Quantum Mechanics

Problem Set 7

Due: 12 October 2016

Reading: Shankar, Chapter 2, Section 5.1; Lecture notes, Chapters 3,4,5.

31. In class we defined canonical transformations \( q, p \to Q, P \) as those which leave the Poisson Brackets invariant

\[
\{f, g\}_{Q,P} = \{f, g\}_{q,p}, \quad \text{Canonical Transform.} \quad (1)
\]

a) Prove from this definition that under a canonical transformation the new coordinates and momenta satisfy

\[
\{Q_k, Q_l\}_{q,p} = 0, \quad \{P_k, P_l\}_{q,p} = 0, \quad \{Q_k, P_l\}_{q,p} = \delta_{kl} \quad (2)
\]

b) Now prove the converse: If the new variables satisfy the P.B. equations of part a), then the P.B. of any pair of dynamical variables \( f(q,p,t), g(q,p,t) \) is invariant under the transformation \( q, p \to Q, P \). To do this write out the P.B. in the original variables and use the chain rule to express derivatives w.r.t. the old variables in terms of derivatives w.r.t. the new variables, and collect terms.

32. In class we discussed generating functions for finite canonical transformations, and we showed that in the limit of infinitesimal transformations they gave

\[
Q_k = q_k + \epsilon \{q_k, G\} + O(\epsilon^2), \quad P_k = p_k + \epsilon \{p_k, G\} + O(\epsilon^2), \quad (3)
\]

where \( G(p,q,t) \) is called the infinitesimal generator. Confirm directly that these definitions imply that \( Q_k, P_k \) satisfy the canonical P.B. relations to first order in \( \epsilon \) for any \( G \). By the previous problem, this proves that the infinitesimal transformation is canonical.

33. a) S, Exercise 2.8.3; b) Exercise 2.8.5

34. In this problem we construct a particular solution of the Hamilton-Jacobi equation from the solution of the equations of motion for a simple harmonic oscillator for which the Hamiltonian is \( H = p^2/2m + m\omega^2q^2/2 \).

a) First find the solution of the equations of motion \( q(t) \) satisfying the initial condition \( q(t_1) = Q \), and the final condition \( q(t_2) = q \).

b) Evaluate the action, \( S \equiv \int_{t_1}^{t_2} dt(m/2)(q^2 - \omega^2q^2) \), by plugging in the solution of part a). Note that since \( q(t) \) is a solution of the equations of motion, by integrating the first term by parts, you can simplify the calculation by first showing that \( S = \int_{t_1}^{t_2} dt(m/2)[q(t_2)\dot{q}(t_2) - q(t_1)\dot{q}(t_1)] \).
c) You will find from part b) that $S$ is a function $S(q, Q, t_2 - t_1)$. Show that the canonical transformation generated by $S(q, Q, t)$ with $q(t) = q, p(t) = p$ and $Q, P$ constant solves the e.o.m. with $q(0) = Q, p(0) = P$.

d) Show by direct substitution that $S(q, Q, t)$ solves the Hamilton-Jacobi equation

$$
\frac{\partial S}{\partial t} = -\frac{1}{2m} \left( \frac{\partial S}{\partial q} \right)^2 - \frac{m\omega^2}{2} q^2
$$

35. (Spreading of a Wave Packet). Consider a particle in a state described by the initial wave function

$$
\langle r | \psi(0) \rangle = \psi(r, 0) = \left( \frac{1}{\pi \Delta_x^2} \right)^{3/4} e^{-r^2/2\Delta_x^2}.
$$

This is a special case of the packet discussed in class where $\Delta_x = \hbar/\Delta$.

a) Calculate the momentum space wave function $\langle p | \psi \rangle$. You will find that it corresponds to the special case $p_0 = 0$ of the motion discussed in class.

b) Using the momentum space wave function obtained above, apply the calculation we did in class to find the coordinate wave function at time $t$, $\langle r | \psi(t) \rangle$.

c) Show that the coordinate probability density as a function of time is proportional to $\exp\{ -r^2/\Delta_x(t)^2 \}$ and find $\Delta_x(t)$.

d) How long would it take for a packet for a particle with the mass of an electron to double in width if its initial width is $10^{-8}$ cm? What if the mass is 1 g and the initial width is $10^{-5}$ cm?