density of states per one spin orientation. If \( a > 0 \), the ferromagnetic state is energetically unfavorable. When it is negative, the the ferromagnetic state is energetically unfavorable. The transition occurs when the coefficient vanishes. Thus the critical value of the coupling constant is

\[ g_c = 1/\nu_F^{(1)}. \]

For \( g > g_c \), polarization is stabilized by the quartic term. An equilibrium value of polarization is determined from the condition

\[
\frac{\partial E}{\partial \zeta} = 0 \rightarrow \\
\frac{2}{3} nE_F \left( 1 - \frac{g}{g_c} \right) \zeta + \frac{4}{81} g_c nE_F \zeta^3 = 0 \rightarrow \\
\zeta = \sqrt{27/2} \left( \frac{g - g_c}{g_c} \right)^{1/2}.
\]

Notice that we replace \( g \) by \( g_c \) in the quartic term, which is justified near the critical point. Unfortunately, the critical value for \( g \) is outside the weak coupling regime and, therefore, the Stoner model cannot be considered as a quantitative theory of ferromagnetism. However, it does provide with a hint as to how ferromagnetism occurs in real system. The Stoner model is also probably the first example of zero-temperature (or quantum) phase transitions, i.e., phase changes in the ground state of the system driven by varying some control parameter.
(in this case, the effective interaction strength). Incidentally, it is also a second-order or continuous phase transition: the energy of the system is continuous through the critical point. The equilibrium value of the order parameter ($\zeta$) is zero for $g < g_c$ and becomes finite but still small for $g > g_c$. A square-root dependence of $\zeta$ on the deviation of the control parameter from its critical value is a characteristic feature of the mean-field theories.

The Stoner model also predicts that the spin susceptibility of the paramagnetic state is enhanced by the repulsive interaction. To this end we need to introduce magnetization (the magnetic moment per unit volume)

$$M = g_L \mu_B \delta n = g_L \mu_B n \zeta,$$

where $g_L$ is the Lande $g$–factor (not to be confused with the coupling constant!) and $\mu_B$ is the Bohr magneton, and switch from the ground state energy to the free energy (at $T = 0$)

$$F = E - MH.$$

Retaining only quadratic in $M$ terms, we have

$$F = E(0) + \frac{a}{(g\mu_B n)^2} M^2 - MH.$$  

An equilibrium magnetization is obtained from the condition

$$\frac{\partial F}{\partial M} = 0 \rightarrow \quad M = \frac{(g\mu_B n)^2}{2a} H.$$

Recalling that the spin susceptibility is defined by the following relation, we read off the susceptibility as

$$\chi = \frac{(g_L \mu_B n)^2}{2a} = \frac{(g_L \mu_B n)^2}{(2/3) n E_F (1 - g \nu_F^{(1)})} = \frac{\chi_0}{1 - g \nu_F^{(1)}} = S \chi_0.$$

where $\nu_F$ is the density of states per two spin orientations, $\chi_0$ is the spin susceptibility of a free electron gas at $T = 0$ (Pauli susceptibility), and

$$S \equiv \frac{1}{1 - g \nu_F^{(1)}}.$$
is called the Stoner enhancement factor. For \( g > 0, \chi > \chi_0 \). Also, \( \chi \) diverges at the critical point. A divergent susceptibility is another characteristic feature of both finite- and zero-temperature phase transitions.

What is the magnetic response of the system at the critical point, where the (linear) susceptibility \( \chi = \infty \)? To answer this, we need to restore the quartic term in the free energy

\[
F = E(0) + \frac{a}{(g_L \mu_B n)^2} M^2 - MH + \frac{1}{81 \mu_B^2} (g_L \mu_B) M^4.
\]

At the critical point, \( g = g_c \) and \( a = 0 \). The equation of state then reads

\[
\frac{\partial F}{\partial M} = -H + \frac{4}{81} \frac{g E_F}{\mu_B^2} M^3 = 0.
\]

Therefore, at the critical point \( M \) scales as \( H^{1/3} \). Note that a magnetic field smears the phase transition because now there is a finite magnetization even above the critical point. The order parameter now changes continuously through the transition which is defined as a critical value of \( g \), where the linear susceptibility diverges.

IV. WIGNER CRYSTAL

As we now understand the properties of a weakly interacting electron gas, let’s turn to the opposite limit, when the interaction energy is much larger than the kinetic one. At \( T = 0 \), the kinetic energy is the Fermi energy so the condition for the strong interaction is \( r_s \gg 1 \). What happens in this limit? The answer was given by Wigner back in 1934. Because the Coulomb energy is very high, electrons would try to be as further away from each other as possible. Ideally, they would all move to the sample boundaries. However, this would create an enormous uncompensated positive charge of ions. The next best thing is to arrange into an electron lattice (Wigner crystal) of spacing comparable to the average inter-electron distance in the liquid phase.

Let’s estimate when the formation of the Wigner crystal is possible. In a liquid, come to each other at arbitrarily small distances, where the Coulomb energy is large. In a crystal, electrons are separated by the lattice spacing, \( a \). The gain in the potential energy

\[
P_L - P_C \approx \frac{e^2}{a}
\]

(for the sake of simplicity I assume that the dielectric constants of ions is unity). On the other hand, kinetic energy in a liquid is \( K_L \approx 1/ma^2 \) whereas that in a crystal \( K_C \approx 1/mr_0^2 \),
where \( r_0 \ll a \) is the rms displacement of an electron about its equilibrium position due to the zero-point motion. For the crystal to be stable, one must require that \( r_0 \ll a \). Thus \( K_C \gg K_L \) and

\[
K_C - K_L \approx 1/mr_0^2.
\]

Crystallization is energetically favorable when the gain in the potential energy exceeds the loss in the kinetic one, i.e.,

\[
P_L - P_C \gg K_C - K_L
\]

or

\[
\frac{e^2}{a} \gg 1/mr_0^2.
\]

(20)

Now I want to show that the condition above is nothing is equivalent to \( r_s \gg 1 \).

To estimate \( r_0 \), consider an oscillatory motion of an electron in a 1D lattice interacting with its nearest neighbors via Coulomb forces. The potential energy of the central electron when it is moved by distance \( x \) from its equilibrium position \( (x = 0) \) is

\[
U(x) = \frac{e^2}{a - x} + \frac{e^2}{a + x}.
\]

Expanding the expression above for \( x \ll a \), we obtain

\[
U(x) = \frac{2e^2}{a} + \frac{1}{2} \frac{e^2}{a^3} x^2.
\]

The harmonic part of the potential is reduced to the canonical form by equating

\[
\frac{1}{2} \frac{e^2}{a^3} x^2 \equiv \frac{1}{2} m \omega_0^2 x^2 \rightarrow \omega_0^2 = \frac{e^2}{ma^3}.
\]

Notice that because \( a^{-3} \simeq n \), \( \omega_0 \) is of order of the plasma frequency \( 4\pi ne^2/m \) which is quite a natural result. Thus the “Debye frequency” of a Wigner crystal is the plasma frequency. The quantum amplitude is related to the frequency via

\[
\frac{1}{mr_0^2} \simeq \omega_0
\]

or

\[
\left( \frac{1}{mr_0^2} \right)^2 \simeq \omega_0^2 \simeq \frac{e^2}{ma^3}.
\]

(21)
Squaring Eq.(40) and using Eq.(21), we find

\[
\left( \frac{e^2}{a} \right)^2 \gg \left( \frac{1}{mr_0^2} \right)^2 = \frac{e^2}{ma^2}
\]

\[e^2ma^2 \gg 1\]

or

\[r_s \gg 1.\]

Thus crystallization is energetically favorable for \(r_s \gg 1\). Quantum Monte Carlo simulations show that the critical value of \(r_s\), at which the liquid crystallizes, is 150 in 3D and 37 in 2D. Why so huge numbers? This one can understand by recalling that crystals melt when the amplitude of the oscillations is still smaller than the lattice spacing. The critical ratio of the amplitude to spacing is called the Lindemann parameter \(\Lambda\). For all lattices, \(\Lambda\) is appreciably smaller than unity (this helps to understand why the melting temperatures are significantly smaller than cohesive energies). Typically, \(\Lambda \approx 0.1 - 0.3\). (Strictly speaking, one has to distinguish between classical and quantum Lindemann parameters—since entropy plays role for the former but not for the latter—but we will ignore this subtlety). With the help of Eq.(21), we find that

\[r_0/a = \frac{1}{r_s^{1/4}}.\]

Wigner crystal melts when

\[r_0/a = \Lambda \rightarrow\]

\[r_s^{\text{c}} = \frac{1}{\Lambda^4} \gg 1.\]

Wigner crystals were observed in layers of electrons adsorbed on a surface of liquid helium (that makes the smoothest substrate one can think of). The search for Wigner crystallization in semiconductor heterostructures is a very active field which so far has not provided a direct evidence for this effect although a circumstantial evidence does exist. The main problem here is that effects of disorder in solid-state structures becomes very pronounced at lower densities so there is no hope to observe Wigner crystallization in its pure form. What one can hope for is to get a distorted crystal \((\text{Coulomb glass})\).
V. FERMI-LIQUID THEORY

A. General concepts

A detailed theory of the Fermi liquid (FL), and its microscopic justification in particular, goes far beyond the scope of this course. Standard references\textsuperscript{3,4,5} provide an exhaustive if not elementary treatment of the FL theory. In what follows, I will do a simplified version of the theory ("FL-lite") and illustrate main concepts on various examples.

