1. Consider $F = F(r, \theta)$. Then the differential is

$$dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta,$$

but we may consider $r$ and $\theta$ to be functions of $x$ and $y$. Therefore their differentials are

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy,$$

$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy.$$

Substituting, we find

$$\left( \frac{\partial F}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial x} \right) dx + \left( \frac{\partial F}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial y} \right) dy.$$

If we now wish to take a derivative of $F$ wrt $x$ holding $y$ constant, we can calculate it by simply setting $dy = 0$ in the differential, obtaining

$$\left. \frac{\partial F}{\partial x} \right|_y = \frac{\partial F}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta} \frac{\partial \theta}{\partial x},$$

just the result expected from the chain rule. The other derivatives requested follow from a similar analysis.

2. We’re given $du = Tds - pdv$.

(a) $(T, v)$. Note that

$$d(u - Ts) = Tds - pdv - Tds - sdT = -pdv - sdT,$$

so $f = u - Ts$ is a function whose differential depends only on $dv$ and $dT$.

(b) $(s, p)$. Consider

$$d(u + pv) = Tds - pdv + pdv + vdp = Tds - vdp,$$

so $h = u + pv$ has a differential which depends only on $ds$ and $dp$.

(c) $(p, T)$. Finally, consider (using results of (b))

$$d(h - Ts) = Tds - vdp - Tds - sdT = -vdp - sdT,$$

so $g = h - Ts = u + pv - Ts$ has a differential which depends only on $dp$ and $dT$. Note in thermodynamics the functionals $f, h$, and $g$ are referred to as the Helmholtz free energy, the enthalpy, and the Gibbs free energy. They represent quantities which are extremal when a thermodynamic system is in equilibrium under conditions when one of the relevant thermodynamic variables is held constant.
3. Start with \( du = T \, ds - pdv \). Think of \( u \) as a function of \( s \) and \( v \). Since this is an exact differential, we know that

\[
T = \left. \frac{\partial u}{\partial s} \right|_V \quad ; \quad -p = \left. \frac{\partial u}{\partial V} \right|_s.
\]

(9)

We can take a \( v \) derivative of the first and an \( s \) derivative of the second to obtain \( \frac{\partial^2 s}{\partial v \partial s} \) and \( \frac{\partial^2 s}{\partial s \partial v} \). But the equality of mixed partial derivatives implies that

\[
\left. \frac{\partial T}{\partial V} \right|_s = -\left. \frac{\partial p}{\partial s} \right|_V,
\]

(10)

one of the Maxwell relations.

4. Proceeding similarly to Prob.3,

(a) \( dh = T \, ds + v \, dp \) \implies \( \left. \frac{\partial T}{\partial p} \right|_s = \left. \frac{\partial V}{\partial s} \right|_p \)

(11)

(b) \( df = -pv \, dv - sdT \) \implies \( \left. \frac{\partial p}{\partial T} \right|_v = \left. \frac{\partial s}{\partial v} \right|_T \)

(12)

(c) \( dg = vdp - sdT \) \implies \( \left. \frac{\partial V}{\partial T} \right|_p = -\left. \frac{\partial s}{\partial p} \right|_T \),

(13)

we get the remaining Maxwell relations.

5. (a) \( \sigma \) is a surface charge density, or charge/area. Since \( xyz \) is a volume, the constant \( a \) must have dimensions of charge/\( L^5 \).

(b) Let’s use method of Lagrange multipliers (Boas p. 214 et seq.). Prescription is to define a new function which is the function to be minimized plus a multiplier \( \lambda \) times the constraint. So

\[
F[x, y, z, \lambda] = axyz + \lambda(x^2 + y^2 + z^2 - b^2).
\]

(14)

Let’s now set all partial derivatives of \( F \) equal to zero:

\[
\frac{\partial F}{\partial x} = ayz + 2\lambda x = 0 \quad ; \quad \frac{\partial F}{\partial y} = axz + 2\lambda y = 0 \quad ; \quad \frac{\partial F}{\partial z} = axy + 2\lambda z = 0 \quad ;
\]

\[
\frac{\partial F}{\partial \lambda} = x^2 + y^2 + z^2 - b^2 = 0.
\]

(15)

Multiplying the first three equations by \( x, y \) and \( z \) respectively, and adding, we get

\[
3axyz + 2\lambda(x^2 + y^2 + z^2) = 0 \quad \Rightarrow \quad axyz = -\frac{2}{3}b^2\lambda.
\]

(16)
Multiplying 1st equation in (15) by $x$ and substituting for $xyz$, we arrive at $x^2 = \lambda/3$. Since the problem is symmetric in $x$, $y$ and $z$, we can immediately say $x^2 = y^2 = z^2 = \lambda/3$. But this means that $x^2 + y^2 + z^2 = \lambda = b^2$, so we have determined the Lagrange multiplier, and can say that the extremum occurs at $x^2, y^2, z^2 = (b^2/3)(1, 1, 1)$.

Now we need to say whether these values are maxima or minima. Let’s assume $a > 0$ without loss of generality. Since $\sigma = axyz$, two extremal values are obviously $\pm b^3a/3^{3/2}$. So without taking second derivatives, we can say that the maxima, where $\sigma = b^3a/3^{3/2}$, occur at $\sqrt{3}(x, y, z)/b = (1, -1, -1), (-1, 1, -1), (1, 1, 1)$ and $(-1, -1, 1)$. 
