Quantum Mechanics

Problem Set 6

Due: 5 October 2016

Reading: Shankar, Finish Chapter 2; Lecture notes, Finish Chapter 3, Sections 4.1-4.4

26. Start with the Lagrangian for a relativistic particle moving in an electromagnetic field \( B = \nabla \times \mathbf{A} \), \( E = -\nabla \phi - \dot{\mathbf{A}}/c \),

\[
L = -mc^2\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}} + \frac{Q}{c} \mathbf{\dot{r}} \cdot \mathbf{A} - Q\phi
\]  

a) Obtain the Hamiltonian for this system, making sure to express it as a function of the canonical variables \( \mathbf{r}, \mathbf{p} \).

b) Show that the transformation \( \mathbf{R} = \mathbf{r}, \mathbf{P} = \mathbf{p} + Q\nabla \Lambda(\mathbf{r}, t) \) is canonical by confirming that it leaves Poisson brackets invariant, and then find a generating function for it.

c) Apply this time dependent canonical transformation to this system to calculate the new Hamiltonian. Show that it describes the particle moving in a gauge transformed potential:

\[
\mathbf{A}' = \mathbf{A} + c\nabla \Lambda(\mathbf{r}, t), \quad \phi' = \phi - \frac{\partial \Lambda}{\partial t}
\]

27. a) S, Exercise 2.7.1; b) S, Exercise 2.7.2; c) S, Exercise 2.7.6.

28. The total momentum \( \sum_k \mathbf{p}_k \) and total angular momentum \( \mathbf{L} = \sum_k \mathbf{r}_k \times \mathbf{p}_k \) are important physical properties of any closed system. For the case of a single particle evaluate all of the Poisson brackets of \( p_x, p_y, p_z, L_x, L_y, L_z \) with each other. Start with the Poisson brackets for the canonical variables \( \mathbf{r}, \mathbf{p} \) and exploit the algebraic identities satisfied by the P.B.’s. Use the antisymmetric symbol \( \epsilon_{klm} \) to present your final answers in compact form.

29. Show that each of the following transformations is canonical, and find the appropriate generating functions:

a) Translations

\[
P_i = p_i; \quad Q_i = q_i + a_i
\]

b) Rotations

\[
P_i = \sum_j R_{ij} p_j; \quad Q_i = \sum_j R_{ij} q_j
\]

where \( RR^T = R^T R = I \), i.e. \( R \) is an orthogonal (real unitary) matrix.
30. An important *time dependent* canonical transformation in nonrelativistic mechanics is a Galilei boost: For a single free particle in three dimensions, it is $R = r + Vt$, $P = p + mV$.

a) Find the generating function for this canonical transformation for finite $V$.

b) By taking $V$ small determine the infinitesimal generator $K$ of Galilei transformations.

c) Show that $K$ is a constant of the motion, even though its Poisson bracket with the Hamiltonian $H = \frac{p^2}{2m}$, which you should determine, is not zero.

d) Find the Poisson brackets of the three components $K_k$ with each other and with the components of angular momentum.