1. **Short Answer.** Must attempt (only) 4 of 6. Circle answers to be graded.

(a) Using the figure sketching the wave function of a particle with the scattering potential a) turned off and b) turned on, state whether the potential was most likely attractive or repulsive, and estimate (quantitatively!) the cross-section of the scattering process in the s-wave approximation in terms of the wave vector $k$ of the scattering particle.

Potential must be attractive in lower figure since wave function has been “pulled in” to well ($\delta_0 > 0$). From the figure, $\delta_0 \simeq \pi/4$, so

$$\frac{d\sigma}{d\Omega} = \sin^2 \delta_0 \frac{4\pi}{k^2} \rightarrow \sigma = \frac{2\pi}{k^2}$$

(b) State the electric dipole selection rules. What are the most likely decay paths to the ground state for a) an $H$ atom starting in a $3d$ state? Draw them on the picture, and explain why the other possibilities are “forbidden”. Do the same starting from b) the $2s$ state.

All electric dipole transitions should have $\Delta \ell = \pm 1$, and $\Delta m = \pm 1, 0$. Therefore the only state to which the $322$ state can decay is the $211$. The $200$ state is infinitely long lived in this approximation since it has nowhere to decay to which satisfies the selection rule.

(c) Describe the three kinds of radiation processes related statistically by Einstein’s detailed balance argument. Discuss the conclusions of this argument, or alternatively state how any two of the rates in the argument are related.
Einstein’s argument relates the rates of spontaneous emission, stimulated emission, and absorption of radiation of a population of atoms in equilibrium with a bath of photons with frequency $\omega_0$. You were supposed to say what each process is. The rate of spontaneous emission is $N_2 A$, rate of stimulated emission is $B n N_2$, and absorption is $n N_1 C$, where $N_2, 1$ is population of upper, lower state and $n$ the number of photons in the cavity. Einstein showed that $A = B = C$.

(d) State the physical meaning of the optical theorem.

Optical theorem: $\sigma = \text{Im}(4\pi f(0))/k$.

The amount of flux scattered out of the incoming beam is $\sigma v_{cl}$. The forward part of the outgoing wave must be diminished by this amount, by conservation of probability.

(e) State the Fermi Golden rule. Be sure to identify each quantity occurring in the expression you give.

\[
R_f = \frac{dP_f}{dt} \simeq \frac{P_f}{T} = \frac{\pi |\langle f | \hat{V}_0 | i \rangle|^2}{2\hbar^2} \rho(\omega_0) \tag{1}
\]

In other words, the rate of making transitions from state $i \rightarrow f$ is proportional to the matrix element of the perturbing potential $V$ squared, times the density of states at an energy $\hbar \omega_0$ equal to the difference between the two levels $E_f - E_i$. This form of the Fermi Golden Rule is for a bunch of two-level atoms exposed to incoherent radiation characterized by some distribution of energies $\rho(\omega)$, e.g. the distribution of radiation in a cavity.

(f) Explain why the quantum total cross-section $\sigma$ for hard-sphere scattering is 4 times the geometrical cross-section.

Like the classical cross section for low-energy classical waves, the QM cross-section for particles scattering from a hard sphere is a factor of 4 larger than the classical cross-section for particles $\pi a^2$, since in QM the particles can “diffract” around the sphere, filling in the classical shadow zone and leading to enhanced scattering.
2. **Precessing spin.** The Hamiltonian for a spin 1/2 particle in a uniform magnetic field \( B = B_0 \hat{z} \) may be written \( H = -\mu \vec{\sigma} \cdot \vec{B} \) where \( B = B_0 \hat{z} \) and \( \vec{\sigma} \) is the Pauli spin vector. At time \( t = 0 \) the spin is along the \( \hat{x} \) direction (an eigenfunction of \( \sigma_x \) with eigenvalue +1).

(a) Solve the time dependent Schrödinger equation to find the spin wave function at time \( t \).

State at \( t = 0 \) is \(| \uparrow_x \rangle \equiv (1/\sqrt{2}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Time evolution operator is

\[
e^{-iHt/\hbar} = e^{i\mu \sigma_z B/\hbar} = \cos \omega t + i \sigma_z \sin \omega t,
\]

where \( \omega = \mu B/\hbar \). So at time \( t \) state is

\[
|\chi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \omega t & i \sin \omega t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \sin \omega t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

(b) Find the expectation value of \( \sigma_x \) at time \( t \). What is the spin precession frequency?

**Hint:** The Pauli matrices are:

\[
\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

\[
\langle \chi | \sigma_x | \chi \rangle = \frac{1}{\sqrt{2}} (\cos \omega t (1 1) - i \sin \omega t (1 -1)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\times \frac{1}{\sqrt{2}} \left( \cos \omega t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \sin \omega t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \cos 2\omega t
\]

so precession frequency is \( 2\omega = 2\mu B/\hbar \).

(c) The field is now changed to

\[
B_x = B_1 \cos \omega t \\
B_y = B_1 \sin \omega t \\
B_z = \text{const.} \gg B_1
\]

3
What value of $\omega$ gives resonant transitions (takes an up spin and converts it to a down spin with maximum probability)?

The large $z$-component of the field makes the energy levels of the spin to zeroth order just $\mu B_z$. Resonant absorption will take place when the frequency of the time-varying magnetic field matches the energy difference between the up and down spin configurations, i.e. $\hbar \omega = 2\mu B_z$.

(d) At $t=0$ all the protons are polarized in the state $| \uparrow \rangle$. What is the probability for a proton at time $t$ to have spin in the $-z$ direction?

