Due: 31 January, 2005
Reading: PH notes
Remarks: On problem set 1 too many people had continued difficulty with matrix algebra. To solidify these crucial concepts, here’s more practice.

1. **Matrix algebra drill.** Given the matrices

\[
A = \begin{pmatrix}
\frac{3}{5} & -\frac{4}{5} \\
\frac{4}{5} & \frac{3}{5}
\end{pmatrix} ; \quad B = \begin{pmatrix}
1 & 1 & -1 \\
-1 & 3 & -1 \\
-1 & 2 & 0
\end{pmatrix} ; \quad C = \begin{pmatrix}
-1 & 2 & 2 \\
2 & 2 & 2 \\
-3 & -6 & -6
\end{pmatrix}
\]  

(a) Find the eigenvalues \(\lambda_i\) and normalized eigenvectors \(v_i\), \(i = 1, 2, 3\) for \(A\), \(B\), and \(C\). State the degeneracy of each eigenvalue.

(b) Find the determinants of \(A\), \(B\), and \(C\).

(c) Recall that, under a change of basis matrices, transform as \(M \rightarrow M' = \hat{U}^{-1} M \hat{U}\) and vectors as \(v \rightarrow v' = \hat{U}^{-1} v\). Find \(\hat{U}\) such that \(A\) is brought into diagonal form by a change of basis. How is the matrix of transformation \(\hat{U}\) related to the eigenvectors of \(A\)? Find the transformed eigenvectors \(\hat{U}^{-1} v_i\). Now answer the same questions for \(C\).

(d) Find the inverses of \(A\) and \(B\), and state why \(C\) is not invertible.

(e) Show that the inverse of the 2D matrix represented by \(M = a_0 \mathbf{1} + \hat{a} \cdot \hat{\sigma}\) is \(M^{-1} = D_0^{-1} (a_0 \mathbf{1} - \hat{a} \cdot \hat{\sigma})\), with \(D_0 = a_0^2 - \hat{a} \cdot \hat{a}\). Here \(\mathbf{1}\) is the identity matrix in 2D and \(\sigma_x\), \(\sigma_y\), and \(\sigma_z\) are the Pauli matrices. [Hint: properties of the Pauli matrices you may find useful: 1) \(\sigma_i^2 = \mathbf{1}\) for any \(i\); 2) \(\sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k\) for \(i \neq j\).]

2. **Phase shift in a vector potential.**

(a) Show that the vector potential outside an infinite solenoid containing a flux \(\Phi\) has the magnitude \(\Phi / 2\pi \rho\), where \(\rho\) is the perpendicular distance from the axis of the solenoid.

(b) Show that if \(\psi_\mathbf{A}\) satisfies

\[
i \hbar \frac{\partial \psi_\mathbf{A}}{\partial t} = \frac{1}{2m} [\mathbf{p} - e \mathbf{A}(\mathbf{r})]^2 \psi_\mathbf{A},
\]

then

\[
\psi_\mathbf{A}(\mathbf{r}, t) = \exp \left( \frac{ie}{\hbar} \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{A} \cdot d\mathbf{s} \right) \psi_{\mathbf{A}=0}(\mathbf{r}, t)
\]

Apparently \(\psi_\mathbf{A}\) depends on an arbitrary initial point \(\mathbf{r}_0\). Comment on this ambiguity.