Scalar Contribution to the Graviton Self-Energy during Inflation

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ABSTRACT
We use dimensional regularization to evaluate the one loop contribution to the graviton self-energy from a massless, minimally coupled scalar on a locally de Sitter background. For noncoincident points our result agrees with the stress tensor correlators obtained recently by Perez-Nadal, Roura and Verdaguer. We absorb the ultraviolet divergences using the $R^2$ and $C^2$ counterterms first derived by ’t Hooft and Veltman, and we take the $D = 4$ limit of the finite remainder. The renormalized result is expressed as the sum of two transverse, 4th order differential operators acting on nonlocal, de Sitter invariant structure functions. In this form it can be used to quantum-correct the linearized Einstein equations so that one can study how the inflationary production of infrared scalars affects the propagation of dynamical gravitons and the force of gravity.

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1 Introduction

The linearized equations for all known force fields do two things:

- They give the linearized force fields induced by sources; and
- They describe the propagation of dynamical particles which carry the force but are, in principle, independent of any source.

This is the classic distinction between the constrained and unconstrained parts of a force field. In electromagnetism it amounts to the Coulomb potential versus photons. In gravity there is the Newtonian potential, plus its three relativistic partners, versus gravitons.

Quantum corrections to the linearized field equations derive from how the 0-point fluctuations of various fields in whatever background is assumed, respond to the linearized force fields. These quantum corrections do not change the dichotomy between constrained and unconstrained fields but they can, of course, modify classical results. Around flat space background there is no effect, after renormalization, on the propagation of dynamical photons or gravitons but there are small corrections to the Coulomb and Newtonian potentials. As might be expected, the long distance effects are greatest for the 0-point fluctuations of massless particles and they take the form required by perturbation theory and dimensional analysis [1, 2],

\[
\left( \frac{\Delta \Phi}{\Phi} \right)_{\text{Coul.}} \sim - \frac{e^2}{\hbar c} \ln \left( \frac{r}{r_0} \right), \quad \left( \frac{\Delta \Phi}{\Phi} \right)_{\text{Newt.}} \sim - \frac{\hbar G}{c^3 r^2},
\]

where \( r \) is the distance to the source, \( r_0 \) is the point at which the renormalized charge is defined, and the other constants have their usual meanings.

Schrödinger was the first to suggest that the expansion of spacetime can lead to particle production by ripping the virtual particles (which are implicit in 0-point fluctuations) out of the vacuum [3]. Following early work by Imamura [4], the first quantitative results were obtained by Parker [5]. He found that the effect is maximized during accelerated expansion, and for massless particles which are not conformally invariant [6], such as massless, minimally coupled (MMC) scalars and (as noted by Grishchuk [7]) gravitons.

The de Sitter geometry is the most highly accelerated expansion consistent with classical stability. For de Sitter background with Hubble constant \( H \) and scale factor \( a(t) = e^{Ht} \) it is simple to show that the number of MMC...
scalars, or either polarization of graviton, created with wave vector $\vec{k}$ is [8],

$$N(t, \vec{k}) = \left( \frac{Ha(t)}{2c\|\vec{k}\|} \right)^2. \quad (2)$$

It is these particles which comprise the scalar and tensor perturbations produced by inflation [9], the scalar contribution of which has been imaged [10]. Of course the same particles also enter loop diagrams to cause an enormous strengthening of the quantum effects caused by MMC scalars and gravitons. A number of analytic results have been obtained for one loop corrections to the way various particles propagate on de Sitter background and also to how long range forces act:

- In MMC scalar quantum electrodynamics, infrared photons behave as if they had an increasing mass [11], and the charge screening very quickly becomes nonperturbatively strong [12], but there is no big effect on scalars [13];
- For a MMC scalar which is Yukawa-coupled to a massless fermion, infrared fermions behave as if they had an increasing mass [14] but the associated scalars experience no large correction [15];
- For a MMC scalar with a quartic self-interaction, infrared scalars behave as if they had an increasing mass (which persists to two loop order) [16];
- For quantum gravity minimally coupled to a massless fermion, the fermion field strength grows without bound [17]; and
- For quantum gravity plus a MMC scalar, the scalar shows no secular effect but its field strength may acquire a momentum-dependent enhancement [18].

The great omission from this list is how inflationary scalars and gravitons affect gravity, both as regards the propagation of dynamical gravitons and as regards the force of gravity. This paper represents a first step in completing the list.

One includes quantum corrections to the linearized field equation by subtracting the integral of the appropriate one-particle-irreducible (1PI) 2-point function up against the linearized field. For example, a MMC scalar $\varphi(x)$
in a background metric \( g_{\mu\nu}(x) \) whose 1PI 2-point function is \(-iM^2(x;x')\), would have the linearized effective field equation,

\[
\partial_{\mu} \left[ \sqrt{g} g^{\mu\nu} \partial_{\nu} \Phi(x) \right] - \int d^4x' M^2(x;x') \Phi(x') = 0.
\] (3)

To include gravity on the list we must therefore compute the graviton self-energy, either from MMC scalars or from gravitons, and then use it to correct the linearized Einstein equation. In this paper we shall evaluate the contribution from MMC scalars; a subsequent paper will solve the linearized effective field equations to determine quantum corrections to the propagation of gravitons and the gravitational response to a point mass.

It should be noted that the vastly more complicated contribution from gravitons was derived some time ago [19]. However, that result is not renormalized, and is therefore only valid for noncoincident points. To use the graviton self-energy in an effective field equation such as (3), where the integration carries \( x'_{\mu} \) over \( x_{\mu} \), one must extract differential operators until the remaining structure functions are integrable. That is the sort of form we will derive, using dimensional regularization to control the divergences and BPHZ counterterms to subtract them.

