Electromagnetic Theory II
Solution Set 13

Due: 21 April 2021

48. J, Problem 10.16. To be clear, in part a) the integral referred to is the integral over angles of the differential cross section, i.e. you are calculating the contribution of shadow scattering to the total cross section. The latter is obtained in part b) via the optical theorem, eq. (10.139). Discuss the relation between the results of parts a) and b).

Solution:

a) We square $\epsilon^* \cdot f$ and sum over polarizations using

$$\sum_{pol} \epsilon_0^* \cdot \epsilon^* \cdot \epsilon_0 = \epsilon_0^* \cdot \epsilon_0 - |\hat{k} \cdot \epsilon_0|^2 = 1 - |\hat{k} \cdot \epsilon_0|^2$$

$$\sigma_{sh} = \frac{k^2}{4\pi^2} \int d\Omega (1 - |\hat{k} \cdot \epsilon_0|^2) \int_{sh} d^2x_\perp d^2x'_\perp e^{-i\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp)}$$

In the short wavelength limit $kR \gg 1$, shadow scattering is dominated by the forwards direction $\hat{k} \approx \hat{k}_0$ so $\hat{k} \cdot \epsilon_0 \approx \hat{k}_0 \cdot \epsilon_0 = 0$. Also in the near forward direction $k_\perp = k \sin \theta$ and $dk_\perp = kd\theta$ so we can identify $k^2 \sin \theta d\theta d\varphi = k_\perp dk_\perp d\varphi = d^2k_\perp$. Then

$$\sigma_{sh} \approx \frac{1}{4\pi^2} \int d^2k \int_{sh} d^2x_\perp d^2x'_\perp e^{-i\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp)} = \int_{sh} d^2x_\perp d^2x'_\perp \delta(\mathbf{r}_\perp - \mathbf{r}'_\perp)$$

which is the Projected Area.

b) In the forward direction, including $\epsilon = \epsilon_0$ the shadow scattering amplitude reads (remembering that the forward direction means $k_\perp = 0$)

$$\epsilon_0^* \cdot f(k = k_0) \approx \epsilon_0^* \cdot \epsilon_0 \frac{ik}{2\pi} \int_{sh} d^2x_\perp = \frac{ik}{2\pi} \text{(Projected Area)}$$

Then according to the optical theorem

$$\sigma_{total} = \frac{4\pi}{k} \text{Im} \epsilon_0^* \cdot f(k = k_0) \approx 2(\text{Projected Area}) \approx 2\sigma_{sh}$$

thus the non-shadow part of the total cross section is approximately equal to the shadow part.

Solution: From the properties of Cherenkov radiation the angle of emission is given by 
\[ \cos \theta = \frac{c}{vn} = \frac{c}{(1.5v)}. \]
So we need to relate \( v/c \) to the particle’s kinetic energy:

\[
K = (\gamma - 1)mc^2, \quad \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{K}{mc^2}, \quad \frac{v^2}{c^2} = 1 - \frac{1}{(1 + K/mc^2)^2} \approx \frac{K^2 + 2Kmc^2}{(mc^2 + K)^2}
\]
\[ \cos \theta = \frac{mc^2 + K}{1.5\sqrt{K(K + 2mc^2)}} \quad (5) \]

For the purposes of this problem take the rest energy of the electron to be 0.5 MeV and the
rest energy of a proton to be 1000 MeV. Then we have

\[
\cos \theta = \begin{cases}
\frac{0.5 + K}{1.5\sqrt{K(K + 1)}} & \text{electrons} \\
\frac{1000 + K}{1.5\sqrt{K(K + 2000)}} & \text{protons}
\end{cases} \quad (6)
\]

Therefore a 1 MeV electron emits Cherenkov light at \( \cos \theta \approx 1/\sqrt{2} \) or at \( \theta \approx 45^\circ \). A 500
Mev proton emits light at \( \cos \theta \approx 1000/\sqrt{500 \cdot 2500} \approx 0.894 \) or at \( \theta \approx 27^\circ \). A 5 GeV proton
emits light at \( \cos \theta \approx 6/(1.5\sqrt{35}) \approx 0.676 \) or at \( \theta \approx 47^\circ \).

