8 Scattering Theory I

8.1 Kinematics

**Problem:** wave packet incident on fixed scattering center \( V(r) \) with finite range.

**Goal:** find probability particle is scattered into angle \( \theta, \phi \) far away from scattering center. Solve S.-eqn. with boundary condition that at \( t = -\infty \) the wave function is a wave packet incident on scattering ctr. Decompose packet into component waves \( e^{i k \cdot r} \), then either this wave is scattered or it’s not. If it’s scattered, at large distances we expect a spherical wave. So at large \( r \) our soln. should have form of linear combs. of

\[
\psi_k(r, t) \simeq \left( e^{i k \cdot r} + f_k(\theta, \phi) \frac{e^{i k r}}{r} \right) e^{-i \omega t},
\]

where \( \hbar \omega = \hbar^2 k^2 / 2m \) is the energy of the asymptotic plane wave before scattering or after it has scattered (elastically). Intuitively 1st term in (1) is unscattered part, 2nd term is scattered wave with angular dependence \( f_k \). \( f_k \) called scattering amplitude. 2nd term

\[
u = f_k e^{i k r} / r
\]

varies as \( 1/r \), so intensity of scattered wave falls off as \( |u|^2 \sim 1/r^2 \) as it must. To verify this, construct probability current

\[
j = \frac{-i \hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) \]

For scattered flux density (replace \( \psi \) by \( u \) above), find

\[
j_{\text{scatt}} = \frac{-i \hbar}{2m} (u^* \nabla u - u \nabla u^*) = \frac{\hbar k}{m r^3} |f(\theta, \phi)|^2
\]

Differential scattering cross section \( \frac{d\sigma}{d\Omega} \)

Def: given incident flux \( n v \) particles per unit area and unit time. (\( n \) is density and \( v \) speed of particles)
Particles collected in detector of area \( A \) at angles \( \theta, \phi \) from incident direction. A therefore subtends \( \delta \Omega = A/r^2 \) steradians. Recall classically

\[
\frac{dN}{dt} = (nv) \cdot \frac{d\sigma}{d\Omega} \cdot \delta \Omega \tag{5}
\]

rate of detecting
particles in \( A \) = incident flux density \cdot differential scattering \cdot subtended solid angle \( \tag{6} \)

From eq. (4) above, have

\[
\frac{dN}{dt} = j_{\text{scatt}} A = \frac{\hbar k}{m} \frac{r}{r^3} |f(\theta, \phi)|^2 A = \frac{\hbar k}{m} \cdot \frac{d\sigma}{d\Omega} \cdot \frac{A}{r^2} \tag{7}
\]

so

\[
\frac{d\sigma}{d\Omega}(\theta, \phi) = |f(\theta, \phi)|^2 \tag{8}
\]

Total scattering cross-section \( \sigma \)

Total cross-section defined by integrating over all angles:

\[
\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega |f(\theta, \phi)|^2 \tag{9}
\]

Reminder–classical analog & origin of name “cross-section”:

Consider \( n \) pt. particles/vol. incident on sphere, radius \( a \), flux \( nv \). Particle scatters if its impact parameter is less than \( a \), so net scattered flux is \( nv \cdot \pi r^2 \), total \( \sigma = \pi a^2 \).
8.2 Optical theorem

A conservation law–relates diminished amplitude of unscattered beam to flux scattered out of beam.

Incident flux density in Fourier component $e^{ikz}$ of incident beam is

$$\nu_{\text{classical}} = \frac{\hbar k}{m}$$  \hspace{1cm} (10)

Role of classical density of particles $n$ played by $|\psi_{\text{beam}}|^2 = 1$. From (9) net scattering rate is

$$nv\sigma = \frac{\hbar k}{m} \int d\Omega |f_k|^2$$  \hspace{1cm} (11)

so outgoing unscattered beam must be diminished by this amount. “Outgoing” means $e^{ikz}$ plus forward scattering part of scattered wave, 

$$\psi = e^{ikz} + \frac{f(0)}{r}e^{ikr}$$  \hspace{1cm} (12)

For angles close to forward scattering,

$$r = \sqrt{z^2 + x^2} \simeq z + \frac{x^2}{2z}$$  \hspace{1cm} (13)

so at large $z$,

$$\psi \sim e^{ikz} \left(1 + \frac{f(0)}{z}e^{ik{x^2/2z}}\right)$$  \hspace{1cm} (14)