1. Motivation

All Fermi systems (metals, degenerate semiconductors, normal He\textsuperscript{3}, neutron stars, etc.) belong to the categories of either moderately or strongly interacting systems. For example, in metals \( r_s \) in the range from 2 to 5. (There are only few exceptions of this rule; for example, bismuth, in which the large value of the background dielectric constant brings the value of \( r_s \) to 0.3 and GaAs heterostructures in which the small value of the effective mass \(-0.07\) of the bare mass–leads to the higher value of the Fermi energy and thus to \( r_s < 1 \) in a certain density range). On the other hand, as we learned from the section on Wigner crystallization, the critical \( r_s \) for Wigner crystallization is very high \(-150\) in 3D and 37 in 2D. Thus almost all Fermi systems occurring in Nature are too strongly interacting to be described by the weak-coupling theory (Hartree-Fock and its improved versions) but too weakly interacting to solidify. In short, since they are neither gases nor solids the only choice left is that they are liquids. A liquid is a system of interacting particles which preserves all symmetries of the gas. Following this analogy, Landau put forward a hypothesis that an interacting Fermi system is qualitatively similar to the Fermi gas\textsuperscript{6}. Although original Landau’s formulation refers to a translationally invariant system of particles interacting via short-range forces, e.g., normal He\textsuperscript{3}, later on his arguments were extended to metals (which have only discrete symmetries) and to charged particles.

Experiment gives a strong justification to this hypothesis. The specific heat of almost all fermionic systems (in solids, one need to subtract off the lattice contribution to get the one from electrons) scales linearly with temperature: \( C(T) = \gamma^* T \). (Some systems demonstrate the deviation from this law and these systems are subject of an active studies for the last 10 years; see more in Non-fermi-liquid behavior). In a free Fermi gas, \( \gamma^* = \gamma = (\pi^2/3) \nu_F = \)
In a band model, when non-interacting electrons move in the presence of a periodic potential, one should use the appropriate value of the density of states at the Fermi level for a given lattice structure. In reality, the coefficient $\gamma^*$ can differ significantly from the band value but the linearity of $C(T)$ in $T$ is well-preserved. In those cases, when one can change continuously the interaction (for example, by applying pressure to normal He$_3$), $\gamma^*$ is found to vary. One is then tempted to assume that the interacting Fermi liquid is composed of some effective particles (quasi-particles) that behave as free fermions albeit their masses are different from the non-interacting values.

**B. Scattering rate in an interacting Fermi system**

The Pauli principle leads to a slow-down of mutual scattering of fermions in a degenerate Fermi system. A qualitative argument is that two fermions can interact effectively if their energies happen to be within the $T$ intervals around the Fermi energy. The probability that one the energies is within this interval is of order $T/E_F$ and, since the particles are independent, the scattering probability is proportional to $(T/E_F)^2$ which is much smaller than unity for $T \ll E_F$.

A precise definition of the scattering rate depends on the quantity measured. In general, different quantities, such as charge and thermal conductivities, contain scattering rates that differ at least by numerical factors. To avoid this complication, we adopt one of the possible definitions of the scattering rate. Consider a scattering process in which an electron with momentum $k$ collides with an electron with momentum $k$ such that the momenta of the final states are $k - q$ and $k + q$, correspondingly. The interaction is assumed to be a screened Coulomb potential

$$U(q, \omega) = \frac{4\pi e^2}{q^2\epsilon(q, \omega)},$$  \hspace{1cm} (22)

where $\epsilon(q, \omega)$ is the dielectric function of the electron gas. The Fermi golden rule for the number of transitions from a given state $k$ is

$$\frac{1}{\tau} = 2\pi \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} |U(q, \varepsilon_k - \varepsilon_{k-q})|^2 \delta(\varepsilon_k + \varepsilon_k - \varepsilon_{k-q} - \varepsilon_{k+q}) f_{0k}(1 - f_{0k-q})(1 - f_{0k+q}),$$ \hspace{1cm} (23)

where $f_{0k} \equiv f_0(\varepsilon_k)$ is the Fermi function. The combination of the Fermi functions in the equation above ensures that the initial state $k$ of the scattering process is occupied while
the final ones are empty. It is convenient to introduce the energy transfer \( \omega = \varepsilon_k - \varepsilon_{k-q} = \varepsilon_{k+q} - \varepsilon_k \) and re-write the \( \delta \)-function as an integral over \( \omega \):

\[
\delta(\varepsilon_k - \varepsilon_{k-q} - \varepsilon_{k+q}) = \int_{\infty} \, d\omega \delta(\varepsilon_k - \varepsilon_{k-q} - \omega) \delta(\varepsilon_{k+q} - \varepsilon_k - \omega).
\]

Then

\[
\frac{1}{\tau} = 2\pi \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^\infty \, d\omega |U(q, \omega)|^2 \delta(\varepsilon_k - \varepsilon_{k-q} - \omega) \delta(\varepsilon_{k+q} - \varepsilon_k - \omega)
\times f_0(\varepsilon_k)(1 - f_0(\varepsilon_k - \omega))(1 - f_0(\varepsilon_k + \omega)),
\]

(24)

We assume now (and justify later) that \( \epsilon(q, \omega) \) can be taken in the static limit (\( \omega = 0 \)) and that typical \( q \ll k_F \), such that we can use the Thomas-Fermi result for \( \epsilon(q \ll k_F, 0) = 1 + \kappa^2/|q|^2 \) with \( \kappa^2 = 4\pi e^2 \nu(E_F) \):

\[
U(q, \omega) \rightarrow U(q, 0) = \frac{4\pi e^2}{q^2 + \kappa^2}.
\]

(25)

We will see that once the interaction potential is assumed to be static, the \( T^2 \) form of \( 1/\tau \) is obtained under very broad assumptions on the potential.

We choose \( k \) to be close to \( k_F \) and assume that \( p \) is near \( k_F \) as well. The arguments of the \( \delta \) functions in \( q \) can be rewritten as

\[
\delta(\varepsilon_k - \varepsilon_{k-q} - \omega) = \delta\left(\frac{k^2}{2m} - \frac{(k-q)^2}{2m} - \omega\right) = \delta\left(\frac{kp}{m} \cos \theta_{pq} - \frac{q^2}{2m} - \omega\right)
\]

\[
\delta(\varepsilon_{k+q} - \varepsilon_k - \omega) = \delta\left(\frac{(k+q)^2}{2m} - \frac{k^2}{2m} - \omega\right) = \delta\left(\frac{pq}{m} \cos \theta_{pq} + \frac{q^2}{2m} - \omega\right),
\]

