1. Finite extent of proton.

Treat proton as ball of radius $r_0$ change density $\rho_0 = e / (4/3\pi r_0^3)$. Outside the ball the potential energy of electron is clearly $-e^2/r$ just as for point proton. But inside $r_0$, the potentials differ! Treat small difference as perturbation:

Gauss’s Law:

$$E = \frac{1}{4\pi \epsilon_0} \begin{cases} \frac{\rho_0 \frac{4}{3} \pi r^3}{r^2} & r < r_0 \Rightarrow \\ \frac{e}{r} & r > r_0 \end{cases}$$  \hspace{1cm} (1)

$$\phi = -\int_{\infty}^{r} dr'E(r') = \begin{cases} -\rho_0 \frac{4}{3} \pi \left( \frac{r^2}{2} - \frac{r_0^2}{2} \right) + \frac{e}{r_0} & r < r_0 \Rightarrow \\ \frac{e}{r} & r > r_0 \end{cases}$$  \hspace{1cm} (2)

$$U = -e\phi = \frac{1}{4\pi \epsilon_0} \begin{cases} -\frac{3e^2}{2r_0} + \frac{e^2 r^2}{2r_0} & r < r_0 \Rightarrow \\ \frac{e^2}{r} & r > r_0 \end{cases}$$  \hspace{1cm} (3)

Perturbation $\hat{V} = U - \left( -\frac{e^2}{r} \right)$ = $\frac{e^2 r^2}{2r_0^3} + \frac{e^2}{r} - \frac{3e^2}{2r_0}$ for $r < r_0$, 0 otherwise  \hspace{1cm} (4)

Evaluate first-order energy shift in ground state:

$$\delta E = \langle 100 | \frac{1}{4\pi \epsilon_0} \left[ \frac{e^2 r^2}{2r_0^3} + \frac{e^2}{r} - \frac{3e^2}{2r_0} \right] \theta(r_0 - r) | 100 \rangle$$  \hspace{1cm} (5)

where

$$|100\rangle = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$  \hspace{1cm} (6)

We need $\langle r^2 \rangle'$, $\langle \frac{1}{r} \rangle'$, and of course $\langle 1 \rangle'$ e.g.

$$\langle r^2 \rangle' = \int_0^{r_0} \frac{d^3r}{\pi a_0^3} e^{-2\pi/a_0 r^2} = \frac{4\pi a_0^2}{\pi} \int_0^{r_0/a_0} dx x^2 e^{-2x}$$  \hspace{1cm} (7)

we used $x = r/a_0$ and since $r_0/a_0 \ll 1$

$$\simeq 4a_0^2 \int_0^{x_0} dx x^4 (1 - 2x + \cdots) \simeq 4a_0^2 \frac{x_0^5}{5} \bigg|_0^{x_0} = \frac{4r_0^5}{5a_0^3}$$  \hspace{1cm} (8)
\[ \langle \frac{1}{r} \rangle' = \int_0^{r_0} \frac{d^3r}{\pi a_0^3} e^{-2r/a_0} \frac{1}{r} = \frac{4\pi}{\pi a_0^2} \int_0^x dx e^{-2x} \approx \frac{4}{a_0} \int_0^x dx (1 - 2x + \cdots) = \frac{2r_0^2}{a_0^2} \quad (9) \]

\[ \langle 1 \rangle' = \int_0^{r_0} \frac{d^3r}{\pi a_0^3} e^{-2r/a_0} \approx \frac{4\pi}{\pi a_0^2} \int_0^x dx x^2 (1 - 2x + \cdots) \approx \frac{4r_0^3}{3a_0^3} \quad (10) \]

So

\[ (4\pi \epsilon_0)\delta E = \frac{e^2}{2r_0} \langle \frac{1}{r} \rangle' + e^2 \langle \frac{1}{r} \rangle' - \frac{3e^2}{2r_0} \langle 1 \rangle' = \frac{e^2}{2r_0} \frac{4r_0^5}{5a_0^2} + \frac{e^2}{2r_0} \frac{2r_0^2}{a_0^2} - \frac{3e^2}{2r_0} \frac{4r_0^3}{3a_0^3} \]

\[ = \frac{e^2}{a_0} \left( \frac{r_0}{a_0} \right)^2 \left[ \frac{4}{10} + 2 - 2 \right] = \frac{4}{10} \left( \frac{r_0}{a_0} \right)^2 \quad (11) \]