This paper contains five sections. In section 2 we give those of the Feynman rules which are needed for this computation, and we describe the geometry of our \( D \)-dimensional, locally de Sitter background. Section 3 derives the relatively simple form for the \( D \)-dimensional graviton self-energy with noncoincident points. We show that this version of the result agrees with the stress tensor correlators recently derived by Perez-Nadal, Roura and Verduguer [20]. Section 4 undertakes the vastly more difficult reorganization which must be done to isolate the local divergences for renormalization. At the end we subtract off the divergences with the same counterterms originally computed for this model in 1974 by ’t Hooft and Veltman [21], and we take the unregulated limit of \( D = 4 \). Our discussion comprises section 5.

## 2 Feynman Rules

In this section we derive Feynman rules for the computation. We start with expressing the full metric as

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu},
\] (4)
where $\bar{g}_{\mu\nu}$ is the background metric, $h_{\mu\nu}$ is the graviton field whose indices are raised and lowered with the background metric, and $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. Expanding the MMC scalar Lagrangian around the background metric we get interaction vertices between the scalar and dynamical gravitons. We take the $D$-dimensional locally de Sitter space as our background and introduce de Sitter invariant bi-tensors which will be used throughout the calculation. We close this section by providing the MMC scalar propagator on the de Sitter background.

2.1 Interaction Vertices

The Lagrangian which describes pure gravity and the interaction between gravitons and the MMC scalar is,

$$\mathcal{L} = \frac{1}{16\pi G} \left[ R - (D-1)(D-2)H^2 \right] - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi g^{\mu\nu} \sqrt{-g} .$$

(5)

where $R$ is Ricci scalar, $G$ is Newton’s constant and $H$ is the Hubble constant.

Computing the one loop scalar contributions to the graviton self-energy consists of summing the 3 Feynman diagrams depicted in Figure 1.

$$-i\langle [\mu \nu \Sigma^{\rho \sigma}] (x; x') \rangle$$

$$= \frac{1}{2} \sum_{I=1}^{2} T_I^{\mu \nu \alpha \beta} (x) \sum_{J=1}^{2} T_J^{\rho \sigma \gamma \delta} (x') \times \partial_\alpha \partial'_\beta \delta_i \Delta (x; x') \times \partial_\gamma \partial'_\delta \delta_i \Delta (x; x')$$

$$+ \frac{1}{2} \sum_{I=1}^{4} F_I^{\mu \nu \rho \sigma \alpha \beta} (x) \times \partial_\alpha \partial'_\beta \delta_i \Delta (x; x') \times \delta^D (x - x')$$

$$+ 2 \sum_{I=1}^{2} C_I^{\mu \nu \rho \sigma} (x) \times \delta^D (x - x') .$$

(6)

Figure 1: Graviton self-energy coupled to the MMC scalars at one loop order.
The 3-point and 4-point vertex factors $T_{I}^{\mu \nu \alpha \beta}$ and $F_{I}^{\mu \nu \rho \sigma \alpha \beta}$ derive from expanding the MMC scalar Lagrangian using (4),

$$-rac{1}{2} \partial_\mu \varphi \partial_\nu \varphi g^{\mu \nu} \sqrt{-g}$$

$$= -\frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi g^{\mu \nu} \sqrt{-g} - \kappa \partial_\mu \varphi \partial_\nu \varphi \left( \frac{1}{2} h g^{\mu \nu} - h^{\mu \nu} \right) \sqrt{-g}$$

$$-\frac{\kappa^2}{2} \partial_\mu \varphi \partial_\nu \varphi \left\{ \left[ \frac{1}{8} h^2 - \frac{1}{4} h^{\rho \sigma} h_{\rho \sigma} \right] g^{\mu \nu} - \frac{1}{2} h h^{\mu \nu} + h^{\mu} h^{\rho} \right\} \sqrt{-g} + O(\kappa^3). \tag{8}$$

The resulting 3-point and 4-point vertex factors are given in the Tables 1 and 2, respectively. The procedure to get the counterterm vertex operators $C_{I}^{\mu \nu \rho \sigma}(x)$ is given in section 4.

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T_{I}^{\mu \nu \alpha \beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{\kappa}{2} \sqrt{-g} g^{\mu \nu} g^{\alpha \beta}$</td>
</tr>
<tr>
<td>2</td>
<td>$+i\kappa \sqrt{-g} g^{\mu (\alpha g^\rho \beta)}$</td>
</tr>
</tbody>
</table>

Table 1: 3-point vertices $T_{I}^{\mu \nu \alpha \beta}$ where $g_{\mu \nu}$ is the de Sitter background metric and $\kappa^2 \equiv 16\pi G$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$F_{I}^{\mu \nu \rho \sigma \alpha \beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{\kappa^2}{4} \sqrt{-g} g^{\mu \nu} g^{\rho \sigma} g^{\alpha \beta}$</td>
</tr>
<tr>
<td>2</td>
<td>$+\frac{\kappa^2}{2} \sqrt{-g} g^{\mu (\rho \alpha} g^{\sigma \beta)}$</td>
</tr>
<tr>
<td>3</td>
<td>$+\frac{\kappa^2}{2} \sqrt{-g} \left[ g^{\mu (\alpha g^\beta \gamma \beta)} g^{\rho \sigma} + g^{\mu \nu} g^{(\alpha g^\rho \sigma)} \right]$</td>
</tr>
<tr>
<td>4</td>
<td>$-2i\kappa^2 \sqrt{-g} g^{\mu (\rho g^\nu (\sigma g^\alpha)}$</td>
</tr>
</tbody>
</table>

Table 2: 4-point vertices $F_{I}^{\mu \nu \rho \sigma \alpha \beta}$ where $g_{\mu \nu}$ is the de Sitter background metric and $\kappa^2 \equiv 16\pi G$