(26)

where \( \theta_{pq} = \angle \{k, q\} \) and \( \theta_{kq} = \angle \{k, q\} \). The energy transfers will be shown to be small: of order \( T \). Therefore, we can neglect \( \omega \) in the arguments of the \( \delta \) functions (but keep it in the arguments of the Fermi functions because there \( \omega \) is divided by \( T \)). The angular integrations can now be readily performed. Write the angular parts of the integral as \( \int d(\cos \theta_{pq}) \int d(\cos \theta_{pq}) \) and integrate in this particular order (that is, first integrate over the direction of \( k \) at fixed \( q \) and then integrate over the direction of \( q \) at fixed \( k \)). With \( \omega = 0 \) in the \( \delta \)-functions, the values of the cosines are \( \cos \theta_{pq} = q/2k \approx q/2k_F \ll 1 \) (since by assumption \( q \ll k_F \) and \( -\cos \theta_{pq} = q/2p \approx q/2k_F \ll 1 \), which is well within the integration interval \( -1 < \cos \theta_{pq}, \cos \theta_{pq} < 1 \). Performing angular integration, we obtain

\[
\frac{1}{\tau} = \frac{1}{32\pi^3 v_F^2} \int_0^\infty \, dp \int_{-\infty}^\infty \, dq \int_{-\infty}^\infty \, d\omega |U(q, 0)|^2
\times f_0(\varepsilon_k)(1 - f_0(\varepsilon_k - \omega))(1 - f_0(\varepsilon_k + \omega)),
\]

(27)

Now, taking into account that \( p \) is near \( k_F \), we approximate \( \int_{-\infty}^\infty \, dq \approx \int d(p - k_F) k_F^2 = \frac{k_F^2}{v_F} \int_{-\infty}^\infty \, d\xi_p \) where \( \xi_p = v_F(p - k_F) \) is the energy of excitation measured from the Fermi
energy. Since the Fermi functions contain only the differences \(\varepsilon_k - E_F \approx \xi_k\), etc., we can replace \(\varepsilon_k\) by \(\xi_k\) and \(\varepsilon_k\) by \(\xi_p\) in there:

\[
\frac{1}{\tau} = \frac{k_F^2}{32\pi^3 v_F^3} \int_0^\infty dq |U(q, 0)|^2 \int_\infty^\infty d\xi_p \int_{-\infty}^{\infty} d\omega f_0(\xi_p)(1 - (1 - f_0(\xi_p + \omega))).
\] (28)

We now see that the energy and momentum integrals are completely decoupled. First we integrate over \(\xi_p\) using

\[
\int_{-\infty}^\infty d\xi_p f_0(\xi_p)(1 - f_0(\xi_p + \omega)) = \frac{\omega}{1 - \exp(-\omega/T)}.
\] (29)

Rescaling \(\omega = yT\), the remaining integral is reduced to

\[
\frac{1}{\tau} = \frac{T^2 m^2}{32\pi^3 v_F} \int_0^\infty dq |U(q, 0)|^2 F(\xi_k/T),
\] (30)

where

\[
F(x) = \int_{-\infty}^\infty dy \frac{y}{\exp(y - x) + 1(1 - \exp(-y))}.
\] (31)

For a particle right on the Fermi surface \((\xi_k = 0)\), we obtain

\[
F(0) = \int_0^\infty dy \frac{y}{\sinh y} = \frac{\pi^2}{4}
\] (32)

and

\[
\frac{1}{\tau} = \frac{T^2 m^2}{128\pi^3 v_F} \int dq |U(q, 0)|^2.
\] (33)

As we had announced, the \(T^2\) law is independent of a particular form of \(U(q, 0)\). For a screened Coulomb potential, we obtain

\[
\int_0^\infty dq |U(q, 0)|^2 = (4\pi e^2)^2 \int_0^\infty dq \frac{1}{(q^2 + \kappa^2)^2} = \frac{(4\pi e^2)^2 \pi}{\kappa^3} = \frac{\pi \kappa}{4\nu^2(E_F)}.
\] (34)

Recalling that \(\nu(E_F) = k_F^2/\pi^2 v_F\), the final result can be written as

\[
\frac{1}{\tau} = \frac{\pi^4 \kappa}{1024 k_F E_F} T^2
\] (35)

or, on restoring \(k_B\) and \(\hbar\),

\[
\frac{1}{\tau} = \frac{\pi^4 \kappa}{1024 k_F E_F} \frac{(k_B T)^2}{\hbar E_F}.
\] (36)

Now, we need to verify assumptions made en route to the final result. We assumed that \(\omega\) can be set to zero in the dielectric function and that typical \(q \ll k_F\). From the last integral over \(q\), we see that typical \(q \sim \kappa \ll k_F\) by assumption of weak e-e interaction, hence the second assumption is satisfied. On the other hand, typical values of \(y\) in (32) are \(\sim 1\), hence
typical $\omega \sim T$. The dielectric function depends on the combination $\omega/v_F q$, hence the static limit is justified if $\omega \ll v_F q \sim v_F \kappa$. Notice that $v_F^2 \kappa^2 \sim v_F^2 e^2 k_F^2 / v_F \sim e^2 n / m \sim \omega_p^2$, where $\omega_p \ll E_F$ is the plasma frequency. Therefore, the $T^2$ law in an electron system is valid for $T \ll \omega_p$.

The integral in Eq. (31) can be calculated exactly with the result

$$ F(x) = \frac{1}{2} \left( x^2 + \pi^2 \right) \frac{1}{1 + e^{-x}}. \quad (37) $$

Then we get instead of Eq. (35) at finite $\xi_k$

$$ \frac{1}{\tau} = \frac{\pi^2}{512} \frac{\kappa^2}{k_F} \frac{\xi_k^2}{E_F} \frac{T^2}{1 + \exp(-\xi_k/T)} \quad (38) $$

C. Quasi-particles

The concept of quasi-particles central to the Landau’s theory of Fermi liquids. The ground state of a Fermi gas is a completely filled Fermi sphere. The spectrum of excited states can be classified in terms of how many fermions were promoted from states below the Fermi surface to the ones above. For example, the first excited state is the one with an electron above the Fermi sphere and the hole below. The energy of this state, measured from the ground state, is $\epsilon = p^2/2m - E_F$. The net momentum of the system is $p$. The next state correspond to two fermions above the Fermi sphere, etc. If the net momentum of the system is $p$, then $p_1 + p_2$, where $p_1$ and $p_2$ are the momenta of individual electrons. We see that in a free system, any excited macroscopic state is a superposition of single-particle states.

This is not so in an interacting system. Even if we promote only one particle to a state above the Fermi surface, the energy of this state would not be equal to $p^2/2m - E_F$ because the interaction will change the energy of all other fermions. However, Landau assumed that excited states with energies near the Fermi level, that is, weakly excited states, can be described as a superposition of elementary excitations which behave as free particles, although the original system may as well be a strongly interacting one. An example of such a behavior are familiar phonons in a solid. Suppose that we have a gas of sodium atoms (which are fermions) which essentially don’t interact because of low density. The elementary excitations in an ideal gas simply coincide with real atoms. Now we condense gas into metals. Individual atoms are not free to move on their own. Instead, they can only
participate in a collective oscillatory motion which is a sound wave. For small frequencies, the sound wave can be thought of consisting of elementary quanta of free particles–phonons. The spectrum of each phonon branch is $\omega_i(q) = s_i q$, where $s_i$ is the speed of sound and the oscillatory energy is

$$E = \sum_i \int \frac{d^3q}{(2\pi)^3} \omega_i n_i,$$

where $n_i$ is the number of excited phonons at given temperature. If the number of phonons is varied, so is the total energy

$$\delta E = \sum_i \int \frac{d^3q}{(2\pi)^3} \omega_i \delta n_i.$$

Quite similarly, Landau assumed that the variation of the total energy of a single-component Fermi liquid (or single band metal) can be written as

$$\delta E = \int \frac{d^3p}{(2\pi)^3} \varepsilon(p) \delta n(p).$$

(39)

(For the sake of simplicity, I also assume that the system is isotropic, i.e., the energy and $n$ depend only on the magnitude but not the direction of the momentum but the argument extends easily to anisotropic systems as well). In this formula, $n(p)$ is the distribution function of quasi-particles which are elementary excitations of the interacting system. In analogy with bosons, these quasi-particles are free. One more –and crucial assumption– is that the quasi-particles of an interacting Fermi system are fermions which is not at all obvious. For example, regardless of the statistics of individual atoms which can be either fermions or bosons, phonons are always bosons. Landau’s argument was that if quasi-particles were bosons they could accumulate without a restriction in every quantum state. That means that an excited state of a quantum system has a classical analog. Indeed, an excited state of many coupled oscillators is a classical sound wave. Fermions don’t have macroscopic states so quasi-particles of a Fermi systems must be fermions (to be precise they should not be bosons; proposal for particles of a statistics intermediate between bosons and fermions–anyons–have been made recently). Thus,

$$n = \frac{1}{\exp \left( \frac{\varepsilon - \mu}{T} \right) + 1},$$

On quite general grounds, one can show that quasi-particles must have spin $1/2$ regardless of (half-integer) spin of original particles$^3, 4$. (Thus, quasi-particles of a system composed
of fermions with spin \( S = 3/2 \) still have spin \( 1/2 \).) Generally speaking, the quasiparticle energy is an operator in the spin space: \( \varepsilon \rightarrow \hat{\varepsilon} \) and, correspondingly,

\[
n \rightarrow \hat{n} = \frac{1}{\exp\left(\frac{\hat{\varepsilon} - \mu}{T}\right) + 1},
\]

