Notes on our Two Loop Sample Calculation in Scalar Field Theory

\[ \mathcal{M} = -i\lambda \left( 1 - \frac{3\lambda}{32\pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) \right) + \frac{i\lambda^2}{32\pi^2} \int_0^1 \! dx \left[ \ln \frac{M^2}{\mu^2 + x(1-x)p_{12}^2} + \text{Perms} \right] \\
- \frac{i\lambda^3}{(32\pi^2)^2} \left\{ 3 \left( \ln \frac{\Lambda^2}{M^2} - 1 \right)^2 + 2 \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) \int_0^1 \! dx \left[ \ln \frac{M^2}{\mu^2 + x(1-x)p_{12}^2} + \text{Perms} \right] \right\} \\
+ \left[ \left( \int_0^1 \! dx \ln \frac{M^2}{\mu^2 + x(1-x)p_{12}^2} \right)^2 + \text{Perms} \right] \right\} \\
- \frac{i\lambda^3}{(32\pi^2)^2} \left\{ 12 \left( \ln \frac{\Lambda^2}{M^2} - 1 \right)^2 + 4 \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) \int_0^1 \! dx \left[ \ln \frac{M^2}{\mu^2 + x(1-x)p_{12}^2} + \text{Perms} \right] \right\} \\
- \frac{i\lambda^3}{(32\pi^2)^2} \int_0^1 \! dx \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{(\mu^2 + (k + p_4)^2)(\mu^2 + (k - p_3)^2)} \ln \frac{M^2}{\mu^2 + x(1-x)k^2} \right] \\
+ \frac{1}{(\mu^2 + (k + p_1)^2)(\mu^2 + (k - p_2)^2)} \ln \frac{M^2}{\mu^2 + x(1-x)k^2} + \text{Perms} \right\} \\
- i\delta_\lambda + \frac{3\lambda^2}{32\pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) + \int_0^1 \! dx \left[ \ln \frac{M^2}{\mu^2 + x(1-x)p_{12}^2} + \text{Perms} \right] \right) \right\} \right\} \\
(1)

The first line gives the tree and one loop diagrams, the second and third lines the double bubble two loop diagrams, the fourth fifth and sixth lines the triangle bubble two loop diagrams and the last line gives the diagrams involving counterterms. Notice the presence of non-polynomial cutoff dependence on lines 2 and 4 due to divergent one-loop sub diagrams of the two loop diagrams.

Cancelling divergences at one loop requires:

\[ \delta_\lambda = \frac{3\lambda^2}{32\pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) + \lambda^3 \xi \]  

(2)

Plugging that result in and dropping terms of \( O(\lambda^4) \) yields

\[ \mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2} \int_0^1 \! dx \left[ \ln \frac{M^2}{\mu^2 + x(1-x)p_{12}^2} + \text{Perms} \right] \\
- \frac{i\lambda^3}{(32\pi^2)^2} \left\{ 15 \left( \ln \frac{\Lambda^2}{M^2} - 1 \right)^2 + \left[ \int_0^1 \! dx \ln \frac{M^2}{\mu^2 + x(1-x)p_{12}^2} \right]^2 + \text{Perms} \right\} \\
- \frac{i\lambda^3}{(32\pi^2)^2} \int_0^1 \! dx \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{(\mu^2 + (k + p_4)^2)(\mu^2 + (k - p_3)^2)} \ln \frac{M^2}{\mu^2 + x(1-x)k^2} \right] \\
+ \frac{1}{(\mu^2 + (k + p_1)^2)(\mu^2 + (k - p_2)^2)} \ln \frac{M^2}{\mu^2 + x(1-x)k^2} + \text{Perms} \right\} \\
+ \frac{18i\lambda^3}{(32\pi^2)^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right)^2 - i\lambda^3 \xi \]  

(3)
Notice the key fact that the non-polynomial divergences have automatically cancelled (with no further thought) after this step. BPHZ show that this always happens: When calculating diagrams up to $n+1$ loop order, substitution of the values of $\delta_Z$, $\delta_\mu$, $\delta_\lambda$ learned up to $n$ loop order automatically removes all of the non-polynomial cutoff dependence in the sum of $n+1$ loop diagrams. Although this is very difficult to prove to all orders, the fact remains that computationally it happens automatically, so there is no need to master the details of the proof to do calculations at any order of perturbation theory!

At this point the only cutoff dependence left is polynomial in the external momenta (for these diagrams just a constant!). It is the job of the $\lambda^3 \xi$ term in $\delta_\lambda$ to remove these last divergences. To determine the cutoff dependence required of $\xi$, we need the cut off dependence of the integral in the third and fourth lines. For this it is enough to set the $p_i = 0$ and evaluate

\[
\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{(\mu^2 + k^2)^2} \ln \frac{M^2}{\mu^2 + x(1-x)k^2} = \frac{1}{8\pi^2} \int_0^1 dx \int_0^A k^3 dk \frac{1}{(\mu^2 + k^2)^2} \ln \frac{M^2}{\mu^2 + x(1-x)k^2} = \frac{1}{16\pi^2} \int_0^1 dx \int_0^A u du \frac{1}{(\mu^2 + u)^2} \ln \frac{M^2}{\mu^2 + x(1-x)u} \\
\sim -\frac{1}{16\pi^2} \int_0^\Lambda du \frac{u}{u} (\ln \frac{u}{M^2} - 2) = -\frac{1}{32\pi^2} (\ln^2 \frac{\Lambda^2}{M^2} - \ln^2 \frac{L^2}{M^2}) + \frac{1}{8\pi^2} \ln \frac{\Lambda^2}{L^2} = -\frac{1}{32\pi^2} \ln^2 \frac{\Lambda^2}{M^2} + \frac{1}{8\pi^2} \ln \frac{\Lambda^2}{M^2} + \text{Finite} \tag{4}
\]