These expressions for interaction vertices are valid for any background metric $g_{\mu \nu}$. In the next subsections we introduce the locally de Sitter space as our background geometry and give the scalar propagator $i \Delta(x; x')$ on it.
2.2 Working on de Sitter Space

We specify our background geometry as the open conformal coordinate sub-

manifold of $D$-dimensional de Sitter space. A spacetime point $x^\mu = (\eta, x^i)$
takes values in the ranges

$$-\infty < \eta < 0 \quad \text{and} \quad -\infty < x^i < +\infty.$$  \hspace{1cm} (9)

In these coordinates the invariant element is,

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = a^2 \eta_{\mu\nu} dx^\mu dx^\nu ,$$  \hspace{1cm} (10)

where $\eta_{\mu\nu}$ is the Lorentz metric and $a = -1/H\eta$ is the scale factor. The
Hubble parameter $H$ is constant for the de Sitter space. So in terms of $\eta_{\mu\nu}$
and $a$ our background metric is

$$g_{\mu\nu} \equiv a^2 \eta_{\mu\nu} .$$  \hspace{1cm} (11)

De Sitter space has the maximum number of space-time symmetries in a
given dimension. For our $D$-dimensional conformal coordinates the $\frac{1}{2}D(D+1)$
de Sitter transformations can be decomposed as follows:

- Spatial transformations - $(D - 1)$ transformations.

$$\eta' = \eta , \quad x'^i = x^i + \epsilon^i .$$  \hspace{1cm} (12)

- Rotations - $\frac{1}{2}(D - 1)(D - 2)$ transformations.

$$\eta' = \eta , \quad x'^i = R^{ij} x^j .$$  \hspace{1cm} (13)

- Dilation - 1 transformation.

$$\eta' = k\eta , \quad x'^i = k x^i .$$  \hspace{1cm} (14)

- Spatial special conformal transformations - $(D - 1)$ transformations.

$$\eta' = \frac{\eta}{1 - 2\theta \cdot \bar{x} + \|\theta\|^2 x \cdot x} , \quad x' = \frac{x^i - \theta^i x \cdot x}{1 - 2\theta \cdot \bar{x} + \|\theta\|^2 x \cdot x} .$$  \hspace{1cm} (15)
It turns out that the MMC scalar contribution to the graviton self-energy is de Sitter invariant. This suggests to express it in terms of the de Sitter length function $y(x; x')$,

$$y(x; x') \equiv a a' H^2 \left[ \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| - i \epsilon)^2 \right]. \quad (16)$$

Except for the factor of $i \epsilon$ (whose purpose is to enforce Feynman boundary conditions) the function $y(x; x')$ is closely related to the invariant length $\ell(x; x')$ from $x^\mu$ to $x'^\mu$,

$$y(x; x') = 4 \sin^2 \left( \frac{1}{2} H \ell(x; x') \right). \quad (17)$$

With this de Sitter invariant quantity $y(x; x')$, we can form a convenient basis of de Sitter invariant bi-tensors. Note that because $y(x; x')$ is de Sitter invariant, so too are covariant derivatives of it. With the metrics $\eta_{\mu\nu}(x)$ and $\eta_{\mu\nu}(x')$, the first three derivatives of $y(x; x')$ furnish a convenient basis of de Sitter invariant bi-tensors [13],

$$\frac{\partial y(x; x')}{\partial x^\mu} = H a \left( y \delta^0_\mu + 2 a' H \Delta x_\mu \right), \quad (18)$$

$$\frac{\partial y(x; x')}{\partial x'^\nu} = H a' \left( y \delta^0_\nu - 2 a H \Delta x_\nu \right), \quad (19)$$

$$\frac{\partial^2 y(x; x')}{\partial x^\mu \partial x'^\nu} = H^2 a \left( y \delta^0_\mu \delta^0_\nu + 2 a' H \Delta x_\mu \delta^0_\nu - 2 a \delta^0_\mu H \Delta x_\nu - 2 \eta_{\mu\nu} \right). \quad (20)$$

Here and subsequently $\Delta x_\mu \equiv \eta_{\mu\nu} (x - x')^\nu$.

Acting covariant derivatives generates more basis tensors, for example [13],

$$\frac{D^2 y(x; x')}{D x^\mu D x'^\nu} = H^2 (2 - y) \eta_{\mu\nu}(x), \quad (21)$$

$$\frac{D^2 y(x; x')}{D x'^\mu D x'^\nu} = H^2 (2 - y) \eta_{\mu\nu}(x'). \quad (22)$$

The contraction of any pair of the basis tensors also produces more basis tensors [13],

$$\eta^{\mu\nu}(x) \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x'^\nu} = H^2 \left( 4y - y^2 \right) = \eta^{\mu\nu}(x') \frac{\partial y}{\partial x'^\mu} \frac{\partial y}{\partial x'^\nu}, \quad (23)$$

7
\( g^{\mu
u}(x) \frac{\partial y}{\partial x^\nu} \frac{\partial^2 y}{\partial x^\mu \partial x^\alpha} = H^2(2 - y) \frac{\partial y}{\partial x^\alpha}, \) \hspace{1cm} (24)

\( g^{\mu
u}(x') \frac{\partial y}{\partial x'^\mu} \frac{\partial^2 y}{\partial x'^\nu \partial x'\sigma} = H^2(2 - y) \frac{\partial y}{\partial x'^\nu}, \) \hspace{1cm} (25)