(40)

where \( \hat{\varepsilon} \) and \( \hat{n} \) are \( 2\times2 \) matrices. If the system is not in the presence of the magnetic field and not ferromagnetic,

\[
\hat{\varepsilon}_{\alpha\beta} = \varepsilon \delta_{\alpha\beta}, \quad \hat{n}_{\alpha\beta} = n \delta_{\alpha\beta}.
\]

In a general case, instead of (39) we have

\[
\delta E = \int \frac{d^3p}{(2\pi)^3} \text{Tr}\hat{\varepsilon}(p) \delta \hat{n}(p),
\]

which, for a spin-isotropic liquid, reduces to

\[
\delta E = 2 \int \frac{d^3p}{(2\pi)^3} \text{Tr}\varepsilon(p) \delta n(p).
\]

The occupation number is normalized by the condition

\[
\delta N = \int \frac{d^3p}{(2\pi)^3} \text{Tr}\delta n(p) = 2 \int \frac{d^3p}{(2\pi)^3} \delta n(p) = 0,
\]

where \( N \) is the total number of real particles.

For \( T=0 \), the chemical potential coincides with energy of the topmost state

\[
\mu = \varepsilon(p_F) \equiv E_F.
\]

Another important property (known as Luttinger theorem) is that the volume of the Fermi surface is not affected by the interaction. For an isotropic system, this means the Fermi momenta of free and interacting systems are the same. A simple argument is that the counting of states is not affected by the interaction, i.e., the relation

\[
N = 2 \frac{4\pi p_F^3}{3 (2\pi)^3}
\]

holds in both cases. A general proof of this statement is given in Ref. [3].

1. Interaction of quasi-particles

Phonons in a solid do not interact only in the first (harmonic) approximation. Anharmonism results in the phonon-phonon interaction. However, the interaction is weak at low
energies not really because the coupling constant is weak but rather because the scattering rate of phonons on each other is proportional to a high power of their frequency: \( \tau_{\text{ph-ph}}^{-1} \propto \omega^5 \).

As a result, at small \( \omega \) phonons are almost free quasi-particles. Something similar happens with the fermions. The nominal interaction may as well be strong. However, because of the Pauli principle, the scattering rate is proportional to \((\varepsilon - E_F)^2\) and weakly excited states interact only weakly. In the Landau theory, the interaction between quasi-particles is introduced via a phenomenological interaction function defined by the proportionality coefficient (more precisely, a kernel) between the variation of the occupation number and the corresponding variation in the quasi-particle spectrum

\[
\delta \varepsilon_{\alpha \beta} = \int \frac{d^3p'}{(2\pi)^3} f_{\alpha \gamma, \beta \delta} (p, p') \delta n_{\gamma \delta} (p') .
\]

(summation over the repeated indices is implied). Function \( f_{\alpha \gamma, \beta \delta} (p, p') \) describes the interaction between the quasi-particles of momenta \( p \) and \( p' \) (notice that these are both initial states of the of the scattering process). Spin indices \( \alpha \) and \( \beta \) correspond to the state of momentum \( p \) whereas indices \( \gamma \) and \( \delta \) correspond to momentum \( p' \). In a matrix form,

\[
\delta \varepsilon = \text{Tr'} \int \frac{d^3p'}{(2\pi)^3} \hat{f} (p, p') \delta \hat{n} (p') ,
\]

(41)

where \( \text{Tr'} \) denotes trace over spin indices \( \gamma \) and \( \delta \). For a spin-isotropic FL, when \( \delta \varepsilon = \delta_{\alpha \beta} \delta \varepsilon \) and \( \delta \hat{n} = \delta_{\alpha \beta} \delta n \) external spin indices (\( \alpha \) and \( \beta \)) can also be traced out and Eq.(41) reduces to

\[
\delta \varepsilon = \int \frac{d^3p'}{(2\pi)^3} f (p, p') \delta n (p') ,
\]

where

\[
f (p, p') \equiv \frac{1}{2} \text{TrTr'} \hat{f} (p, p') .
\]

For small deviations from the equilibrium, \( \delta n (p') \) is peaked near the Fermi surface. Function \( \hat{f} (p, p') \) can be then estimated directly on the Fermi surface, i.e., for \( |p| = |p'| = p_F \). Then \( \hat{f} \) depends only on the angle between \( p \) and \( p' \). The spin dependence of \( \hat{f} \) can be established on quite general grounds. In a spin-isotropic FL, \( \hat{f} \) can depend only on the scalar product of spin operators but on the products of the individual spin operators with some other vectors. Thus the most general form of \( \hat{f} \) for a spin-isotropic system is

\[
\nu^* \hat{f} (p, p') = F^a (\theta) \hat{I}^a + F^a (\theta) \hat{\sigma} \cdot \hat{\sigma}' ,
\]

where

\[
F^a (\theta) = \left( \begin{array}{c}
F_{11}^a (\theta) \\
F_{12}^a (\theta) \\
F_{21}^a (\theta) \\
F_{22}^a (\theta)
\end{array} \right) ,
\]

and

\[
\nu^* = \left( \begin{array}{cccc}
\nu_{11} & \nu_{12} & \nu_{13} & \nu_{14} \\
\nu_{21} & \nu_{22} & \nu_{23} & \nu_{24} \\
\nu_{31} & \nu_{32} & \nu_{33} & \nu_{34} \\
\nu_{41} & \nu_{42} & \nu_{43} & \nu_{44}
\end{array} \right) .
\]
where $\hat{I}$ is the unity matrix, $\hat{\sigma}$ is the vector of three Pauli matrices, $\theta$ is the angle between $\mathbf{p}$ and $\mathbf{p}'$, and the density of states was introduced just to make functions $F^s(\theta)$ and $F^a(\theta)$ dimensionless. (Star in $\nu^*$ means that we have used a renormalized value of the effective mass so that $\nu^* = m^*k_F/\pi^2$, but this is again just a matter of convenience.) Explicitly,

$$\nu^* f_{\alpha\gamma,\beta\delta} (\mathbf{p}, \mathbf{p}') = F^s(\theta) \delta_{\alpha\beta} \delta_{\gamma\delta} + F^a(\theta) \hat{\sigma}_{\alpha\beta} \cdot \hat{\sigma}_{\gamma\delta}. \quad (42)$$

In general, the interaction function is not known. However, if the interaction is weak, one can relate $f$ to the pair interaction potential. In a microscopic theory, it can be shown (see Appendix A) that if particles interact via a weak pair-wise potential $U(q)$ such that $U(0)$ is finite (which excludes, e.g., a bare Coulomb potential), then to the first order in this potential

$$f_{\alpha\gamma,\beta\delta} (\mathbf{p}, \mathbf{p}') = \delta_{\alpha\beta} \delta_{\gamma\delta} \left[ U(0) - \frac{1}{2} U(|\mathbf{p} - \mathbf{p}'|) \right] - \frac{1}{2} \hat{\sigma}_{\alpha\beta} \cdot \hat{\sigma}_{\gamma\delta} U(|\mathbf{p} - \mathbf{p}'|).$$

On the Fermi surface, $|\mathbf{p} - \mathbf{p}'| = 2p_F \sin \theta/2$. Comparing this expression with Eq.(42), we find that

$$F^s(\theta) = \nu \left[ U(0) - \frac{1}{2} U(2p_F \sin(\theta/2)) \right] \quad (43)$$

$$F^a(\theta) = -\nu \frac{1}{2} U(2p_F \sin(\theta/2)).$$

Notice that repulsive interaction ($U > 0$) corresponds to the attraction in the spin-exchange channel ($F^a < 0$).