So the remaining cutoff dependence of $M$ is thus

\[
-\frac{i\lambda^3}{(32\pi^2)^2} \left( 15 \left( \ln \frac{\Lambda^2}{M^2} - 1 \right)^2 - 6 \ln^2 \frac{\Lambda^2}{M^2} + 24 \ln \frac{\Lambda^2}{M^2} \right) + \frac{18i\lambda^3}{(32\pi^2)^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right)^2 \\
= -\frac{i\lambda^3}{(32\pi^2)^2} \left( -9 \ln^2 \frac{\Lambda^2}{M^2} + 30 \ln \frac{\Lambda^2}{M^2} - 3 \right) \tag{5}
\]

Thus $\xi$ has the cutoff dependence

\[
\xi = \frac{1}{(32\pi^2)^2} \left( 9 \ln^2 \frac{\Lambda^2}{M^2} - 30 \ln \frac{\Lambda^2}{M^2} + 3 \right) \tag{6}
\]

So to this order $\delta_\lambda$ has the cutoff dependence

\[
\delta_\lambda = \frac{3\lambda^2}{32\pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) + \frac{\lambda^3}{(32\pi^2)^2} \left( 9 \ln^2 \frac{\Lambda^2}{M^2} - 30 \ln \frac{\Lambda^2}{M^2} + 3 \right) \tag{7}
\]

Looking back at the original Lagrangian, we recall that $\lambda + \delta_\lambda = Z^2 \lambda_0$. Thus to two loop order, we have the relation

\[
Z^2 \lambda_0 = \lambda \left[ 1 + \frac{3\lambda}{32\pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) + \frac{\lambda^2}{(32\pi^2)^2} \left( 9 \ln^2 \frac{\Lambda^2}{M^2} - 30 \ln \frac{\Lambda^2}{M^2} + 3 \right) \right] \tag{8}
\]
\[ \frac{1}{Z^2 \lambda_0} = 1 - \frac{3 \lambda}{32 \pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) + \frac{\lambda}{(32\pi^2)^2} \left( 30 \ln \frac{\Lambda^2}{M^2} - 3 \right) + O(\lambda^2) \]  
(9)

which we can rewrite as

\[ \frac{1}{Z^2 \lambda_0} = 1 - \frac{3}{32 \pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) + \frac{\lambda}{(32\pi^2)^2} \left( 30 \ln \frac{\Lambda^2}{M^2} - 3 \right) + O(\lambda^2) \]  
(10)

which can also be inverted as:

\[ \frac{1}{\lambda} = \frac{1}{Z^2 \lambda_0} + \frac{3}{32 \pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) - \frac{\lambda}{(32\pi^2)^2} \left( 30 \ln \frac{\Lambda^2}{M^2} - 3 \right) - O(\lambda^2) \]  
(11)

\[ \frac{1}{\lambda} = \frac{1}{Z^2 \lambda_0} + \frac{3}{32 \pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) - \frac{\lambda_0}{(32\pi^2)^2} \left( 30 \ln \frac{\Lambda^2}{M^2} - 3 \right) - O(\lambda_0^2) \]  
(12)

The replacement of \( \lambda \) by \( \lambda_0 \) in the last line is valid because \( \lambda = \lambda_0 + O(\lambda_0^2) \). We have not evaluated \( Z \) to two loops, but we have seen that \( Z = 1 + O(\lambda_0^2) \) because the one loop correction is pure mass renormalization. Furthermore, there is only a single power of \( \ln \frac{\Lambda^2}{M^2} \) in the two loop correction, so replacing it by 1 in the first term only modifies the coefficients 30 and -3 in the third term:

\[ \frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{3}{32 \pi^2} \left( \ln \frac{\Lambda^2}{M^2} - 1 \right) - \frac{\lambda_0}{(32\pi^2)^2} \left( c_1 \ln \frac{\Lambda^2}{M^2} + c_2 \right) - O(\lambda_0^2) \]  
(13)

As we shall see when we study the renormalization group, the absence of a \( \lambda_0 \ln^2 \frac{\Lambda^2}{M^2} \) on the right side of this equation is a universal feature of renormalization. Indeed the first two terms include all of the “leading log” contributions to charge renormalization. That is all terms \( \lambda_0^n \ln^{n+1} \frac{\Lambda^2}{M^2} \) with \( n > 0 \) are absent from the right side.