Now want to compute probability flux in forward direction. Rapidly varying part of (14) is $e^{ikz}$, so

$$\nabla \psi \simeq ik\psi \hat{z}$$  \hspace{1cm} (15)

$$\psi^* \nabla \psi \simeq ik\psi^* \psi = ik \left(1 + \frac{f^*(0)}{z}e^{-ik{x^2/2z}}\right) \left(1 + \frac{f(0)}{z}e^{ik{x^2/2z}}\right)$$  \hspace{1cm} (16)

$$\simeq ik \left(1 + \frac{2}{z} \Re f(0)e^{ik{x^2/2z}}\right)$$  \hspace{1cm} (17)

\footnote{Peebles calls this $\psi_f$ for forward, but I’m too lazy, so you have to remember I’m always talking about small angles}
So to lowest order flux density in forward direction is

$$ j = -\frac{i\hbar}{2m}(\psi^* \nabla \psi - \text{c.c.}) $$

(18)

$$ \simeq \frac{\hbar k}{m} \left( 1 + \frac{2}{z} \text{Re } f(0)e^{i\frac{kx^2}{2z}} \right) $$

(19)

The first term is the same as the incident flux density, the second is the reduction of the outgoing flux by forward scattering. If we go out far enough along $z$, even for small angles the $x$ can get large, so we need to integrate all the scattered flux (2nd term) in the forward direction:

$$ \left( \text{reduction of flux by forward scattering} \right) = -v_{cl} \int_{0}^{\infty} 2\pi x dx \frac{2}{z} \text{Re } f(0)e^{i\frac{kx^2}{2z}} $$

(21)

see footnote $\rightarrow$ = $v_{cl} \pi \frac{2}{z} \text{Im } f(0) \frac{2z}{k} \equiv v_{cl}\sigma$

(22)

since $v\sigma$ is the definition of the total flux removed from the beam by scattering! So

$$ \sigma = \frac{4\pi}{k} \text{Im } f(0) $$

“Optical theorem”

(23)

**Interpretation:**

Incident plane wave brings in probability current density in $z$ direction. Some of it gets scattered into various directions. Must give rise to decrease in current density **behind target** ($\theta = 0$) due to **destructive interference** of incident plane wave and scattered wave in forward direction. So total flux scattered ($v\sigma$) related to forward scattering amplitude $f(0)$.

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This is a mathematically very ill-defined procedure, and you shouldn’t believe it until you see we get the same result by **phase shift analysis** later. But the integration goes as follows:

$$ \int_{0}^{\infty} 2x dx \ e^{ikx^2/(2x)} = \frac{2z}{ik} \int_{0}^{\infty} du \ e^{u} = -\frac{2iz}{k} \ e^{u} \bigg|_{0}^{\infty} $$

(20)

We now want to argue that we can neglect the exponential at its upper limit because it is rapidly oscillating, i.e. we want to be able to set $e^{i\infty} \rightarrow 0$. Peebles argues physically: that a real beam will consist of a finite spread of incident directions and cut off the increase in the rate of oscillations as $z \rightarrow \infty$. That the oscillations at $\infty$ then average out to zero is guaranteed by the **Riemann-Lesbegue lemma**.

\[3\]
8.3 Born approximation.

Valid for *weak scattering* or *fast* particles!

Want to solve S.’s eqn for scattering potential $V(r)$ with boundary condition

$$\psi \to e^{ik \cdot r} + f \frac{e^{ikr}}{r} \quad (24)$$

for suff. large $r$. We’re looking first for solutions with

$$E = \frac{\hbar^2 k^2}{2m} \quad \text{"scattering states"} \quad (25)$$

i.e., *not* bound states where the particle is trapped by the target. So S.-eqn. looks like

$$\nabla^2 \psi + k^2 \psi = \frac{2mV}{\hbar^2} \psi \equiv \epsilon U \psi \quad (26)$$

Now seek power series expansion of $\psi$ (in powers of scattering potential):

$$\psi(r) = e^{ik \cdot r} + \epsilon \psi^{(1)} + \epsilon^2 \psi^2 + \ldots \quad (27)$$

plugging into (26) gives

$$\epsilon(\nabla^2 \psi^{(1)} + k^2 \psi^{(1)}) + \epsilon^2(\nabla^2 \psi^{(2)} + k^2 \psi^{(2)}) =
\quad = \epsilon U e^{ik \cdot r} + \epsilon^2 U \psi^{(1)} + \ldots \quad (28)$$

as usual equate powers of $\epsilon$:

$$\nabla^2 \psi^{(1)} + k^2 \psi^{(1)} = U e^{ik \cdot r} \quad (29)$$

$$\nabla^2 \psi^{(2)} + k^2 \psi^{(2)} = U \psi^{(1)}, \quad \text{etc.} \quad (30)$$

\vdots \quad (31)

See that we get sequence of coupled eqns., each coupled to previous one on rhs. This is called *Born’s series*.