D. General strategy of the Fermi-liquid theory

One may wonder what one can achieve introducing unknown phenomenological quantities, such as the interaction function or its charge and spin components. FL theory allows one to express general thermodynamic characteristic of a liquid (effective mass, compressibility, spin susceptibility, etc.) via the angular averages of Landau functions–$F^s(\theta)$ and $F^a(\theta)$. Some quantities depend on the same averages and thus one express such quantities via each other. The relationship between such quantities can be checked by comparison with the experiment.
E. Effective mass

As a first application of the Landau theory, let’s consider the effective mass. In a Galilean-invariant system, the flux of particles does not depend on the interaction, so one can calculate the same flux in terms of either original particles with velocity \( v = p/m \) or in terms with quasiparticles with velocity \( \partial \epsilon / \partial p \)

\[
\text{Tr} \int \frac{d^3p}{(2\pi)^3} \frac{p}{m} \hat{n} = \text{Tr} \int \frac{d^3p}{(2\pi)^3} \frac{\partial \epsilon}{\partial p} \hat{n},
\]

where \( m \) is the bare mass. Taking the variation of both sides of this equation, we obtain

\[
\text{Tr} \int \frac{d^3p}{(2\pi)^3} \frac{p}{m} \delta \hat{n} = \text{Tr} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\partial \delta \epsilon}{\partial p} \hat{n} + \frac{\partial \epsilon}{\partial p} \delta \hat{n} \right].
\]

(44)

Now,

\[
\frac{\partial \delta \epsilon}{\partial p} = \text{Tr}' \int \frac{d^3p'}{(2\pi)^3} \frac{\partial f(p,p')}{\partial p} \delta n(p').
\]

For a spin-isotropic liquid, reduces to

\[
\int \frac{d^3p}{(2\pi)^3} \frac{p}{m} \delta n = \int \frac{d^3p}{(2\pi)^3} \frac{\partial \epsilon}{\partial p} \delta n + \int \frac{d^3p}{(2\pi)^3} \left[ \int \frac{d^3p'}{(2\pi)^3} \frac{\partial f(p,p')}{\partial p} \delta n(p') \right] n(p).
\]

In the second term re-label the variables \( p \to p', p' \to p \), use the fact that \( f(p,p') = f(p',p) \) and integrate by parts

\[
\int \frac{d^3p}{(2\pi)^3} \frac{p}{m} \delta n = \int \frac{d^3p}{(2\pi)^3} \frac{\partial \epsilon}{\partial p} \delta n - \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} f(p,p') \frac{\partial n(p')}{\partial p'} \delta n(p).
\]

Because this equation should be satisfied for an arbitrary variation \( \delta n(p) \),

\[
\frac{p}{m} = \frac{\partial \epsilon}{\partial p} - \int \frac{d^3p'}{(2\pi)^3} f(p,p') \frac{\partial n(p')}{\partial p'}.
\]

Near the Fermi surface, the quasi-particle energy can be always written as

\[
\epsilon(p) = v_F^*(p - p_F)
\]

so that

\[
\frac{\partial \epsilon}{\partial p} = v_F^* \hat{p} = \frac{p_F}{m^*} \hat{p},
\]

where \( \hat{p} \) is the unit vector in the direction of \( p \), and thus

\[
\frac{p}{m} = \frac{p_F}{m^*} \hat{p} - \int \frac{d^3p'}{(2\pi)^3} f(p,p') \frac{\partial n(p')}{\partial p'}.
\]
Now, \( n(p) = \theta(p_F - p) \) near the Fermi surface so that

\[
\frac{\partial n(p')}{\partial p'} = -\delta(p - p_F) \hat{p}
\]

and

\[
\frac{p}{m} = \frac{p_F \hat{p}}{m^*} + \frac{p_F^2}{2\pi^2} \int \frac{d\Omega'}{4\pi} f(p, p') \hat{p}'.
\]

Setting \(|p| = p_F\) and multiplying dotting the equation above into \(\hat{p}\), we obtain

\[
\frac{1}{m} = \frac{1}{m^*} + \frac{1}{\pi^2} \int \frac{d\Omega}{4\pi} f(\theta) \cos \theta
\]

or

\[
\frac{m^*}{m} = 1 - \frac{m^* p_F}{2\pi^2} \int \frac{d\Omega}{4\pi} f(\theta) \cos \theta.
\]

This is the Landau’s formula for the effective mass. Noticing that \( f(\theta) = \frac{1}{2} T \) \( T' \hat{f}(\theta) = \frac{2}{\pi^2} F^*(\theta) \) \( F^*(\theta) = \frac{2\pi^2}{m^* p_F} F^*(\theta) \) the last equation can be re-written in term of the dimensionless Landau function as

\[
\frac{m^*}{m} = 1 + \int \frac{d\Omega}{4\pi} F^*(\theta) \cos \theta \equiv 1 + F^*_1.
\]

Although we do not know the explicit form of \( f(\theta) \), some useful conclusions can be made already from the most general form [Eq.(45)]. If \( \langle f(\theta) \cos \theta \rangle \) is negative, \( 1/m^* > 1/m \rightarrow m^* < m \); conversely, if \( \langle f(\theta) \cos \theta \rangle \) is positive, \( m^* > m \). For the weak short-range interaction, we know that \( m^* > m \), whereas for a weak long-range (Coulomb) interaction \( m^* < m \). Now we see that both of these cases are described by the Landau’s formula. Recalling the weak-coupling form of \( F^*(\theta) \) [Eq.(43)], we find

\[
F^*_1 = \int \frac{d\Omega}{4\pi} F^*(\theta) \cos \theta = \nu \int \frac{d\Omega}{4\pi} \left[ U(0) - \frac{1}{2} U \left( 2p_F \sin \frac{\theta}{2} \right) \right] \cos \theta
\]

\[
= -\nu \int \frac{d\Omega}{4\pi} \frac{1}{2} U \left( 2p_F \sin \frac{\theta}{2} \right) \cos \theta.
\]

If \( U \) is repulsive and picked at small \( \theta \) where \( \cos \theta \) is positive, \( F_1 < 0 \) and \( m^* < m \). This case corresponds to a screened Coulomb potential. If \( U \) is repulsive and depends on \( \theta \) only weakly (short-range interaction), the angular integral is dominated by values of \( \theta \) close to \( \pi \), where \( \cos \theta < 0 \). Then \( m^* > m \). In general, forward scattering reduces the effective mass, whereas large-angle scattering enhances it.
F. Spin susceptibility

1. Free electrons

We start with free electrons. An electron with spin $s$ has a magnetic moment $\mu = 2\mu_B s$, where $\mu_B = e\hbar/2mc$ is the Bohr magneton. Zeeman splitting of energy levels is $\Delta E = E_\uparrow - E_\downarrow = \mu_B H - (-\mu_B H) = 2\mu_B H$. The number of spin-up and -down electrons is found as an integral over the density of states

$$n_{\uparrow,\downarrow} = \frac{1}{2} \int_0^{E_F \pm \mu_B H} d\varepsilon \nu(\varepsilon).$$

The Fermi energy now also depends on the magnetic field but the dependence is only quadratic (why?), whereas the Zeeman terms are linear in $H$. For weak fields, one can neglect the field-dependence of $E_F$. Total magnetic moment per unit volume–magnetization–is found by expanding in $H$

$$M = \mu_B (n_\uparrow - n_\downarrow) = \frac{\mu_B}{2} \left[ \int_0^{E_F + \mu_B H} d\varepsilon \nu(\varepsilon) - \int_0^{E_F - \mu_B H} d\varepsilon \nu(\varepsilon) \right] = \mu_B^2 H \nu_F.$$

Spin susceptibility

$$\chi = \frac{\partial M}{\partial H} = \mu_B^2 \nu_F$$

is finite and positive, which means that spins are oriented along the external magnetic field.

