1. Uncertainty principle

1a) A microscope uses a single lens of diameter \( d \) to image an object on a slide a distance \( D \gg d \) away using light of wavelength \( \lambda \). What is the classical spatial resolution \( \Delta x \) of this device, roughly? Imagine that the imaging occurs with the use of one photon. Estimate how much transverse momentum \( \Delta p \) (i.e., in the plane of the slide) is transferred to the object, and verify the uncertainty principle.

1b) Estimate the ground state energy of the H atom based on the uncertainty principle. Hint: take the radius of the orbit \( a_0 \) to which the atom is “confined” to be a variational parameter, i.e. write down the kinetic energy of the electron plus the potential energy. To find the kinetic energy you will need the momentum, & to find the momentum you will use the uncertainty principle. Then minimize the total energy with respect to the radius, and you should find an energy close to Bohr’s ground state energy.)

1c) Redo this calculation to estimate the ground state energy \( E_0 \) and average radius \( a \) of the ground state ”orbit” in a 2-electron atom of nuclear charge \( +Ze \).

2. Delta function

The Dirac delta function is defined via a set of functions \( \delta_n(x) \) with the property

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} dx \, \delta_n(x) f(x) = f(0)
\]  

for sufficiently regular functions \( f \). Then one writes

\[
\lim_{n \to \infty} \delta_n(x) = \delta(x) \quad \text{and} \quad \int_{-\infty}^{\infty} dx \, \delta(x) f(x) = f(0). 
\]  

The \( \delta_n \) are called regularizations of the \( \delta \)-function.

2a) Show the sequence \( \delta_n^{(1)}(x) = (n/\pi)^{1/2} e^{-nx^2} \) is a regularization of the \( \delta \)-function. Plot \( \delta_n^{(1)}(x) \) for \( n = 4, 20, 100 \).

2b) Prove, using only the definition of \( \delta(x) \) in (2), that

\[
\delta(ax) = \delta(x)/|a| \quad \text{and} \quad \delta(x^2 - a^2) = [\delta(x - a) + \delta(x + a)]/(2|a|). 
\]
2c) Calculate the Fourier transform $\tilde{f}(p)$ of $f(x)$ with respect to $p$ for $f(x) = \delta(x)$ and $f(x) = 1/(x^2 + 1)$. If you like you may use Maple with the command `inttrans[fourier](expr,x,p)` or `with(inttrans):fourier(expr,x,p)`. The $\delta$-function is called `Dirac(x)` by Maple.