1. Short answer.

- Explain the photoelectric effect

Shine light on metal surface, only photons of sufficiently high energy (frequency) able to kick an electron out. Increasing intensity of low-energy light doesn’t help. A single electron can make a transition out of the solid only if a single photon carries sufficient energy $\hbar \omega$.

- Explain the significance of $\hbar$ in quantum mechanics, and give an example of a place where it shows up.

Planck’s const. is a quantum (smallest possible unit) of angular momentum. It shows up, for example, in the orbits of the Bohr atom, where an electron may not have any value of angular momentum, but only an integer number of $n\hbar$.

- Discuss the uncertainty principle briefly

If you make a measurement of an object’s position and momentum simultaneously, you cannot measure them both arbitrarily precisely. The product of the uncertainty in knowledge of the object’s position $\Delta x$ and in momentum $\Delta p$ fulfills the Heisenberg relation

$$\Delta x \Delta p \gtrsim \hbar$$  \hspace{1cm} (1)

- Explain the difference between the 2 versions of Schrödinger’s equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

and

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi$$

The first is the general version, the second applies for ”stationary states”, i.e. if $\psi$ is an eigenfunction of the Hamiltonian. These special states change in time in a trivial way, $\psi_n(x,t) = \psi(x) \exp(-iE_n t/\hbar)$. 

1
• What are the units of \( P(x,t) \), the probability density in 1 dimension? Justify your answer.

Must be 1/volume, since \( \int dx P(x,t) = 1 \).

• Calculate the commutator \([p_x,x^2]\)

\[
[p_x,x^2] \psi = (-i\hbar \nabla)(x^2 \psi) - x^2(-i\hbar \nabla)\psi \\
= -i\hbar 2x \psi - ix^2 \psi' + i\hbar x^2 \psi' \\
= -i\hbar 2x \psi.
\]

So \([p_x,x^2] = -i\hbar 2x \).

• Calculate the expression for the Bohr levels of the hydrogen atom from the Bohr-Ehrenfest quantization condition.

See notes.

2. Consider a wave packet defined by

\[
\psi(x) = \int_{-\infty}^{\infty} dk f(k) e^{ikx-\omega t} \tag{2}
\]

with \( \omega = \hbar k^2/2m \) and \( f(k) \) given by

\[
f(k) = \begin{cases} 
0 & k < -\Delta k/2 \\
\alpha & -\Delta k/2 < k < \Delta k/2 \\
0 & \Delta k/2 < k
\end{cases}
\tag{3}
\]

(a) Find the form of \( \psi(x) \) at \( t = 0 \).

\[
\psi(x) = \int dk f(k) e^{ikx} = \alpha \int_{-\Delta k/2}^{\Delta k/2} dk e^{ikx} \\
= \frac{\alpha}{ix} (e^{ix\Delta k/2} - e^{-ix\Delta k/2}) \\
= \frac{\alpha}{ix} 2i \sin \frac{\Delta kx}{2}.
\]

(b) Find the value of \( \alpha \) for which \( \psi(x) \) is properly normalized.

Need \( \int dx |\psi|^2 = 1 \), so

\[
1 = 4\alpha^2 \int dx \sin^2 \frac{\Delta kx}{2} \\
= 4\alpha^2 \pi \Delta k/2
\]
so \( a = \sqrt{1/(2\pi \Delta k)} \). I would have given you this integral on the formula sheet on a real test.

(c) How is this related to the choice of \( a \) for which

\[
\int_{-\infty}^{\infty} dk |f(k)|^2 = 1
\]

From the definition of \( f \), the integral means \( a = 1/\sqrt{\Delta k} \). Put another way, the way the Fourier transform \( f \) is defined here, \(|f|^2\) is not normalized to 1 but to \( 1/2\pi \).

(d) Show that for a reasonable definition of \( \Delta x \), the size of the packet given by your answer in a), \( \Delta k \Delta x > 1 \).

A good estimate of the size of the packet might be the first time \( \sin \Delta kx/2 \) goes to zero, which occurs at \( x \simeq \Delta x/2 = 2\pi/\Delta k \). So \( \Delta x \Delta k \simeq 4\pi \) for this packet.

