1. **Dipole transitions.** The electric dipole approximation to a radiative transition between states of a system of charged particles assumes the velocities of particles are small, so the size of the radiating system is negligible compared to the wavelength of the radiation, and it neglects the interaction of the system with the magnetic field of the radiation (exactly the opposite of what we did when studying hyperfine transition—why?). The energy of a particle with charge $q$ at position $r$ in an electric field $\mathbf{E}$ is

$$V = -q r \cdot \mathbf{E}.$$  \hspace{1cm} (1)

In the electric dipole approximation, the variation of the electromagnetic field across the system is neglected, so the electric field at the position of the system is

$$\mathbf{E}(t) = E_0 \vec{\epsilon} \cos \omega t,$$  \hspace{1cm} (2)

where $E_0$ is the real amplitude of the wave, $\omega$ the angular frequency of the radiation, and $\vec{\epsilon}$ is a unit polarization vector. The term in the Hamiltonian describing the interaction of a single particle with the radiation is

$$V = -qE_0 \vec{\epsilon} \cdot r \cos \omega t.$$  \hspace{1cm} (3)

In the first order electric dipole approx. to the photoionization of atomic hydrogen, the initial electronic wavefunction is $\psi_i \propto e^{-r/a_0}$, i.e. an electron in the ground state of H, and the final state is approximated as a plane wave

$$\psi_f \propto e^{i k f \cdot r},$$ \hspace{1cm} (4)

i.e. an electron zipping away from the atom with momentum $\hbar \mathbf{k}$.

* Use first-order $t$-dependent perturbation theory to find the angular distribution of the ejected electron. That is, find the probability of observing $\mathbf{k}_f$ in a given direction relative to initial $\mathbf{E}$-field polarization $\vec{\epsilon}$. Don’t bother to compute any factors (normalization constants, etc.) which are independent of direction.

(Hint: the computation can be simplified by using polar coordinates with $z$ axis along $\mathbf{k}_f$. A useful result from spherical trig. is that if $\mathbf{r}$ has spherical angles $\theta$ and $\phi$ in this coordinate system, and $\vec{\epsilon}$ has angles $\theta'$, $\phi'$, then the angle $\Theta$ between $\mathbf{r}$ and $\vec{\epsilon}$ satisfies

$$\cos \Theta = \cos \theta' \cos \theta + \sin \theta' \sin \theta \cos(\phi - \phi').$$  \hspace{1cm} (5)

If you find yourself doing long messy integrals, think again...)
2. Selection rules. The “selection rules” which govern the allowed changes of parity and angular momentum in an electric dipole radiative transition can be derived easily. The electric dipole moment of a system of charges is

\[ d = \sum_i q_i r_i \]  

(6)

so the potential is

\[ V = -E_0 \vec{\epsilon} \cdot \vec{d} \cos \omega t. \]  

(7)

Neglecting spins, the states of the full problem can be classified in terms of the quantum numbers of the unperturbed system (atom with no radiation): \( L^2, L_z, \) and parity \( \Pi: \)

\[ |i\rangle = |n, \ell, m, \pi \rangle. \]  

(8)

(a) The parity operator reverses the dipole moment:

\[ \Pi d = -d \Pi. \]  

(9)

By considering the matrix elements of this relation in the basis states of (8), show that the matrix elements among the eigenstates of \( H_0 \) induced by the perturbation \( V \) vanish unless the initial and final atomic states satisfy the parity selection rule for an electric dipole transition,

\[ \pi' = -\pi. \]  

(10)

(b) Let

\[ d_\pm = d_x + id_y, \]  

(11)

where the \( d_i \) are the components of \( d \). Evaluate the commutators

\[ [L_z, d_z], \]  

(12)

\[ [L_z, d_\pm], \]  

(13)

\[ [L_\pm, d_z], \]  

(14)

\[ [L_\pm, d_\pm] \]  

(15)

(c) Use the commutation relations to show that \( d_\pm|\ell, m\rangle \) is an admixture of states \( |\ell', m'\rangle \) with \( m' = m + 1. \)

(d) Find the electric dipole selection rule (which transitions have nonzero rates) for the quantum numbers \( m \) and \( m' \) in the transition \( |\ell, m\rangle \rightarrow |\ell', m'\rangle \) when \( \vec{\epsilon} \) is parallel to \( \hat{x}. \)