\( g^{\mu
u}(x) \frac{\partial^2 y}{\partial x^\nu \partial x^\rho} \frac{\partial^2 y}{\partial x^\mu \partial x^\sigma} = 4H^4g^{\rho\sigma}(x') - H^2 \frac{\partial y}{\partial x^\nu} \frac{\partial y}{\partial x^\sigma}, \) \hspace{1cm} (26)

\[ g^{\rho\sigma}(x') \frac{\partial^2 y}{\partial x^\rho \partial x'^\nu} \frac{\partial^2 y}{\partial x^\sigma \partial x'^\mu} = 4H^4g^{\mu\nu}(x) - H^2 \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu}. \] \hspace{1cm} (27)

### 2.3 Scalar Propagator on de Sitter

From the MMC scalar Lagrangian of (5) we see that the propagator obeys

\[ \partial_\mu \left[ \sqrt{-g} g^{\mu\nu} \partial_\nu \right] i\triangle(x; x') = \sqrt{-g} i\triangle(x; x') = i\delta^D(x - x') \] \hspace{1cm} (28)

The scalar propagator on de Sitter is expressed in terms of the de Sitter length function \( y(x; x') \) [22],

\[ i\triangle(x; x') = A(y(x; x')) + k \ln(aa'). \] \hspace{1cm} (29)

Here the constant \( k \) is given as,

\[ k \equiv \frac{H^{D-2}}{(4\pi)^{D\over 2}} \frac{\Gamma(D-1)}{\Gamma(D\over 2)}, \] \hspace{1cm} (30)

and the function \( A(y) \) has the expansion,

\[ A(y) = \frac{H^{D-2}}{(4\pi)^{D\over 2}} \left\{ \frac{\Gamma(D\over 2)}{y} \frac{4}{D-1} - \frac{\Gamma(D\over 2 + 1)}{y} - \frac{\pi}{2} \cot \left( \frac{\pi D}{2} \right) \frac{\Gamma(D-1)}{\Gamma(D\over 2)} \right\} + \sum_{n=1}^{\infty} \left[ \frac{1}{n} \frac{\Gamma(n+D-1)}{\Gamma(n+D\over 2)} \left( \frac{y}{4} \right)^n - \frac{1}{n-D\over 2} + \frac{\Gamma(n+D+1)}{\Gamma(n+2)} \left( \frac{y}{4} \right)^{n-D\over 2 + 2} \right]. \] \hspace{1cm} (31)

The infinite series terms of \( A(y) \) vanish for \( D = 4 \), so they only need to be retained when multiplying a potentially divergent quantity, and even then one only needs to include a handful of them. This makes loop computations manageable.

We note that the MMC scalar propagator (29) has the de Sitter breaking term, \( k \ln(aa') \). However the graviton self-energy only involves the terms like \( \partial_\alpha \partial_\beta i\triangle(x; x') \). Being differentiated twice, once with respect to \( x \) and then with respect to \( x' \), the de Sitter breaking term drops out. So the scalar contribution to the graviton self-energy remains de Sitter invariant.
3 One Loop Graviton Self-energy

In this section we calculate the primitive diagrams. It turns out the graviton self-energy with coincident point vanishes in $D = 4$ dimensions. We reach the relatively simple form for the $D$-dimensional graviton self-energy with noncoincident points. We check that this result agrees with the stress tensor correlator recently derived by Perez-Nadal, Roura and Verdaguer [20].

3.1 No Contributions from the 4-point Vertices

The 4-point contributions,

$$\sum_{I=1}^{4} F^{\mu\nu\rho\sigma}_{I}(x) \times \partial_\alpha \partial'_\beta i \Delta(x;x') \times \delta^{D}(x-x'), \quad (32)$$

vanish on de Sitter space. Because the sum contains a factor of $(D-4)$ and there is no divergence from the coincidence limit of the propagator in dimensional regularization, the 4-point diagrams make no net contribution to the final, renormalized result.

To check this, we calculate the 4-point contributions explicitly. In the coincidence limit $x' = x$ we have

$$\lim_{x' \to x} \frac{\partial y(x;x')}{\partial x^\mu} = \lim_{x' \to x} H a(y\delta^0_\mu + 2a' H \Delta x_\mu) = 0, \quad (33)$$

$$\lim_{x' \to x} \frac{\partial y(x;x')}{\partial x'^\nu} = \lim_{x' \to x} H a'(y\delta^0_\nu - 2a H \Delta x_\nu) = 0, \quad (34)$$

$$\lim_{x' \to x} \frac{\partial^2 y(x;x')}{\partial x^\mu \partial x'^\nu} = \lim_{x' \to x} H^2 a a'(y\delta^0_\mu \delta^0_\nu + 2a' H \Delta x_\mu \delta^0_\nu - 2a \delta^0_\mu \Delta x_\nu - 2\eta_{\mu\nu}) = -2H^2 g_{\mu\nu}. \quad (35)$$

Therefore,

$$\lim_{x' \to x} \partial_\mu \partial'_\nu i \Delta(x;x') = \lim_{x' \to x} \left\{ A'' \frac{\partial y}{\partial x^\nu} \frac{\partial y}{\partial x'^\mu} + A' \frac{\partial^2 y}{\partial x^\mu \partial x'^\nu} \right\}$$

$$= 0 + A'(y = 0) \times \left[ -2H^2 g_{\mu\nu} \right] \quad (36)$$

From the definition of $A(y)$, $A'(y)$ is

$$A'(y) = \frac{1}{4} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ -\Gamma\left(\frac{D}{2}\right)\left(\frac{4}{y}\right)^\frac{D}{2} - \Gamma\left(\frac{D}{2}+1\right)\left(\frac{4}{y}\right)^{\frac{D}{2}-1} \right. \right.$$

$$+ \frac{1}{2} \Gamma\left(\frac{D}{2}+2\right)\left(\frac{4}{y}\right)^{\frac{D}{2}-2} + \cdots - 0 + \frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)} \right\}, \quad (37)$$
Now we recall that in dimensional regularization any $D$-dependent power of 
vanishes. Then we see that when $y = 0$, the first three terms of this (37) 
vanish in the dimensional regularization scheme. Thus,

$$A'(y = 0) = \frac{1}{4} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{\Gamma(\frac{D}{2} + 1)},$$

and therefore we have

$$\partial_\mu \partial'_\nu i \triangle(x; x') \delta^D(x - x') = \lim_{x' \to x} \partial_\mu \partial'_\nu i \triangle(x; x') = -\frac{1}{2} \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{\Gamma(\frac{D}{2} + 1)} \eta_{\mu\nu}.\quad (39)$$

