9 Linear algebra

Read: Boas Ch. 3.

9.1 Properties of and operations with matrices

$M \times N$ matrix with elements $A_{ij}$

$$A = \begin{bmatrix}
A_{11} & A_{12} & \ldots & A_{1j} & \ldots & A_{1N} \\
A_{21} & A_{22} & \ldots & A_{2j} & \ldots & A_{2N} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
A_{i1} & A_{i2} & \ldots & A_{ij} & \ldots & A_{iN} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
A_{M1} & A_{M2} & \ldots & A_{Mj} & \ldots & A_{MN}
\end{bmatrix} \tag{1}$$

Definitions:

- Matrices are equal if their elements are equal, $A = B \iff A_{ij} = B_{ij}$.
- $(A + B)_{ij} = A_{ij} + B_{ij}$
- $(kA)_{ij} = kA_{ij}$ for $k$ const.
- $(AB)_{ij} = \sum_{\ell=1}^{N} A_{i\ell}B_{\ell j}$. Note for multiplication of rectangular matrices, need $(M \times N) \cdot (N \times P)$.
- Matrices need not “commute”. $AB$ not nec. equal to $BA$. $[A, B] = AB - BA$ is called “commutator of $A$ and $B$. If $[A, B] = 0$, $A$ and $B$ commute.
- For square mats. $N \times N$, $\det A = |A| = \sum_{\pi} \sgn \pi A_{1\pi(1)}A_{2\pi(2)}\ldots A_{N\pi(N)}$, where sum is taken over all permutations $\pi$ of the elements $\{1, \ldots N\}$. Each term in the sum is a product of $N$ elements, each taken from a different row of $A$ and from a different column of $A$, and $\sgn \pi$. Examples:

$$\begin{vmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}, \tag{2}$$

$$\begin{vmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{vmatrix} = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} - A_{11}A_{22}A_{32} \tag{3}$$

- $\det AB = \det A \cdot \det B$ but $\det(A + B) \neq \det A + \det B$. For practice with determinants, see Boas.
• Identity matrix $I$: $IA = A \forall A$. $I_{ij} = \delta_{ij}$.

• Inverse of a matrix. $A \cdot A^{-1} = A^{-1}A = I$.

• Transpose of a matrix $(A^T)_{ij} = A_{ji}$.

• Formula for finding inverse:

$$A^{-1} = \frac{1}{\det A} C^T,$$

where $C$ is “cofactor matrix”. An element $C_{ij}$ is the determinant of the $N - 1 \times N - 1$ matrix you get when you cross out the row and column (i,j), and multiply by $(-1)^i(-1)^j$. See Boas.

• Adjoint of a matrix. $A^\dagger$ is adjoint of $A$, has elements $A^\dagger_{ij} = A^*_ji$, i.e. it’s conjugate transpose. Don’t worry if you don’t know or have forgotten what conjugate means.

• $(AB)^T = B^T A^T$

• $(AB)^{-1} = B^{-1} A^{-1}$

• “Row vector” is $1 \times N$ matrix: $[a \ b \ c \ldots \ n]$

• “Column vector” is $M \times 1$ matrix:

$$\begin{pmatrix}
e \\ f \\ g \\ \vdots \\ m
\end{pmatrix}$$

(5)

• Matrix is “diagonal” if $A_{ij} = A_{ii} \delta_{ij}$.

• “Trace” of matrix is sum of diagonal elements: $\text{Tr } A = \sum_i A_{ii}$. Trace of produce is invariant under cyclic permutations (check!):

$$\text{Tr } ABC = \text{Tr } BCA = \text{Tr } CAB.$$ (6)

9.2 Solving linear equations

Ex.

$$\begin{aligned}
x - y + z &= 4 \\
2x + y - z &= -1 \\
3x + 2y + 2z &= 5
\end{aligned}$$
may be written

\[
\begin{bmatrix}
1 & -1 & 1 \\
2 & 1 & -1 \\
3 & 2 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
-1 \\
5
\end{bmatrix}.
\]