3. A particle in an infinite square well (of width \( a \)) has as its initial wave function an equal mixture of the first two stationary states:

\[
\Psi(x,0) = C[\psi_1(x) + \psi_2(x)]
\]

(a) Normalise \( \Psi(x,0) \). (That is, find \( C \).)

First we’ll need to find \( \psi_1(x), \psi_2(x) \), the 1st 2 stationary states. From notes or just from guessing with the standing wave boundary conditions at \( x = \pm a/2 \), they are

\[
\psi_1 = \sqrt{\frac{2}{a}} \cos \frac{\pi x}{a} \\
\psi_2 = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}
\]

corresponding to energy eigenvalues \( -\hbar^2 \nabla^2 / 2m \psi = E_\alpha \psi_\alpha \)

\[
E_1 = \frac{\hbar^2 \pi^2}{2ma^2} \\
E_2 = \frac{4\hbar^2 \pi^2}{2ma^2}
\]

Time dependence: we will need to know how these 2 states evolve in time. Since they are eigenstates of \( H \), we must have \( H_\alpha \psi_\alpha = E_\alpha \psi_\alpha = i\hbar(\partial/\partial t) \psi_\alpha \), which has the solution

\[
\psi_\alpha(x,t) = \psi_\alpha(x)e^{-iE_\alpha t/\hbar}
\]

3
Now, on to business. The normalization condition is

\[ 1 = \int dx |\Psi|^2 \]

\[ = C^2 \int dx (|\psi_1|^2 + |\psi_2|^2 + \psi_1^* \psi_2 + \psi_2^* \psi_1) \]

\[ = C^2 (1 + 1 + 0 + 0) \]

where the last 2 terms vanish because they are odd in \( x \). So \( C^2 = 1/2 \).

(b) Find \( \Psi(x, t) \) and \( |\Psi(x, t)|^2 \). Express the latter in terms of \( \sin \) and \( \cos \) using \( e^{i\theta} = \cos \theta + i \sin \theta \). Use \( \omega = \pi^2 \hbar / 2ma^2 \).

From above, \( \Psi(x, t) = C(\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}) \), and

\[ |\Psi(x, t)|^2 = C^2 (|\psi_1|^2 + |\psi_2|^2 + 2\psi_1 \psi_2 \cos 3\omega t) \]

where I used the info given, that \( E_1 = \hbar \omega \), \( E_2 = 4\hbar \omega \). Note the oscillation in time.

(c) Compute \( \langle x \rangle \). Notice that it oscillates in time. What is the frequency of the oscillation? What is the amplitude?

\[ \langle x \rangle = C^2 \int dx (|\psi_1|^2 + |\psi_2|^2 + 2\psi_1 \psi_2 \cos 3\omega t)x \]

\[ = \frac{2}{a} \int_{a/2}^{-a/2} dx \cos \frac{\pi x}{a} \sin \frac{2\pi x}{a} = \frac{16a}{9\pi^2} \sin 3\omega t, \]

where note the 1st two terms vanished because they were even in \( x \), but now last one wasn’t. So this is a particle which is sloshing back and forth in the well. Again I would have given you this integral on a real test.

(d) Compute \( \langle p \rangle \).

\[ \langle p \rangle = C^2 \int dx (\psi_1^* + \psi_2^*) (-i\hbar \nabla) (\psi_1 + \psi_2) \]

\[ = -\frac{4}{3a} i\hbar \left( e^{i(E_2-E_1)t/\hbar} - e^{-i(E_2-E_1)t/\hbar} \right) \]

\[ = \frac{4\hbar}{3a} \sin 3\omega t \]

(e) Find the expectation value of the Hamiltonian operator, \( H \), in terms of \( E_1 \) and \( E_2 \).

\[ \langle H \rangle = C^2 \int (\psi_1^* + \psi_2^*) H (\psi_1 + \psi_2) \]

\[ = C^2 \int (\psi_1^* + \psi_2^*) (E_1 \psi_1 + E_2 \psi_2) \]

\[ = C^2 \int (E_1 |\psi_1|^2 + E_2 |\psi_2|^2) = (E_1 + E_2)/2, \]
where in the 2nd to last step I used $\int dx \psi_1^* \psi_2 = 0$, etc. from the properties of sin, cos, or recall that eigenfunctions belonging to different eigenvalues are orthogonal. In the last step I used the normalization of $\psi_1, \psi_2$. 