So the 4-point contributions become

$$\sum_{I=1}^{4} F^I_{\mu
u\rho\sigma \alpha\beta}(x) \times \partial_\alpha \partial'_{\gamma} i \triangle(x; x') \times \delta^D(x - x')$$

$$= -\frac{1}{2} \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{\Gamma(\frac{D}{2} + 1)} \eta_{\alpha\beta} \times i\kappa^2 \sqrt{-g} \left\{- \frac{1}{4} g^{[\rho \sigma} g^{\mu \nu]} g^{[\alpha \beta]} + \frac{1}{2} g^{(\rho \sigma)(\mu \nu)} g^{[\alpha \beta]} \\
+ \frac{1}{2} \left[ g^{(\alpha \beta)(\mu \nu)} g^{\rho \sigma} + g^{(\mu \nu)} g^{(\rho \sigma)(\alpha \beta)} - 2 g^{(\rho \sigma)(\mu \nu)} g^{(\alpha \beta)} \right] \right\}$$

$$= -\frac{1}{2} \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{\Gamma(\frac{D}{2} + 1)} i\kappa^2 \sqrt{-g} \left\{- \frac{1}{4} g^{[\rho \sigma} g^{\mu \nu]} + \frac{1}{4} g^{(\rho \sigma)[\mu \nu]} \right\} (D - 4). \quad (40)$$

We explicitly see that with no divergence factors multiplied by $(D - 4)$, this 
becomes zero in $D = 4$.

### 3.2 Contributions from the 3 point vertices

In this section we divide our discussion into two subsections: first for the 
graviton self-energy in terms of $A(y)$ and second for the decomposition of it 
into two parts.

#### 3.2.1 3-point contributions in terms of $A(y)$

In this section we calculate the graviton self-energy with noncoincident points,

$$-g^{\mu \rho} T_{\mu \nu \rho \sigma \alpha \beta}(x; x') \times \partial_{\alpha \partial'_{\gamma} i \triangle(x; x') \times \partial_\beta \partial'_{\delta} i \triangle(x; x'). \quad (41)$$
Using the de Sitter background MMC scalar propagator (29), we can express the 3-point contributions in terms of the de Sitter invariant function $A(y)$ given in equation (31). The key computation is,

$$\partial'_i i \triangle(x; x') = A' \frac{\partial y}{\partial x'^\gamma} + kH \alpha'd_y,$$

$$\partial_\alpha \partial'_{\gamma} i \triangle(x; x') = A'' \frac{\partial y}{\partial x'^\alpha} \frac{\partial y}{\partial x'^\gamma} + A' \frac{\partial^2 y}{\partial x^\alpha \partial x'^\gamma} + 0.$$ (42)

$$\partial_\alpha \partial'_{\gamma} i \triangle(x; x') = A'' \frac{\partial y}{\partial x'^\alpha} \frac{\partial y}{\partial x'^\gamma} + A' \frac{\partial^2 y}{\partial x^\alpha \partial x'^\gamma} + 0.$$ (43)

We note that the de Sitter breaking term drops out being differentiated twice and therefore the graviton self-energy turns out de Sitter invariant.

Multiplying by one more propagator with derivatives and contracting with the vertex factors we can arrange the 3-point contribution in terms of the five basis tensors given in section 2.2:

$$-i [\mu \nu \Sigma^{\rho \sigma}]_{3\text{-point}}(x; x') = \sqrt{-g} \sqrt{-g'} \left\{ \frac{\partial^2 y}{\partial x_\mu \partial x'_{(\rho}} \frac{\partial^2 y}{\partial x'_{\sigma)}} \partial_\nu \gamma \alpha(\gamma) 
+ \frac{\partial y}{\partial x_{(\mu}} \frac{\partial y}{\partial x'_{\rho}} \frac{\partial y}{\partial x'_{\sigma)}} \partial_\nu \gamma \beta(\gamma) 
+ \frac{\partial y}{\partial x_{(\mu}} \frac{\partial y}{\partial x'_{\rho}} \frac{\partial y}{\partial x'_{\sigma)}} \partial_\nu \gamma \gamma(\gamma) 
+ \sqrt{-g'} g'^{\rho \sigma} H^4 \partial_\nu \gamma \delta(\gamma) 
+ [g''^{\mu \nu} \partial_\nu \gamma \partial_\rho \partial_\sigma + \partial_\nu \gamma \partial_\rho \partial_\sigma g'^{\mu \nu}] H^2 \partial_\nu \gamma \epsilon(\gamma) \right\}. \quad \text{(44)}$$

