Problem 1. Symmetries of the Dirac Lagrangian. Consider the Dirac Lagrangian (3.34):
\[ \mathcal{L}_{\text{Dirac}} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi. \]

(a) Show that the transformation
\[ \psi(x) \to e^{i\alpha} \psi(x) \]
is a symmetry of the Lagrangian. Find the corresponding Noether current and check that it is indeed conserved.

(b) Now consider the transformation
\[ \psi(x) \to e^{i\alpha \gamma^5} \psi(x) \]
and again find the corresponding Noether current. Is it conserved? Consider two cases: \( m = 0 \) and \( m \neq 0 \).

Problem 2. Parity. In this problem we will consider parity transformations, i.e. mirror reflections in spacetime:
\[ x^\mu \to x'^\mu = (t, -\vec{x}). \]

(a) Let \( \psi(x) \) satisfy the Dirac equation
\[ (i \gamma^\mu \partial_\mu - m) \psi(x) = 0. \]

Show that
\[ \psi'(x') \equiv \gamma^0 \psi(x) \]
satisfies the Dirac equation in the parity-reflected world:
\[ (i \gamma^\mu \partial'_\mu - m) \psi'(x') = 0, \]
where
\[ \partial'_\mu = \frac{\partial}{\partial x'^\mu} = \left( \frac{\partial}{\partial t}, -\frac{\partial}{\partial \vec{x}} \right). \]
(b) Now we can derive the $P$-parity assignments of the bilinears in the table on page 71, simply by comparing each one to its cousin in the parity-reflected world. Show that

\[
\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x), \quad (5)
\]

\[
i\bar{\psi}'(x')\gamma^5\psi'(x') = -i\bar{\psi}(x)\gamma^5\psi(x), \quad (6)
\]

\[
\bar{\psi}'(x')\gamma^\mu\psi'(x') = g^{\mu\nu}\bar{\psi}(x)\gamma^\nu\psi(x), \quad (7)
\]

\[
\bar{\psi}'(x')\gamma^\mu\gamma^5\psi'(x') = -g^{\mu\nu}\bar{\psi}(x)\gamma^\nu\gamma^5\psi(x), \quad (8)
\]

\[
\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = g^{\mu\nu}g^{\rho\sigma}\bar{\psi}(x)\sigma^{\rho\sigma}\psi(x), \quad (9)
\]

where in the last three equations I have used $g^{\mu\nu}$ (no sum over $\mu$) instead of the confusing $(-1)^\mu$ defined in the text.

**Problem 3. Allowed interactions.** If $\phi$ is a real scalar field and $\psi$ is a Dirac field, which of the following terms can appear in the Lagrangian density of a sensible field theory in $3 + 1$ dimensions? Explain why or why not. Which ones can appear in principle, but become irrelevant at sufficiently low energies? In each case, the (real) coefficients $c_i$ are chosen to have dimensions such that the term in the Lagrangian density has mass dimension 4.

(a) $c_1 \phi^7(x)$;
(b) $c_2 \partial_\mu \phi(x) \bar{\psi}(x)\gamma^\mu\psi(x)$;
(c) $c_3 \phi(x) \bar{\psi}(x)\psi(x)$;
(d) $c_4 \phi(x)i\bar{\psi}(x)\gamma^5\psi(x)$;
(e) $c_5(\bar{\psi}(x)\psi(x))^2$;
(f) $c_6 \phi^2(x)\bar{\psi}(x)$;
(g) $c_7 \phi(x)\bar{\psi}(x + \delta)\psi(x + \delta)$;
(h) $c_8 \phi(x)\gamma^\mu\partial_\mu\psi(x)$.

**Problem 4. Symmetry factors in $\varphi^4$ theory.** Consider the second-order term in the expansion (4.43)

\[
\langle 0| T \left\{ \phi(x)\phi(y) \frac{(-i)^2}{2!} \frac{\lambda}{4!} \int d^4z \phi(z)\phi(z)\phi(z)\phi(z) \int d^4u \phi(u)\phi(u)\phi(u)\phi(u) \right\} |0\rangle.
\]

(a) Classify all topologically distinct classes of diagrams.

(b) By enumerating all possible contractions, calculate the symmetry factor for each class. **Hint:** Check that the sum of the symmetry factors is equal to $9!!/(2!4!4!) = 945/1152$.

(c) Check your answer from (b) by considering the symmetries of the diagrams and thus give an independent justification of your previous answers.

(d) Extra credit: Can you now handle the third-order term in (4.43)?