Reading: Class notes and references linked to the class diary (e.g., arXiv:1006.0653).

**Problem 1.** $s_{\text{min}}$ variable. Consider a collider event in which a certain number $N_{\text{vis}}$ of visible particles were produced and their total energy $E$ and momentum $\vec{P}$ were measured. In other words, we combine the individually measured energies $E_i$ and momenta $\vec{P}_i$ of the visible particles into a single 4-vector $(E, \vec{P})$ describing the whole collection of visible particles:

$$E \equiv \sum_{i=1}^{N_{\text{vis}}} E_i, \quad \vec{P} \equiv \sum_{i=1}^{N_{\text{vis}}} \vec{P}_i.$$  

Suppose that in this event we also observe transverse momentum imbalance, i.e., we measure some missing transverse momentum $\vec{P}_T$. We hypothesize that this missing momentum is due to the production of $N_{\text{inv}}$ invisible particles, with unknown masses $M_1, M_2, M_3, \ldots M_{N_{\text{inv}}}$. 

We now ask the question, what is the minimum possible center-of-mass energy in this event? Minimize the $\sqrt{s}$ of the event as a function of the momenta $\vec{q}_i$ of the invisible particles, subject to the constraint

$$\sum_{i=1}^{N_{\text{inv}}} \vec{q}_i = \vec{P}_T$$

and show that

$$\sqrt{s_{\text{min}}} = \sqrt{E^2 - P_z^2} + \sqrt{M_{\text{miss}}^2 + \vec{P}_T^2},$$

where

$$M_{\text{miss}} \equiv \sum_{i=1}^{N_{\text{inv}}} M_i$$

is the combined mass of all invisible particles.

*Hint: If you are having trouble with the general case, start with $N_{\text{inv}} = 1$ and then move up from there.*

**Problem 2.** Transverse mass in single $W$ production. Consider single production of $W$-like bosons in the presence of initial state radiation, e.g. a jet with some transverse momentum $\vec{U}_T$. The $W$ decays to a massless lepton with measured momentum $\vec{P}$ and an invisible particle with some unknown mass $M_\chi$. Calculate the value of $P_T$ in the following two special cases: In the CM frame of the event, the $W$ and the jet are purely in the transverse plane (have no $z$-component of the momentum) and the lepton is emitted

a) in the direction of the jet $\vec{U}_T$

b) in the direction opposite to the jet $\vec{U}_T$ (i.e. in the direction of the $W$).
Show that

\[ P_{T}^{\pm} = \frac{M_{W}^{2} - M_{\chi}^{2}}{2M_{W}^{2}} \left( \sqrt{M_{W}^{2} + U_{T}^{2}} \pm U_{T} \right) \]

For those two situations (\( \pm \)), find the transverse \( W \) mass \( M_{TW} \) as a function of the two measured momenta \( \vec{P}_{T} \) and \( \vec{U}_{T} \) and a hypothesized mass \( \tilde{M}_{\chi} \) for the missing particle. Let us take the true masses to be \( M_{W} = 400 \text{ GeV} \) and \( M_{\chi} = 200 \text{ GeV} \). Plot the two functions \( M_{T,W}(\tilde{M}_{\chi}) \) for \( U_{T} = 0 \); for \( U_{T} = 100 \text{ GeV} \) and for \( U_{T} = 1000 \text{ GeV} \).