The energy per unit length emitted in a frequency interval \( d\omega \) is

\[
d\frac{dE}{dx} = |F| = \frac{Q^2}{4\pi\epsilon_0c^2}\omega d\omega (1 - \cos^2 \theta) = \frac{\alpha \hbar \omega}{c} d\omega (1 - \cos^2 \theta). \quad (7)
\]

Each quantum has energy \( \hbar \omega \) so the number of quanta emitted per unit length in \( d\omega \) is

\[
d\frac{dN}{dx} = \frac{\alpha}{c} d\omega (1 - \cos^2 \theta) \\
dN = \frac{\alpha}{c} (\omega_{\text{max}} - \omega_{\text{min}})(1 - \cos^2 \theta) = \frac{2\pi\alpha}{n} \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\lambda_{\text{max}}\lambda_{\text{min}}} = \frac{2\pi\alpha}{36000 \times 10^{-8}\text{cm}} (1 - \cos^2 \theta) \\
= \frac{\pi\alpha \times 10^5}{9\text{cm}} (1 - \cos^2 \theta) \approx 255(1 - \cos^2 \theta)\text{cm}^{-1} \\
= 255(0.5, 0.201, 0.543)\text{cm}^{-1} = (127, 51, 138)\text{cm}^{-1} \quad (8)
\]


Solution: For nonrelativistic motion \( |\beta| \ll 1 \) so the Lienard Wiechert fields lead to the
power angular distribution

\[
d\frac{dP}{d\Omega} \approx \frac{\alpha \hbar}{4\pi} |\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{\beta}})|^2 \quad (9)
\]
a) For \( \mathbf{r} = a \hat{z} \cos \omega t, \dot{\mathbf{\beta}} = -(a \omega_0^2/c) \hat{z} \cos \omega t \) and 

\[
| \mathbf{n} \times (\mathbf{n} \times \hat{z})|^2 = |\mathbf{n} \cdot \hat{z} - \hat{z}|^2 = 1 - (\mathbf{n} \cdot \hat{z})^2 = \sin^2 \theta
\]

\[
\langle \frac{dP}{d\Omega} \rangle \approx \frac{\alpha h a^2 \omega_0^4}{4 \pi 2c^2} \sin^2 \theta
\]

\[
P = \frac{2\pi \alpha h a^2 \omega_0^4}{4 \pi 2c^2} \int_{-1}^{1} dz (1 - z^2) = \frac{\alpha h a^2 \omega_0^4}{3c^2} \frac{4}{3} = \frac{\alpha h}{3c^2} a^2 \omega_0^4
\]

where we used \( \langle \cos^2 \omega_0 t \rangle = 1/2. \)

b) For uniform circular motion in the \( xy \) plane we have

\[
\mathbf{r}(t) = R(\hat{x} \cos \omega_0 t + \hat{y} \sin \omega_0 t), \quad \dot{\mathbf{\beta}} = -\frac{\omega_0^2 R}{c}(\hat{x} \cos \omega_0 t + \hat{y} \sin \omega_0 t)
\]

\[
| \mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{\beta}})|^2 = \frac{\omega_0^4 R^2}{c^2} (1 - (\mathbf{n} \cdot \hat{x} \cos \omega_0 t + \mathbf{n} \cdot \hat{y} \sin \omega_0 t)^2)
\]

\[
= \frac{\omega_0^4 R^2}{c^2} (1 - \sin^2 \theta \cos^2 (\varphi - \omega_0 t))
\]

\[
\langle | \mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{\beta}})|^2 \rangle = \frac{\omega_0^4 R^2}{c^2} \left(1 - \frac{1}{2} \sin^2 \theta \right)
\]

\[
\langle \frac{dP}{d\Omega} \rangle \approx \frac{\alpha h R^2 \omega_0^4}{4 \pi c^2} \left(1 - \frac{1}{2} \sin^2 \theta \right) = \frac{\alpha h R^2 \omega_0^4}{3 \pi 2c^2} (1 + \cos^2 \theta)
\]

\[
P = \frac{2\pi \alpha h R^2 \omega_0^4}{4 \pi 2c^2} \int_{-1}^{1} dz (1 + z^2) = \frac{\alpha h R^2 \omega_0^4}{4c^2} \frac{8}{3} = \frac{2\alpha h}{3c^2} R^2 \omega_0^4
\]

The factor of 2 compared to part a) is explained by thinking of circular motion as the superposition of two orthogonal linear oscillators.

These results are of course very familiar to us from our earlier treatment of radiation by localized sources. We plot the angular distributions \( 1 \pm \cos^2 \theta \) in the following figure.

For part a) the maximum intensity occurs at \( \theta = \pi/2 \) because then the observer sees the maximum oscillation. For part b), thinking of the circular motion as two orthogonal linear oscillations, the maximum is at \( \theta = 0, \pi \), because then the observer sees two oscillators at their maximum. At \( \theta = \pi/2 \) the observer sees only one oscillator, and the intensity matches the maximum of part a).