Here the coefficients are functions of $A(y)$:

$$\alpha(\gamma) = \kappa^2 \left[ -\frac{1}{2} (A')^2 \right], \quad \text{(45)}$$

$$\beta(\gamma) = \kappa^2 \left[ -A'A'' \right], \quad \text{(46)}$$

$$\gamma(\gamma) = \kappa^2 \left[ -\frac{1}{2} (A'')^2 \right], \quad \text{(47)}$$

$$\delta(\gamma) = \kappa^2 \left[ -\frac{1}{8} \right] \left[ (A')^2(4y - y^2)^2 + 2A'A''(2y)(4y - y^2) 
+ (A')^2[4(D - 4) - (4y - y^2)] \right], \quad \text{(48)}$$

$$\epsilon(\gamma) = \kappa^2 \left[ \frac{1}{4} \right] \left[ (4y - y^2)(A'')^2 + 2(2 - y)A'A'' - (A')^2 \right]. \quad \text{(49)}$$
We can check this result agrees with the stress tensor correlators recently derived by Perez-Nadal, Roura and Verdaguer [20]. We compare our result for the MMC scalar contribution to the graviton self-energy, (44) with the stress tensor correlators $F_{\mu\nu\rho\sigma}$ in the massless limit which is from [20],

$$F_{\mu\nu\rho\sigma} = \partial_\mu \partial'_\nu G^+ \partial_\sigma \partial'_\sigma G^+ + \partial_\mu \partial'_\sigma G^+ \partial_\nu \partial'_\rho G^+ - \eta_{\mu\nu} \partial_\sigma \partial'_\rho G^+ \partial_\sigma \partial'_\rho G^+ - \eta_{\rho\sigma} \partial_\mu \partial'_\nu G^+ \partial_\sigma \partial'_\rho G^+ + \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \partial_\sigma \partial'_\nu G^+ \partial_\rho \partial'_\rho G^+ .$$

(50)

For comparison we first note the conversion of the essential variables. Our biscalar function $y(x; x')$ corresponds to $Z(x; x')$ with the relation,

$$Z = 1 - \frac{y}{2} .$$

(51)

In the massless limit, the function $G^+$ in equation (50) becomes

$$G^+ = G(Z) = \frac{1}{(4\pi)^{D/2}} \sum_{n=0}^{\infty} \frac{\Gamma(D-1+n)}{\Gamma(D/2+n)} \frac{1}{n!} \left(1 + \frac{Z}{2}\right)^n .$$

(52)

Recalling the hypergeometric function,

$$\, _2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} \frac{z^n}{n!} ,$$

(53)

we see that $G(Z)$ can be written as

$$G(Z) = G(y) = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(D/2)} \frac{\Gamma(0)}{\Gamma(D/2)} \, _2F_1(D-1, 0; D/2; 1 - \frac{y}{4}) .$$

(54)

Now we use one of the transformation formulas for the hypergeometric functions (See for example, 9.131 of Gradshteyn and Ryzhik [32]) to write $G$ as the series expansion of $y/4$

$$G(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \left\{ \frac{\Gamma(D/2)}{D/2 - 1} \frac{4}{y} y^{D/2 - 1} + \frac{\Gamma(D/2 + 1)}{D/2 - 2} \left(4 - y\right)^{D/2 - 2} - \Gamma(0) \frac{\Gamma(D-1)}{\Gamma(D/2)} \right. + \left. \sum_{n=1}^{\infty} \frac{\Gamma(n+D-1)}{n} \frac{y^n}{\Gamma(n+D)} \left(4 - y\right)^{n+D/2 - 2} \right\} .$$

(55)
Here we recover $H$ for which they used $H = 1$ unit. We see that this is almost the same as the function $A(y)$ except the singular term

$$-\Gamma(0) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})},$$

which we regulated using the de Sitter breaking term as

$$-\pi \cot\left(\frac{\pi D}{2}\right) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}. $$

Noting that the stress tensor correlator (50) only involves derivatives of $G(y)$ this constant term does not effect the answer. Then we can check that the stress tensor correlator (50) agrees with our result for the MMC scalar contributions to the graviton self-energy:

$$F_{\mu\nu\rho\sigma} = \partial_\mu \partial'_\nu A(y) \partial_\rho \partial'_\sigma A(y) + \partial_\mu \partial'_\sigma A(y) \partial_\nu \partial'_\rho A(y) - \frac{1}{2} g_{\mu\nu} g_{\rho\sigma} \partial_\alpha \partial'_\alpha A(y) \partial_\alpha \partial'_\alpha A(y),$$

$$= -\frac{4}{\kappa^2} \times \sqrt{-\bar{g}} \sqrt{-\bar{g}'} \times -i\mu\nu^{\rho\sigma}_{\text{3-point}}(x; x').$$

### 3.2.2 Separation into spin 0 and spin 2 parts

Now provided by the gauge symmetry of our Lagrangian (5), it is convenient to decompose the graviton self-energy into spin 0 and spin 2 parts:

$$-i\mu\nu^{\rho\sigma}_{\text{3-point}}(x; x') = \sqrt{-\bar{g}} \sqrt{-\bar{g}'} \left\{ \begin{array}{l} \text{[transverse, spin 0 operator]} F_1(y) \\ \text{[transverse, traceless, spin 2 operator]} F_2(y) \end{array} \right\}. $$

Here the structure functions $F_1(y)$ and $F_2(y)$ are scalar functions of $y$. Actually they are arbitrary scalar functions, but because the result for the graviton self-energy is de Sitter invariant we can assume the scalar is a function of the de Sitter invariant length, $y$.