If we want to solve to *linear* order in $\epsilon$, take only (29). Solve by Fourier transform; and define
\[ \psi_k = \int \psi^{(1)}(\mathbf{r}) e^{-i \mathbf{k} \cdot \mathbf{r}} d^3 \mathbf{r} \] (32)

and the inverse
\[ \psi^{(1)}(\mathbf{r}) = \int \psi_k e^{i \mathbf{k} \cdot \mathbf{r}} \frac{d^3 k}{(2\pi)^3} \] (33)

So multiply (29) by \( e^{-i \mathbf{k}' \cdot \mathbf{r}} \) and integrate lhs by parts\(^4\) to get
\[ (k^2 - k'^2) \psi_k' = \int U(\mathbf{r}) e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} d^3 \mathbf{r} \] (35)

which yields immediately
\[ \psi^{(1)}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3 k' \frac{e^{i \mathbf{k}' \cdot \mathbf{r}}}{k^2 - k'^2} \int d^3 r' U(\mathbf{r}') e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}'} \] (36)
\[ = \frac{1}{(2\pi)^3} \int d^3 r' U(\mathbf{r}') e^{i \mathbf{k} \cdot \mathbf{r}'} \int d^3 k' \frac{e^{i \mathbf{k}' \cdot (\mathbf{r} - \mathbf{r}')}}{k^2 - k'^2} \frac{I(\mathbf{r} - \mathbf{r}')} \] (37)

Now we spend some tedious but hopefully useful time showing how to calculate integrals of this type.

First evaluate \( I(\mathbf{r} - \mathbf{r}') \). Use polar coords., polar axis along \( \mathbf{y} \equiv \mathbf{r} - \mathbf{r}' \), \( \theta \) is angle between \( \mathbf{y} \) and \( \mathbf{k}' \):
\[ k' \cdot \mathbf{y} = k' y \cos \theta \] (38)
\[ dk' = k'^2 dk' \sin \theta d\theta d\phi. \] (39)

So angular part of \( I \) is
\[ \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \, e^{ik'y \cos \theta} = 2\pi \int_{-1}^1 d \cos \theta e^{ik'y \cos \theta} \] (40)
\[ = 4\pi \frac{\sin k'y}{k'y} \] (41)

so left with
\[ I = 4\pi \int_0^\infty \frac{k'^2 dk'}{k^2 - k'^2} \frac{\sin k'y}{k'y} \] (42)

Trick: note integrand is even in \( k' \), write \( I \) as
\[ I = \frac{2\pi}{i} \int_\infty^{\infty} \frac{k'dk'}{k^2 - k'^2} \frac{e^{ik'y}}{y} \] because cos part odd in \( k' \), therefore doesn’t contribute (43)
\[ \int (\nabla^2 \psi(\mathbf{r})) e^{-i \mathbf{k}' \cdot \mathbf{r}} d^3 \mathbf{r} = -k'^2 \psi_k' \] (34)

\(^4\)Explicitly, use
\[ \int (\nabla^2 \psi(\mathbf{r})) e^{-i \mathbf{k}' \cdot \mathbf{r}} d^3 \mathbf{r} = -k'^2 \psi_k' \] (34)
This is an improper integral due to the singularity at $k = \pm k'$. However we use physics to regularize: so far we haven’t used condition that we want w. fctn. $\psi$ at $\infty$ to look like scattered wave. This corresponds in integral scattering formulation to specifying contour for $I$ in complex plane. One possibility is to handle singularities by following contour shown below.

Use Cauchy integral formula, complete the contour in upper half-plane, shrink the contour to the pole at $+k$, to get:

$$I = \frac{2\pi e^{iky}}{i} \cdot \frac{1}{2} \oint \frac{dk'}{k - k'} = -2\pi^2 \frac{e^{iky}}{y}$$

This procedure may seem rather arbitrary at 1st sight, but a look at other choices suggests it’s right thing to do. If we take contour above both poles, get zero! If we take contour opposite to figure (above $k$ and below $-k$) or below both poles, get a contribution $I \propto e^{-iky}$. Recall $y = |r - r'|$; if $r$ is in asymptotic region, $r \gg r'$, $I \propto e^{-ikr}$, which is ingoing, not outgoing scattered wave. So choice of contour effectively implements boundary condition.