Since binding energy of unperturbed ground state is \( e^2/(4\pi \epsilon_0 \cdot 2a_0) = 13.6 \text{ eV} \equiv B \), fractional change in \( B \) due to finite extent of proton is

\[ \frac{\delta B}{B} = \frac{8}{10} \left( \frac{r_0}{a_0} \right)^2 \approx \left( \frac{10^{-15}m}{10^{-16}m} \right)^2 \approx 10^{-10} \text{ small!} \quad (12) \]

b) Although \( \delta E_{n=1} \) is small, it’s a good deal larger than \( \delta E \) in other hydrogenic states since the electron “spends so much less time” near the origin \( r = 0 \). Find, e.g.

\[ n = 2, \ l = 1, \ \frac{\delta B}{B} \approx \frac{1}{120} \left( \frac{r_0}{a_0} \right)^4 \quad (13) \]

Due to “centrifugal barrier” for \( l > 0 \) states.

2. Bead on wire loop with “dimple”

a) We said in class that for a particle confined to a circular loop with otherwise free motion, the Hamiltonian is that for the quantum rotor, \( H_0 = L_z^2/2I \), with \( I = mR^2 \) the moment of inertia of a point particle orbiting at radius \( R \). Clearly the eigenfunctions of \( H_0 \) are those of angular momentum, \( e^{in\theta} \), and since \( L_z = -i\hbar \partial_\theta \), the energies are \( E_n = \hbar^2 n^2/(2mR^2) \). Griffiths wants to express the eigenfunctions in terms of \( x \), which he defines as the displacement along the loop, so \( \theta = x/R \equiv 2\pi x/L \), where \( L \) is the circumference of the loop. The prefactor of \( \sqrt{1/L} \) is required to ensure \( \int_0^L dx |\psi_n|^2 = 1 \).

b) Using Griffith’s formulae for degenerate 2-state perturbation theory, with \( a \to n \), \( b \to -n \), and noting that \( \psi_n^* = \psi_{-n} \), we have

\[ W_{aa} = W_{bb} = -\frac{2}{L} \int_{-L/2}^{L/2} dx e^{-x^2/a^2} \approx -\frac{V_0}{L} \int_{-\infty}^{\infty} dx e^{-x^2/a^2} = -\frac{V_0}{L} a\sqrt{\pi} \]

\[ W_{ab} = -\frac{V_0}{L} \int_{-L/2}^{L/2} dx e^{-x^2/a^2} e^{-4\pi inx/L} \approx -\frac{V_0}{L} \int_{-\infty}^{\infty} dx e^{-x^2/a^2} e^{-4\pi inx/L} = -\frac{V_0}{L} a\sqrt{\pi} e^{-(2\pi na/L)^2} \]
where in the last step we completed the square in the exponent to factor out the exponential term out of the Gaussian integral. Eq. 6.27 of Griffiths then gives first order correction

\[ E^1_\pm = W_{aa} \pm |W_{ab}| = -\sqrt{\frac{V_0a}{L}} \left(1 \mp e^{-\left(\frac{2\pi na}{L}\right)^2}\right) \]

c) Eq. 6.22 of Griffiths ⇒ the linear combinations which diagonalize \( H' \) are \( \alpha \psi_n + \beta \psi_{-n} \), with \( \beta = \alpha(E^1_+ - W_{aa})/W_{ab} = \mp \alpha \), so what Griffiths calls the “good” eigenstates (the ones which diagonalize \( H' \)) are

\[
\psi_\pm = \frac{1}{\sqrt{2}}(\psi_n \pm \psi_{-n}) = \sqrt{\frac{2}{L}} \left\{ \begin{array}{c}
\cos \frac{2\pi nx}{L} \\
i \sin \frac{2\pi nx}{L}
\end{array} \right.
\]

\[
E^1_+ = \langle \psi_+ | H' | \psi_+ \rangle = \frac{2}{L} \left(-V_0\right) \int_{-L/2}^{L/2} e^{-x^2/a^2} \cos^2 \frac{2\pi nx}{L} \, dx \approx -\sqrt{\frac{V_0a}{L}} \left(1 + e^{-\left(\frac{2\pi na}{L}\right)^2}\right)
\]

\[
E^1_- = \langle \psi_+ | H' | \psi_+ \rangle = \frac{2}{L} \left(-V_0\right) \int_{-L/2}^{L/2} e^{-x^2/a^2} \sin^2 \frac{2\pi nx}{L} \, dx \approx -\sqrt{\frac{V_0a}{L}} \left(1 - e^{-\left(\frac{2\pi na}{L}\right)^2}\right),
\]

where I used Maple to evaluate the integrals after replacing the limits \( \pm L/2 \) with \( \pm \infty \).

d) Note from c) that the eigenfunctions \( \psi_\pm \) have definite parity, so we can use the parity operator \( \Pi \) to apply the theorem. Works!