Notice that if, for some reason the $g$-factor of electrons were not equal to 2, then Eq.(46) would change to

$$\chi = g\mu_B^2 \nu_F$$

2. Fermi liquid

In a Fermi liquid, the energy of a quasi-particle in a magnetic field is changed not only due to the Zeeman splitting (as in the Fermi gas) but also because the occupation number is changed. This effect brings in an additional term in the energy as the energy is related to the occupation number. This is expressed by following equation

$$\delta \hat{\varepsilon} = -\mu_B \mathbf{H} \cdot \hat{\mathbf{\sigma}} + T Y' \int \frac{d^3 p'}{(2\pi)^3} \hat{f}(p', p') \delta \hat{n}(p').$$
The first term is just the Zeeman splitting, the second one comes from the interaction. Now,

$$\delta \hat{n} = \frac{\partial \hat{n}}{\partial \varepsilon} \delta \hat{\varepsilon},$$

and we have an equation for $\delta \hat{\varepsilon}$

$$\delta \hat{\varepsilon} = -\mu_B \mathbf{H} \cdot \hat{\sigma} + \text{Tr} \int \frac{d^3 p'}{(2\pi)^3} f (\mathbf{p}, \mathbf{p}') \frac{\partial \hat{n}}{\partial \varepsilon'} \delta \hat{\varepsilon} (\mathbf{p}').$$

Replacing $\frac{\partial \hat{n}}{\partial \varepsilon'}$ by the delta-function and setting $|\mathbf{p}| = p_F$

$$\delta \hat{\varepsilon} (p_F \hat{p}) = -\mu_B \mathbf{H} \cdot \hat{\sigma} - \text{Tr} \nu_F^* \int \frac{d\Omega'}{4\pi} \hat{f} (p_F \hat{p}, p_P \hat{p}') \delta \hat{\varepsilon} (p_F \hat{p}'). \quad (48)$$

Try a solution of the following form

$$\delta \hat{\varepsilon} = -\frac{\mu_B}{2} g \mathbf{H} \cdot \hat{\sigma}, \quad (49)$$

where $g$ has the meaning of an effective $g-$ factor. For free electrons, $g = 2$. Recalling that $\nu_F^* f = F_s^*(\theta) \hat{I} + F^a(\theta) \hat{\sigma} \cdot \hat{\sigma}'$ and substituting (49) into (49), we obtain

$$-\frac{\mu_B}{2} g \mathbf{H} \cdot \hat{\sigma} = -\mu_B \mathbf{H} \cdot \hat{\sigma} - \text{Tr} \int \frac{d\Omega'}{4\pi} \left[ F_s^*(\theta) \hat{I} + F^a(\theta) \hat{\sigma} \cdot \hat{\sigma}' \right] \left( -\frac{\mu_B}{2} g \mathbf{H} \cdot \hat{\sigma}' \right).$$

The term containing $F_s$ in the integral vanishes because Pauli matrices are traceless: $\text{Tr} \hat{\sigma} = 0$. The term containing $F^a$ is re-arranged using the identity $\text{Tr}(\hat{\sigma} \cdot \hat{\sigma}') \hat{\sigma}' = 2 \hat{\sigma}$ upon which we get

$$\frac{g}{2} = 1 - \frac{g}{2} F_0^a,$$

where

$$F_0^a = \int \frac{d\Omega}{4\pi} F^a(\theta).$$

Therefore,

$$g = \frac{2}{1 + F_0^a}.$$

Substituting this result into Eq.(47) and replacing $\nu_F$ by its renormalized value $\nu_F^*$, we obtain the spin susceptibility of a Fermi liquid

$$\chi^* = \frac{\mu_B^2 \nu_F^*}{1 + F_0^a} = \chi \frac{1 + F_1^s}{1 + F_0^a},$$

where $\chi$ is the susceptibility of free fermions. Notice that $\chi^*$ differs from $\chi$ both because the effective mass is renormalized and because the g-factor differs from 2. When $F_0^a = -1,$
the g-factor and susceptibility diverge signaling a ferromagnetic instability. However, even if \( F_0^a = 0 \), \( \chi^* \) is still renormalized in proportion to the effective mass. Recall that [see Eq.(43)] in the weak-coupling limit \( F^a(\theta) = -\frac{1}{2} U (2p_F \sin \theta/2) \). Thus, for repulsive interactions \( g > 2 \) which signals a ferromagnetic tendency. This is another manifestation of general principle that repulsively interacting fermions tend to have spin aligned to minimize the energy of repulsion. For normal He\(^3\), \( F_0^a \approx -2/3 \) over a wide range of temperatures.

G. Zero sound

All gases and liquids support sound waves. Even ideal gases have finite compressibilities and therefore finite sound velocities

\[
s = \sqrt{\frac{\partial P}{\partial \rho}}. \]

For example, in an ideal Boltzmann gas \( P = nT = \rho T/m \) and

\[
s = \sqrt{\frac{T}{m}} = \frac{1}{\sqrt{3}} v_T, \]

where \( v_T = \sqrt{3T/m} \) is the rms thermal velocity. In an ideal Fermi gas, \( P = (2/3)E \) where \( E = (2/5) nE_F \) is the ground state energy and

\[
s = \frac{1}{\sqrt{3}} v_F. \]

It seems that interactions are not essential for sound propagation. This conclusion is not true. In a sound wave, all thermodynamic characteristics—density \( n(r,t) \), pressure \( P(r,t) \), temperature \( T(r,t) \) (if the experiment is performed under adiabatic conditions), etc.—vary in space and time in sync with the sound wave. Thus we are dealing with a non-equilibrium situation. To describe a non-equilibrium situation by a set of local and time-dependent quantities, one needs to maintain local and temporal equilibrium. When the sound wave arrives to an initially unperturbed region, this region is driven away from its equilibrium state. Collisions between molecules (or fermions in a Fermi gas) has to be frequent enough to establish a new equilibrium state before the sound wave leaves the region. For sound propagation, the characteristic spatial scale is the wavelength \( \lambda \) and characteristic time is the period \( 2\pi/\omega \). If the mean free path and time are \( l \) and \( \tau \), respectively, the conditions of establishing local and temporal equilibrium are

\[
\lambda \gg l
\]
Therefore, collisions (interactions) are essential to ensure that the sound propagation occurs under quasi-equilibrium conditions. For classical gases and liquids these conditions are satisfied for all not extremely high frequencies. For example, the mean free path of molecules in air at $P = 1$ atm and $T = 300$ K is about $l \simeq 10^{-5}$ cm and $s \simeq 300$ m/s$=3\times10^4$ cm/s. Condition $\lambda \gg l$ takes the form

$$\omega \ll \frac{s}{l} \approx \frac{3 \times 10^4 \text{cm/s}}{10^{-5} \text{cm}} = 3 \times 10^9 \text{s}^{-1}.$$ 

In a liquid, $l$ is of order of the inter-molecule separation $\approx \text{few} \times 10^{-7}$ cm. As long as $\lambda \gg 10^{-7}$ cm, equilibrium is maintained.

In a Fermi liquid, the situation is different because $l$ and $\tau$ increase as temperature goes down. Local equilibrium is maintained as long as $\tau (T) \omega \ll 1$. Now, $\tau (T) = 1/AT^2$ so at fixed temperature sound of frequency $\omega \gg \tau^{-1} (T) = AT^2$ cannot proceed in a quasi-equilibrium manner.

What happens when we start at small frequency and then increase it? As long as $\tau (T) \omega \ll 1$, a Fermi liquid supports normal sound (in the FL theory it is called first sound). The velocity of this sound is linked to the sound velocity in the ideal Fermi gas but differs from because of renormalization

$$s_1 = \frac{v_F}{\sqrt{3}} \left(1 + F_0^*\right)^{1/2} \left(1 + F_1^*\right)^{1/2},$$

The first sound corresponds to the oscillations of fermion number density in sync with the wave. Because the number density fixes the radius of the Fermi sphere, $p_F$ oscillates in time and space. As the product $\omega \tau$ increases, sound absorption does so too, at when $\omega \tau$ becomes impossible. However, as $\omega \tau$ increases further and becomes $\gg 1$, another type of sound wave emerges. These waves are called zero sound. These waves can propagate even at $T = 0$, when $\tau = \infty$ as they do not require local equilibrium. Another difference between zero- and first sound is that in the first sound-wave the shape of the Fermi surface remains spherical whereas its radius changes. Instead, in the zero-sound wave the shape of the Fermi surface is distorted. In a simplest zero-sound wave, the occupation number is

$$\delta n = \delta (\varepsilon - E_F) \nu \left(\hat{k}\right) e^{i(\hat{k} \cdot \vec{r} - \omega t)}$$

$$\nu \left(\hat{k}\right) = C \frac{\cos \theta}{s_0 / v_F - \cos \theta}.$$
where \( s_0 > v_F \) is the zero-sound velocity. The Fermi surface is an egg-shaped spheroid, with the narrow end pointing in the direction of the wave propagation. The condition \( s_0 > v_F \), always satisfied for zero sound, means absence of Landau damping.