(7)

Symbolically, \( \mathbf{A} \cdot \mathbf{r} = \mathbf{k} \). What we want is \( \mathbf{r} = A^{-1}\mathbf{k} \). So we find the determinant \( \det A = 12 \), and the cofactor matrix

\[
C = \begin{bmatrix}
4 & -7 & 1 \\
4 & -1 & -5 \\
0 & 3 & 5
\end{bmatrix},
\]

(8)

then take the transpose and construct \( A^{-1} = \frac{1}{\det A} C^T \):

\[
A^{-1} = \frac{1}{12} \begin{bmatrix}
4 & 4 & 0 \\
-7 & -1 & 3 \\
1 & -5 & 5
\end{bmatrix},
\]

(9)

so

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \frac{1}{12} \begin{bmatrix}
4 & 4 & 0 \\
-7 & -1 & 3 \\
1 & -5 & 5
\end{bmatrix}
\begin{bmatrix}
4 \\
-1 \\
5
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-1 \\
2
\end{bmatrix}.
\]

(10)

So \( x = 1, y = -1, z = 2 \). Alternate method is to use Cramer’s rule (see Boas p. 93):

\[
x = \frac{1}{12} \begin{vmatrix}
4 & -1 & 1 \\
-1 & 1 & -1 \\
5 & 2 & 2
\end{vmatrix}
= 1,
\]

(11)

and “similarly” for \( y \) and \( z \). Here, the 12 is \( \det A \), and the determinant shown is that of the matrix of coefficients \( A \) with the \( x \) coefficients (in this case) replaced by \( \mathbf{k} \).

Q: what happens if \( \mathbf{k} = 0 \)? Eqs. are homogeneous \( \Rightarrow \det A = 0 \).

### 9.3 Rotation matrices

Here’s our old example of rotating coordinate axes. \( \mathbf{r} = (x, y) \) is a vector. Let’s call \( \mathbf{r}' = (x', y') \) the vector in the new coordinate system. The two are related by
the equations of coordinate transformation we discussed in week 4 of the course. These may be written in matrix form in a very convenient way (check):

\[
\begin{bmatrix}
x'
\end{bmatrix}
\begin{bmatrix}
y'
\end{bmatrix}
=
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
\]

(12)

where \( R_\theta \) is a rotation matrix by \( \theta \). Note the transformation preserves lengths of vectors \(|\vec{r}'| = |\vec{r}| \) as we mentioned before. This means the rotation matrix is orthogonal:

\[ R_\theta^T R_\theta = I. \]  

(13)

These matrices have a special property (“group property”), which we can show by doing a second rotation by \( \theta' \):

\[
R_\theta R_{\theta'} = \begin{bmatrix}
\cos \theta' & \sin \theta' \\
-\sin \theta' & \cos \theta'
\end{bmatrix}
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
= \begin{bmatrix}
\cos \theta' \cos \theta - \sin \theta' \sin \theta & \cos \theta' \sin \theta + \sin \theta' \cos \theta \\
-\sin \theta' \cos \theta - \cos \theta' \sin \theta & \cos \theta' \cos \theta - \sin \theta' \sin \theta
\end{bmatrix}
\]

(14)

\[
= \begin{bmatrix}
\cos(\theta + \theta') & \sin(\theta + \theta') \\
-\sin(\theta + \theta') & \cos(\theta + \theta')
\end{bmatrix} = R_{\theta + \theta'}.
\]

(15)

Thus the transformation is linear. More general def. \( A(c_1 B + c_2 D) = c_1 AB + c_2 AD \).
9.4 Matrices acting on vectors

\[
\begin{bmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_N \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2j} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{ij} & \cdots & A_{iN} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{Nj} & \cdots & A_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix},
\]

or more compactly

\[ v'_i = \sum_{j=1}^{N} A_{ij} v_j. \]  

Another notation you will often see comes from the early days of quantum mechanics. Write the same equation

\[ |v'\rangle = A|v\rangle. \]

So this is a “ket”, or column vector. A “bra”, or row vector is written as the adjoint of the column vector\(^1\):

\[ \langle v | = (|v\rangle)^\dagger \equiv (v^*_1 \ v^*_2 \ \ldots \ v^*_N) \]

\[ = (v_1 \ v_2 \ \ldots \ v_N) \quad \text{if } v_i \text{ are real}. \]

N.B. In this notation the scalar product of $|v\rangle$ and $|w\rangle$ is $\langle v | w \rangle$, and the length of a vector is given by $|v|^2 = \langle v | v \rangle$.