We can guess, from the flat space case which is summarized in section 4, the de Sitter generalization of the spin 0 operator as

$$[\text{spin 0 operator}]$$

$$= [\Box g_{\mu\nu} - D_\mu D_\nu + (D-1)H^2g_{\mu\nu}, [\Box g'_{\rho\sigma} - D'_\rho D'_\sigma + (D-1)H^2g'_{\rho\sigma}].$$
The question arises as to what the de Sitter generalization of the spin 2 operator is. It is hard to guess because it involves mixed index groups: $\mu\nu$ is an index group associated to $x$ and $\rho\sigma$ is another index group associated to $x'$, and unlike $\eta_{\mu\nu}$ which is the same everywhere, $\varphi_{\mu\nu}(x) \neq \varphi_{\mu\nu}(x')$. So our strategy is to construct a transverse, traceless spin 2 operator using the Weyl tensor. We recall that the Weyl tensor is traceless. It is not transverse in general but in the linearized order which we actually consider, it is transverse. Varying the linearized Weyl tensor with respect to the graviton field $h_{\mu\nu}$ we can form a transverse, traceless differential operator,

$$[\text{spin 2 operator}] = \frac{\partial \delta C^{\alpha\beta\gamma\delta}(x)}{\partial h_{\rho\sigma}} \left[ \frac{\partial^2 y}{\partial x^\rho \partial x'^\sigma} \frac{\partial^2 y}{\partial x^\beta \partial x'^\gamma} \frac{\partial^2 y}{\partial x^\gamma \partial x'^\rho} \frac{\partial^2 y}{\partial x^\delta \partial x'^\beta} \right].$$

(62)

Here the linearized Weyl tensor, $\delta C^{\alpha\beta\gamma\delta}(x)$ is given as

$$\delta C^{\alpha\beta\gamma\delta}(x) = -\frac{1}{2} \left[ D_\gamma D_\beta h_{\alpha\delta} - D_\gamma D_\alpha h_{\delta\beta} - D_\delta D_\beta h_{\alpha\gamma} + D_\delta D_\alpha h_{\gamma\beta} \right] - \frac{1}{2(D-2)} \left[ -g_{\gamma\beta} [D^\lambda D_\delta h_{\lambda\alpha} + D_\alpha D^\lambda h_{\lambda\delta} - \Box h_{\delta\alpha} - D_\delta D_\alpha h] 
+ g_{\alpha\delta} [D^\lambda D_\gamma h_{\lambda\beta} + D_\beta D^\lambda h_{\lambda\gamma} - \Box h_{\gamma\beta} - D_\gamma D_\beta h] - g_{\alpha\gamma} [D^\lambda D_\beta h_{\lambda\delta} + D_\delta D^\lambda h_{\lambda\beta} - \Box h_{\delta\beta} - D_\delta D_\beta h] \right]$$

$$- \frac{1}{(D-2)(D-1)} [\varphi_{\alpha\gamma} \varphi_{\beta\delta} - \varphi_{\alpha\delta} \varphi_{\beta\gamma}] [\Box h - D^\tau D^\tau h_{\tau\omega}],$$

(63)

and we get $\delta C^{\alpha'\beta'\gamma'\delta'}(x')$ by replacing $\alpha, \beta, \gamma, \delta \rightarrow \alpha'\beta'\gamma'\delta'$ respectively and calculating all $h$'s and $\varphi$'s at $x'$. Further we can act $\frac{\partial}{\partial h_{\mu\nu}}$ on $\delta C^{\alpha\beta\gamma\delta}(x)$ to get

$$\frac{\partial}{\partial h_{\mu\nu}} \delta C^{\alpha\beta\gamma\delta}(x)$$

$$= 1) \frac{1}{2} \left[ \delta^{(\mu}_{\alpha} \delta^{\nu)}_{\delta} D_\gamma D_\beta - \delta^{(\mu}_{\delta} \delta^{\nu)}_{\beta} D_\gamma D_\alpha - \delta^{(\mu}_{\alpha} \delta^{\nu)}_{\gamma} D_\delta D_\beta + \delta^{(\mu}_{\delta} \delta^{\nu)}_{\gamma} D_\delta D_\alpha \right]$$

$$2) - \frac{1}{2(D-2)} \left[ -g_{\alpha\delta} [\delta^{(\mu}_{\gamma} D^{\nu)} D_\beta - \delta^{(\mu}_{\beta} D^{\nu)} D_\gamma D_\delta + \delta^{(\mu}_{\beta} \delta_{\gamma} D^{\nu)} D_\delta D_\alpha] \right]$$

14
\[ +\mathcal{F}_{\bar{\beta}\bar{\delta}}[\delta_{\gamma}^{(\mu D^{\nu})} D_{\alpha} - \delta_{\alpha}^{(\mu D^{\nu})} \Box - \mathcal{F}^{\mu\nu} D_{\gamma} D_{\alpha}] + \mathcal{F}_{\alpha\gamma}[\delta_{\delta}^{(\mu D^{\nu})} D_{\beta} - \delta_{\beta}^{(\mu D^{\nu})} \Box - \mathcal{F}^{\mu\nu} D_{\delta} D_{\beta}] - \mathcal{F}_{\beta\gamma}[\delta_{\gamma}^{(\mu D^{\nu})} D_{\alpha} - \delta_{\alpha}^{(\mu D^{\nu})} \Box - \mathcal{F}^{\mu\nu} D_{\gamma} D_{\alpha}] \]

\[ \mathcal{F}^{\mu\nu} \equiv \frac{1}{(D-2)(D-1)}[\mathcal{F}_{\alpha\gamma}\mathcal{F}_{\bar{\beta}\bar{\delta}} - \mathcal{F}_{\alpha\bar{\delta}}\mathcal{F}_{\bar{\beta}\gamma}] [\mathcal{D}^{(\mu D^{\nu})} - \mathcal{F}^{\mu\nu} \Box]. \] (64)

To write this simpler let us define \( \mathcal{D}_{\alpha\beta\gamma\delta}(x) \) as

\[ \mathcal{D}_{\alpha\beta\gamma\delta}(x) \equiv \frac{1}{2} \left[ \delta_{\alpha\gamma}^{(\mu D^{\nu})} D_{\beta} - \delta_{\beta\delta}^{(\mu D^{\nu})} D_{\alpha} - \delta_{\delta\gamma}^{(\mu D^{\nu})} D_{\alpha} - \delta_{\alpha\gamma}^{(\mu D^{\nu})} D_{\beta} + \delta_{\beta\delta}^{(\mu D^{\nu})} D_{\alpha} \right]. \] (65)