As well as \( s_1 \), the zero-sound velocity \( s_0 \) depends on Landau parameters \( F_0^s \) and \( F_1^s \), although the functional dependence is more complicated. At the same time, \( F_1^s \) can be extracted from the effective mass (measured from the specific heat) and \( F_0^s \) can be extracted from the compressibility. Having two experimentally determined parameters \( F_0^s \) and \( F_1^s \), one can substitute them into theoretical expressions for \( s_1 \) and \( s_0 \) and compare them with measured sound velocities. The agreement is quite satisfactory. This comparison provides a quantitative check for the Fermi-liquid theory.

VI. NON-FERMI-LIQUID BEHAVIORS

Majority of simple metals confirm to the Fermi-liquid behavior. \( \text{He}^3 \) is a classical Fermi-liquid system. However, there is quite a few of situation when the expected Fermi-liquid behavior is not expected. Sometimes it’s because the system is disordered and the Fermi-liquid theory is constructed for a translationally invariant system. Even more interestingly, there is quite a few of systems, most notably HTC superconductors in their normal phase and heavy fermion materials, where effects of disorder can be ruled out yet the Fermi-liquid behavior is not observed. In what follows, I will give a brief overview of deviations from the Fermi-liquid behavior.

A. Dirty Fermi liquids

1. Scattering rate

According the Fermi-liquid theory, the scattering rate of quasi-particles behaves as \( T^2 \). Since these are inelastic collisions, they break quantum-mechanical phase and one might have expected that the phase-breaking rate, observed in the experiments on weak-localization goes also as \( T^2 \) at low enough temperatures, when phonons are already frozen out. In fact, a \( T^2 \) is never observed at low temperature. Instead, the phase-breaking rate in thin films goes as \( T \) and in thin wires as \( T^{2/3} \). What went wrong? Well, we forgot to take into account that in a disordered metal electrons move along diffusive trajectories rather than along straight
lines. This is going to change $T^2$–law. One can understand very easily why diffusive motion enhances the scattering rate of electron-electron interaction. Diffusing electrons move slowly and thus spend more time lingering around each other hence the interaction is effectively enhanced.

In what follows, I will give a simple argument (due to Altshuler and Aronov) to why and how the scattering rate is changed in the presence of disorder. In a clean system, electrons exchange energies of order $\Delta \varepsilon \simeq T$ in the course of interaction. By the uncertainty principle, the duration of the collision event (not to be confused with the mean free time!) is given by
\[
\Delta t \simeq \frac{1}{\Delta \varepsilon} \simeq \frac{1}{T}.
\]
If
\[
\Delta t \ll \tau_1 \rightarrow T\tau_1 \gg 1
\]
where $\tau_1$ is the impurity mean free time, electrons don’t have time to scatter at impurities during a single act of interaction and interaction between them proceeds as if there is no disorder. In this case, we are back to the FL result $\tau^{-1} \propto T^2$. In the opposite limit, when
\[
\Delta t \gg \tau_1 \rightarrow T\tau_1 \ll 1,
\]
electrons experience many collisions with impurities while interacting with each other. To see how this is going to affect the scattering rate, let’s come back to the calculation of $\tau^{-1}$ in a clean system at $T = 0$ (Notes I). There, we showed that
\[
\frac{1}{\tau} \simeq \int_0^\infty d\omega \int_{-\omega}^0 d\epsilon_1 \int dq q^2 w(q) \int_{-1}^{+1} d(\cos \theta) \delta (\omega - qv_F \cos \theta) \int_{-1}^{+1} d(\cos \theta') \delta (\omega - qv_F \cos \theta').
\]
Angular integral of each of the delta-function gives a factor $1/qq_F$. Re-writing this factor as
\[
\frac{q^{-1}}{v_F}
\]
we see that it has units of time. The meaning of this time is the duration of the interaction act in which electrons change their momenta by $q$. Thus $\Delta t = q^{-1}v_F$. In a clean system, the $q$ integral is convergent at the lower limit and simply gives a constant prefactor in front of the $e^2$–dependence, which comes from the double energy integral. In a dirty system, momentum trasfer will take much longer. Time and space are related via the diffusion law
\[
q^{-1} = \sqrt{D\Delta t} \rightarrow \\
\Delta t = \frac{1}{Dq^2}.
\]
Now instead of the factor $q^{-2}$ from the delta-functions, we have $\Delta t^2 = 1/D^2q^4$. The matrix element is finite at $q \to 0$. However, the integral now diverges at the lower limit

$$\int dq q^2 = \int dq q = \frac{1}{q}. \quad (50)$$

To regularize the divergence we have to recalling that the uncertainty in the transferred energy has to be smaller than the transferred energy itself

$$\Delta t^{-1} < \omega \rightarrow$$

$$Dq^2 > \omega \rightarrow$$

$$q > \sqrt{\omega/D}$$

Thus the lower limit in Eq.(50) is $\sqrt{\omega/D}$ and

$$\int_{\sqrt{\omega/D}} dq q^2 \propto \frac{1}{\sqrt{\omega}}$$

The energy integral now becomes

$$\frac{1}{\tau} = A' \int_0^\epsilon d\omega \int_{-\omega}^0 d\epsilon_1 \frac{1}{\sqrt{\omega}} \simeq A' \epsilon^{3/2}$$

instead of the FL- form $\epsilon^2$. Repeating the same steps in 2D, we find for the $q-$ integral

$$\int dq q^4 \propto \frac{1}{\omega}$$

and thus

$$\frac{1}{\tau} = A' \epsilon.$$

Upon thermal averaging, this gives us a linear-in-$T$ dependence of the dephasing rate.?

Does a change from the $T^2$ to the $T-$ dependences means a breakdown of the FL? Not really. Restoring all dimensional prefactors and for the dimensionless coupling constant $\kappa/k_F \simeq 1$, the scattering rate in a dirty FL can be written as

$$\frac{1}{\tau} \simeq \frac{1}{T \tau_i E_F}$$

(recall that $T \tau_i \ll 1$ an thus $\tau^{-1} \gg T^2/E_F$) or

$$\frac{1}{\tau} = \frac{T}{E_F \tau_i}. \quad (51)$$

A quasi-particle is a well-defined excitation as long as the width of energy level $\Gamma = 1/2\tau$ is much smaller than the energy itself. For thermal quasi-particles, typical energy (counted
from the Fermi level) $\langle \varepsilon \rangle \simeq T$. Using Eq.(51), we see that the condition $\Gamma \ll \langle \varepsilon \rangle$ is satisfied as long as $E_F \tau_i \gg 1$ which coincides with the general validity of the approach to disordered systems. Therefore, quasi-particles are well-defined even in a disordered metal provided that it is still a *good* metal in a sense that $E_F \tau_i \gg 1$.

2. *T*-dependence of the resistivity

One can often see or hear the statement that the resistivity of a Fermi liquid goes as $T^2$. As such, this statement is incorrect. A FL is by construction a translationally invariant system. Electron-electron collisions in such a system conserve the total momentum and thus the current, which is proportional to the momentum, cannot be relaxed by these collisions.\(^7\) Hence, the resistivity of FL per se is infinite. When the FL is coupled to an external system (lattice, impurities), the resulting resistivity is going to be determined by what the FL is coupled to. We know, for example, that the resistivity of Fermi gas interacting with lattice vibrations goes as $T$ and $T^5$ in the regime of high and low temperatures, respectively. A FL would show the same dependence. Electron-electron collisions can lead to a finite resistivity only if Umklapp processes are involved. If the Fermi surface lies within the Brillouin zone and is separated from the BZ boundaries by a finite momentim $q_{\text{min}}$, then at lower temperatures the probability of an Umklapp scattering is exponentially small $\propto \exp (\frac{-v_F q_{\text{min}}}{T})$ because it’s proportional to the number of electrons with momentum $q_{\text{min}}$ above the Fermi surface. Consequently, $\rho \propto \exp (\frac{-v_F q_{\text{min}}}{T})$ which is quite different from the expected $T^2$. Such a behavior is indeed observed in ultra-pure alkaline metals, whose (almost spherical) Fermi surfaces lie within the Brillouin zone. On the other hand, if the Fermi surface intersects the boundaries of the Brillouin zone, Umklapp processes are allowed at all temperatures and go at the rate comparable to that of normal processes. In this case, indeed $\rho \propto T^2$.