From $t$-dependent perturbation theory, the amplitude for a spin to be in the down state at some later time is

$$c_{\downarrow} \simeq \frac{1}{i\hbar} \int_0^t dt' \langle \downarrow | -\mu_x B_1 \cos \omega t' - \mu_y B_1 \sin \omega t' | \uparrow \rangle \times e^{i(E_{\downarrow} - E_{\uparrow})t/\hbar}$$

Now use $\mu = (g/2m_p)cS = (g/4m_p)c\sigma$, and matrix elements $\langle \downarrow | \sigma_x | \uparrow \rangle = 1$ and $\langle \downarrow | \sigma_y | \uparrow \rangle = i$ to find

$$c_{\downarrow} = -\frac{1}{i\hbar} \frac{g B_1}{2m_p c} \frac{1}{2} \int_0^t dt' e^{i(\omega + \omega_0)t}$$

$$= \frac{-g B_1}{i\hbar 4m_p c} \frac{e^{i(\omega + \omega_0)t/\hbar} - 1}{i(\omega + \omega_0)},$$

where $\omega_0 = (E_{\downarrow} - E_{\uparrow})/\hbar$. Probability to find proton spin down is then $|c_{\downarrow}(t)|^2$. 

4
3. **Born approximation.** A particle of mass $m$ is scattered by a potential $V(r) = V_0 \exp(-r/a)$.

(a) Find the differential scattering cross section in the first Born approximation.

\[
f(\theta) = -\frac{m}{2\pi\hbar^2} \int d^3r' V(r') e^{iq \cdot r'}
\]
\[
= -\frac{mV_0}{2\pi\hbar^2} (2\pi) \int r'^2 \, dr' e^{-r'/a} \int_0^\pi d\theta \sin \theta e^{iqr' \cos \theta}
\]
\[
= -\frac{2mV_0}{\hbar^2 q} \int_0^\infty dr' r' \sin(qr') e^{-r'/a}
\]
\[
= -\frac{4mV_0a^3}{\hbar^2(1 + q^2a^2)^2},
\]

where $q = k|\hat{k} - \hat{r}| = 2k \sin(\theta/2)$. So

\[
\frac{d\sigma}{d\Omega} = |f|^2 = \frac{16m^2V_0^2a^6}{\hbar^4(1 + 4k^2a^2 \sin^2 \theta/2)^4}
\]

(b) Sketch the angular dependence for small and large $k$, where $k$ is the wave number of the particle being scattered. At what $k$ value does the scattering start to become significantly non-isotropic (estimate)?

If $ka \ll 1$, the $\sin \theta/2$ in the denominator won’t matter. So roughly speaking if $k \ll a^{-1}$, there will be no angular dependence of the cross section. One could have guessed this without doing the calculation for part a), because as we discussed, the scattering is always isotropic when $k$ is much smaller than the inverse range of the scattering (only length in the problem).
(c) The criterion for the validity of the first Born approximation is that the 1st-order correction to the incident plane wave be small, \( |\psi^{(1)}(\mathbf{r})| \ll |e^{ik\mathbf{r}}| \). Using the integral form of the Schrödinger equation, translate this condition into a condition on \( V_0 \), and simplify it for small \( k \), \( ka \ll 1 \). Hint: you may compare the incoming wave function and the first Born scattering correction to it at \( \mathbf{r} = 0 \).

Integral form of the Schrödinger equation is (see helpful formulae)

\[
\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2} \int e^{ik|\mathbf{r}-\mathbf{r}'|} V(\mathbf{r}')\psi(\mathbf{r}')d^3r',
\]

and the first Born correction is obtained by inserting \( \psi_0 = e^{ik\mathbf{r}} \) for \( \psi \), so

\[
\left| \frac{\Delta\psi^{(1)}(0)}{\psi^{(0)}(0)} \right| = \left| \frac{V_0m}{2\pi\hbar^2} \int \frac{e^{ikr'}}{r'} e^{-r'/a} e^{ik\mathbf{r'}} d^3r' \right|
\]

\[
= \left| \frac{V_0m}{\hbar^2} \int_0^\infty r'dr' \int_0^\pi \sin \theta' d\theta' e^{ikr'} e^{-r'/a} e^{ik\mathbf{r'}} \right|
\]

\[
= \left| \frac{V_0m}{\hbar^2} \int_0^\infty r'dr' \int_0^\pi \sin \theta' d\theta' e^{ikr'-r'/a+ikr'\cos\theta'} \right|
\]

\[
= \left| \frac{V_0m}{\hbar^2} \int_0^\infty r'dr' \left( -ie^{-r'/a} e^{2ikr'} - 1 \right) \right|
\]

\[
= \frac{2m|V_0|a^2}{\hbar^2\sqrt{1 + 4k^2a^2}}
\]

so the criterion that the Born approximation be valid is

\[
\frac{2m|V_0|a^2}{\hbar^2\sqrt{1 + 4k^2a^2}} \ll 1
\]

and in the limit \( ka \ll 1 \),

\[
|V_0| \ll \frac{\hbar^2}{2ma^2}
\]

(d) What is the limit on \( V_0 \) which comes from this criterion in the high-\( k \) limit, i.e. \( ka >> 1 \)?

Compare with answer to c). In limit \( ka \gg 1 \), find

\[
|V_0| \ll \frac{\hbar^2}{ma} = \frac{\hbar^2(ka)}{ma^2}
\]

So that if \( ka \gg 1 \), the Born approximation is actually valid over a much larger regime.