Continuing, find

$$\psi^{(1)}(r) = -\frac{1}{4\pi} \int d^3r' U(r') e^{ik|r-r'|} e^{ik\hat{r}\cdot r'}$$

Finally, look at $\psi$ in asymptotic region, $r \gg r'$: use

$$|r - r'| \approx r + \hat{r} \cdot r'$$

$$e^{i\hat{r} \cdot r'} \approx \frac{1}{r} e^{i\hat{r} \cdot r'}$$

define $k_s \equiv k\hat{r}$

to write

$$\psi^{(1)}(r) \approx -\frac{1}{4\pi} \left( \int d^3r' U(r') e^{i(k-k_s)\cdot r'} \right) \frac{e^{ik\hat{r}}}{r}$$

---

If you don’t like the integral formula, note this choice is equivalent to replacing (29) by

$$\nabla^2 \psi^{(1)} + (k^2 + i\epsilon k) \psi^{(1)} = U e^{ikr}$$

i.e. shifting the zeros of the integrand of $I$ to $k' = \pm (k + i\epsilon/2)$.
Now can compare directly to Eq. (1), to identify (use $U = 2mV/\hbar^2$).

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int d^3r' V(r') e^{i(k-k_s) \cdot r'}$$  \hspace{1cm} (52)

$k$ along incident direction
$k_s$ along scattered direction

This is Born approximation. Diff. scatt. cross sec. will be $d\sigma/d\Omega = |f|^2$.

Low-energy limit

We consider only elastic scattering, $E_f = E_i = \hbar^2 k^2 / 2m$. \footnote{So in scattering process direction changes from $k$ to $k_s$, but momentum magnitude $\hbar k$ doesn’t change.} Note main contribution to integral in (53) is from $r'$’s inside range of scattering potential $r_0$ (meaning $V(r')$ small for $|r'| \gg r_0$.) Thus if we consider low-energy scattering such that $kr_0 \ll 1$, argument of exponential in (53) is $\simeq 1$, so find immediately

$$d\sigma/d\Omega \simeq \frac{m^2}{4\pi^2\hbar^4} \left( \int d^3r' V(r') \right)^2, \hspace{1cm} kr_0 \ll 1$$  \hspace{1cm} (53)

Isotropic!

8.4 Screened Coulomb scattering

Scattering by bare Coulomb potential very singular, not obvious we can treat it properly with formalism introduced here ($V(r)$ was assumed to have finite range). However we’ll treat a screened Coulomb or Yukawa\textsuperscript{7} potential,

$$V(r) = -e^2 \frac{e^{-\alpha r}}{r}$$  \hspace{1cm} (54)

which has range $1/\alpha$, and show that in limit $\alpha \to 0$ we indeed recover Rutherford differential cross section for Coulomb scattering.

Born approx. for Yukawa potential.

\textsuperscript{6}Yukawa wrote it down as a phenomenological model for the strong force in nuclei to explain $p-p$ scattering experiments, and was led to propose the existence of the pion as a mediating particle. The form is also characteristic of the Coulomb interaction between charges in a charged medium capable of screening, hence the title of this section. It’s not a bad model for atomic scattering, since a fast electron penetrating the electronic cloud surrounding the nucleus sees more and more charge the closer it gets.
Define $q = k - k_s$. Work here is only in calculating integral

$$
I \equiv \int d^3r e^{-iqr - \alpha r} = \int dr re^{-\alpha r} \int d\Omega e^{-iqr} = \int dr re^{-\alpha r} \cdot -i2\pi(e^{iqr} - e^{-iqr})/qr = \frac{4\pi}{q} \text{Im} \frac{1}{\alpha - iq} = \frac{4\pi}{\alpha^2 + q^2}
$$

(55)

So diff. scatt. cross section is

$$
\frac{d\sigma}{d\Omega} \simeq \frac{m^2}{4\pi^2\hbar^4} I = \frac{4m^2e^4}{\hbar^4(\alpha^2 + q^2)^2}
$$

(56)

$$
\rightarrow \frac{4m^2e^4}{q^4\hbar^4}
$$

(57)

$$
\rightarrow \frac{e^4}{4m^2v^4\sin^4(\theta/2)}
$$

(58)

(59)

(60)

(61)

where $v = \hbar k/m$ is classical velocity, and we used geometry to relate momentum transfer $q$ to incoming momentum $k$, $q = 2k \sin \theta/2$, as seen from picture.

Eq. 62 is in fact famous Rutherford formula for Coulomb scattering so we did get “right” answer after all. Note great irony here: classical result obtained correctly using leading-order Born approximation, so $\hbar$ doesn’t play major role! Yet measurement of Rutherford cross-section proved existence of pointlike nucleus, and it was problem of stable orbits around nucleus which led to Bohr’s quantum mechanics!