Although most of the conventional metals belong to the second category, cases of unambiguous determinations of the $T^2$- dependence as coming from the e-e interactions are quite rare\(^7\). The problem is that the prefactor in the phonon ($T^5$) is numerically very large (explain why!) and this dependens extends to quite low temperatures where saturation of the T-dependence occurs. Also, other processes can mimic the $T^2$ law. For example, when the FS is strongly anisotropic, e.g., an ellipsoid with vastly different semi-major axes $k^\parallel_F \gg k^\perp_F$, the linear-in-$T$ dependence is observed for $T \gg s k^\parallel_F$ whereas the $T^5$ law sets in for $T \ll s k^\perp_F$.\(^{42}\)
In the intermediate range, $sk_F^\perp \ll T \ll sk_F^\parallel$, the resistivity is quadratic in $T^2$. Also, interference between electron-impurity and electron-phonon scattering results in a $T^2$ term in the resistivity. So far, the most well-established observations of the $T^2$ law and its identification with the e-e mechanism are limited to a very special class of compounds—\textit{heavy-fermion} materials, which are alloys of rare-earth metals (UPt$_3$, etc.) [see the two comprehensive reviews by Greg Stewart on this subject]. Conduction electrons in these materials come from very narrow $f$–bands and the effective masses are enormously high –100-200 of the bare electron masses, whereas the densities are at the normal metal scale. Consequently, the e-e scattering rate

$$\frac{1}{\tau} \approx \frac{T^2}{E_F} \approx \frac{m^*T^2}{n^{2/3}},$$

which is proportional to the mass is also very high and dominate the resistivity. The Fermi surfaces of these materials tend to be very complicated which means that there is no problem with Umklapp scattering.

What happens at even low temperatures when the dominant scattering mechanism is the impurity scattering. A scattering rate of a Fermi gas electron at an impurity is $T$–independent and thus the resistivity saturates at the residual value. Is this also true in a FL? The answer is NO. First of all, as we learned when studying weak-localization, the resistivity of films and wires does not really saturate at $T \to 0$ but continues to increase. Suppose that we applied a relatively weak magnetic field—just enough to kill the weak-localization effect without causing any more changes—so this effect is suppressed. Is that’s all? No, because now we have to consider scattering of interacting quasi-particles at impurities. Back in 1979, Altshuler and Aronov predicted that the conductivity of such a system acquires an additional temperature dependence of the type similar to the one coming from weak localization

$$\delta\sigma = -\frac{e^2}{\hbar} C \ln \frac{1}{T\tau_i},$$

(52)

which is valid for $T\tau_i \ll 1$ when log is positive. In contrast to weak localization, prefactor $C$ is, generally speaking, not universal but depends on the strength of the electron-electron interaction. Eq.(52) is known as “interaction correction” or “Altshuler-Aronov correction”. In the model of weakly interacting electron gas, $C$ is a universal positive number and thus sign of the correction coincides with the weak-localization one. However, if spin-exchange processes are strong, so that the metal is close to the ferromagnetic instability, $C < 0$, and
thus the interaction *increases* the conductivity. Up until recently, only positive (localizing) signs of $C$ were observed. Altshuler-Aronov theory was developed for the case of low temperatures, in a sense that $T \tau_i \ll 1$, i.e., when electrons move diffusively during a single act of interaction. Although this theory was verified in a number of independent derivations and countless number of experiments, it was only recently realized that the condition $T \tau_i \ll 1$ is a sufficient but not necessary for the occurrence of the effect. In the opposite limit, $T \tau_i \gg 1$, electrons interact with one impurity at a time, thus the problem can be formulated as a scattering theory for interacting electrons. We have already seen an application of such a scattering theory to the 1D case (see Section “Friedel oscillations”). There, we found that a resonant scattering at a Friedel oscillation surrounding a potential barrier leads to a non-analytic (log) dependence of the transmission coefficient on $\varepsilon - E_F$. In 2D (and 3D), it is more appropriate to speak of a scattering amplitude $f(\theta)$ (not to be confused with the Landau interaction function!). A modification of the reasoning for the 2D case, leads to the result that $f$ acquires a non-analytic dependence $\sqrt{\varepsilon - E_F} \simeq \sqrt{T}$ in a narrow angular interval around $\pi |\theta - \pi| \simeq \sqrt{\varepsilon - E_F} \simeq \sqrt{T}$. Thus the correction to the cross-section and hence to conductivity goes as $T$

$$\delta \sigma = -\frac{e^2}{\hbar} C' T \tau_i.$$  

The sign and magnitude of $C'$ depends on the interaction strength; for strong interactions, the correction is of the metallic, i.e., anti-localizing sign. Thus a metal with interactions is a better metal than without! This effect is a prime suspect for the recently observed metallic T-dependences in semiconductor heterostructures which are associated with the phenomena of “metal-insulator transition in 2D” occurring at odds with the single-parameter scaling theory of localization.

Appendix A: Born approximation for the Landau function

If fermions interact via a static potential, the Landau function coincides with the scattering amplitude. The direct and exchange diagrams for the first-order scattering amplitude are shown in Fig. ???. The wavy line represents the interaction potential, $U(q)$. Algebraically,

$$f_{\alpha\gamma;\beta\delta}(p, p') = \delta_{\alpha\gamma} \delta_{\beta\delta} U(0) - \delta_{\alpha\delta} \delta_{\beta\gamma} U(p - p').$$  

(A1)
Now, we introduce the Pauli matrices $\hat{\sigma}^i \ (i = 1 \ldots 3)$ which form a complete basis. The completeness relation can be expressed as

$$\hat{\sigma}_{\alpha\gamma} \cdot \hat{\sigma}_{\beta\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta} \quad \text{(A2)}$$

which means that

$$\delta_{\alpha\delta}\delta_{\beta\gamma} = \frac{1}{2}\delta_{\alpha\gamma}\delta_{\beta\delta} + \frac{1}{2}\hat{\sigma}_{\alpha\gamma} \cdot \hat{\sigma}_{\beta\delta}.$$ 

Substituting this result into Eq. (A1), we obtain

$$f_{\alpha,\gamma;\beta,\delta}(p, p') = \delta_{\alpha\gamma}\delta_{\beta\delta} \left[ U(0) - \frac{1}{2} U(p - p') \right] - \frac{1}{2}\hat{\sigma}_{\alpha\gamma} \cdot \hat{\sigma}_{\beta\delta} U(p - p'). \quad \text{(A3)}$$

To prove the completeness relation, notice that any $2 \times 2$ matrix can be decomposed over the Pauli matrices as

$$\hat{M} = aI + \sum_{i=1 \ldots 3} b_i \hat{\sigma}^i.$$ 

Since $\text{Tr} \hat{\sigma}^i = 0$ and $\text{Tr} \hat{\sigma}^i \hat{\sigma}^j = 2\delta_{ij}$, we see that

$$a = \frac{1}{2} \text{Tr} \hat{M}$$

and

$$b_i = \frac{1}{2} \text{Tr} \hat{M} \hat{\sigma}^i$$

or

$$2\hat{M} = \text{Tr}(\hat{M})I + \sum_i \hat{\sigma}^i \text{Tr}(\hat{M} \hat{\sigma}^i)$$

Picking $\alpha\gamma$ component of $\hat{M}$, we obtain

$$2M_{\alpha\gamma} = \sum_{\beta} M_{\beta\beta} \delta_{\alpha\gamma} + \sum_{i} \sigma^i_{\alpha\gamma} \sum_{\beta\delta} M_{\delta\beta} \sigma^i_{\delta\beta}$$

or

$$2 \sum_{\beta\delta} \delta_{\alpha\delta}\delta_{\beta\gamma} M_{\beta\beta} = \delta_{\alpha\gamma} \sum_{\beta\delta} M_{\delta\beta}\delta_{\beta\delta} + \sum_{i} \sigma^i_{\alpha\gamma} \sum_{\beta\delta} M_{\delta\beta} \sigma^i_{\delta\beta}$$

Since the relation above must be valid for any $\hat{M}$, the completeness relation, Eq. (A2), is evident.

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2 Mahan, *Many-body physics*


9 E. Abrahams, S. V. Kravchenko, and M. P. Sarachik, Rev. Mod. Phys. 73, 251 (2001).