9.5 Similarity transformations

Suppose a matrix $B$ rotates $|r\rangle$ to $|r_1\rangle$, $|r_1\rangle = B|r\rangle$. Now we rotate the coordinate system by some angle as well, so that the vectors in the new system are $|r'\rangle$ and $|r'_1\rangle$, e.g. $|r'_1\rangle = R|r_1\rangle$. What is the matrix which relates $|r'\rangle$ to $|r'_1\rangle$, i.e. the transformed matrix $B$ in the new coordinate system?

\[ |r'_1\rangle = R|r_1\rangle = RB|r\rangle = RBR^{-1}R|r\rangle = (RBR^{-1})(R|r\rangle) = (RBR^{-1})|r'\rangle, \]

so the matrix $B$ in the new basis is

\[ B' = RBR^{-1}. \]

\(^1\)The name comes from putting bra and ket together to make the word bracket; no one said physicists were any good at language
This is called a similarity transformation of the matrix $B$.

To retain:

- similarity transformations preserve traces and determinants: $\text{Tr } M = \text{Tr } M'$, $\det M = \det M'$.
- matrices $R$ which preserve lengths of real vectors are called *orthogonal*, $RR^T = 1$ as we saw explicitly.
- matrices which preserve lengths of *complex* vectors are called *unitary*. Suppose $|v'\rangle = U|v\rangle$, require $UU^\dagger = 1$, then
  \[
  \langle v'|v' \rangle = (U|v\rangle)^\dagger U|v\rangle = \langle v|U^\dagger U|v\rangle = \langle v|v \rangle.
  \]
  A similarity transformation with unitary matrices is called a unitary transformation.
- If a matrix is equal to its adjoint, it’s called self-adjoint or *Hermitian*.

Examples

1. 
\[
A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 5 \\
3 & 5 & 6
\end{bmatrix}
\]

is *symmetric*, i.e. $A = A^T$, also Hermitian because it is real.

2. 
\[
A = \begin{bmatrix}
0 & 2 & 3 \\
-2 & 0 & 5 \\
-3 & -5 & 0
\end{bmatrix}
\]

is *antisymmetric*, and anti-self-adjoint, since $A = -A^T = -A^\dagger$.

3. 
\[
A = \begin{bmatrix}
1 & -i \\
i & 2
\end{bmatrix}
\]

is *Hermitian*, $A = A^\dagger$. 

4. 

\[ A = \begin{bmatrix} i & 1 \\ -1 & 2i \end{bmatrix} \]  

(27)

is *anti*Hermitian, \( A = -A^\dagger \). Check!

5. 

\[ A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \]  

(28)

is unitary, \( A^{-1} = A^\dagger \).

6. 

\[ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \]  

(29)

is orthogonal, \( A^{-1} = A^T \). Check!

### 9.5.1 Functions of matrices

Just as you can define a function of a vector (like \( \vec{r}^3 \)), you can define a function of a matrix \( M \), e.g. \( F(M) = aM^2 + bM^5 \) where \( a \) and \( b \) are constants. The interpretation here is easy, since powers of matrices can be understood as repeated matrix multiplication. On the other hand, what is \( \exp(M) \)? It must be understood in terms of its Taylor expansion, \( \exp(M) = \sum_n M^n/n! \). Note that this makes no sense unless *every* matrix element sum converges.

Remark: note \( e^A e^B \neq e^{A+B} \) unless \( [A, B] = 0 \)! Why? (Hint: expand both sides.)