Then the terms of the first line in 2) become

\[ \mathcal{D}_{\alpha\beta\gamma\delta}(x) \equiv g^{\kappa\lambda} \mathcal{D}_{\kappa\lambda\gamma\delta}(x) = D^{(\mu D^{\nu})} - \mathcal{F}^{\mu\nu} \Box. \] (66)

and the terms in 3) become

\[ \mathcal{D}_{\alpha\beta\gamma\delta}(x) \equiv g^{\kappa\lambda} \mathcal{D}_{\kappa\lambda\gamma\delta}(x) = D^{(\mu D^{\nu})} - \mathcal{F}^{\mu\nu} \Box. \] (67)

We also write

\[ \frac{\partial^2 y}{\partial x_{\alpha} \partial x'_{\alpha'}} \equiv T^{\alpha\alpha'}. \] (68)

Then we can write the spin 2 part a little simpler:

\[ [\text{spin 2 operator}] F_2(y) = \left\{ \mathcal{D}_{\alpha\beta\gamma\delta} - \frac{1}{(D-2)} \left[ -\mathcal{F}_{\alpha\delta} \mathcal{D}_{\beta\gamma} + \mathcal{F}_{\beta\delta} \mathcal{D}_{\alpha\gamma} - \mathcal{F}_{\alpha\gamma} \mathcal{D}_{\beta\delta} + \mathcal{F}_{\beta\gamma} \mathcal{D}_{\alpha\delta} \right] + \frac{1}{(D-2)(D-1)} [\mathcal{F}_{\alpha\gamma}\mathcal{F}_{\beta\delta} - \mathcal{F}_{\alpha\delta}\mathcal{F}_{\beta\gamma}] [\mathcal{D}^{\mu\nu}] \right\} \]

\[ \left\{ \mathcal{D}_{\alpha'\beta'\gamma'\delta'} - \frac{1}{(D-2)} \left[ -\mathcal{F}_{\alpha'\delta'} \mathcal{D}_{\beta'\gamma'} + \mathcal{F}_{\beta'\delta'} \mathcal{D}_{\alpha'\gamma'} - \mathcal{F}_{\alpha'\gamma'} \mathcal{D}_{\beta'\delta'} + \mathcal{F}_{\beta'\gamma'} \mathcal{D}_{\alpha'\delta'} \right] + \frac{1}{(D-2)(D-1)} [\mathcal{F}_{\alpha'\gamma'}\mathcal{F}_{\beta'\delta'} - \mathcal{F}_{\alpha'\delta'}\mathcal{F}_{\beta'\gamma'}] [\mathcal{D}^{\mu\nu}] \right\} \]

\[ T^{\alpha\alpha'} T^{\beta\beta'} T^{\gamma\gamma'} T^{\delta\delta'} F_2(y) \]. (69)
With the graviton self-energy separated into two pieces

\[-i [\mu \nu \Sigma_{\rho \sigma}]_{3\text{-point}}(x; x') = \sqrt{-g} \sqrt{-g'} \left\{ [\text{Spin 0 Operator}] F_1(y) + [\text{Spin 2 Operator}] F_2(y) \right\}, \quad (70)\]

the next step is to find the structure functions \( F_1(y) \) and \( F_2(y) \) by comparing (70) with the original expression, (44). In order to compare them, we need to arrange (70) in terms of the five basis tensors. To accomplish that we have to act the spin 0 and spin 2 operators on \( F_1 \) and \( F_2 \), respectively. Acting the spin 0 operator on \( F_1 \) gives,

\[
[\text{spin 0 operator}] F_1(y) = \frac{\partial^2 y}{\partial x^\mu \partial x'(\rho \partial x^\sigma \partial x^\nu)} \left[ \alpha_1(F_1) \right] \\
+ \frac{\partial y}{\partial x^\mu \partial x'(\rho \partial x^\sigma \partial x^\nu)} \left[ \beta_1(F_1) \right] \\
+ \frac{\partial y}{\partial x^\mu \partial x'(\rho \partial x^\nu \partial x^\sigma \partial x^\nu)} \left[ \gamma_1(F_1) \right] \\
+ g_{\mu \nu} g'_{\rho \sigma} H^4 [\delta_1(F_1)] \\
+ \left[ g_{\mu \nu} \frac{\partial y}{\partial x'^\rho \partial x^\sigma} + \frac{\partial y}{\partial x^\nu \partial x'^\rho \partial x^\sigma} \right] H^2 [\epsilon_1(F_1)] . \quad (71)
\]

Here the coefficients are

\[
\alpha_1(F_1) = 2 F''_1, \quad \beta_1(F_1) = 4 F'''_1, \quad \gamma_1(F_1) = F''''_1, \quad (72) \quad (73) \quad (74)
\]

\[
\delta_1(F_1) = (4y - y^2)^2 F'''_1 + 2(D+1)(2-y)(4y-y^2)F''''_1 - 4(4y-y^2)F''_1' + (D^2-3)(2-y)^2 F_1'' + (D-1)^2 (2-y) F_1'' + (D-1)^2 F_1 , \quad (75)
\]

\[
\epsilon_1(F_1) = -(4y-y^2) F'''_1'' - (D+3)(2-y) F''''_1 + (D+1) F''''_1 . \quad (76)
\]

Acting the spin 2 operator on \( F_2 \) gives,

\[
[\text{spin 2 operator}] F_2(y) = \frac{\partial^2 y}{\partial x^\mu \partial x'(\rho \partial x^\sigma \partial x^\nu)} \left[ \alpha_2(F_2) \right] \\
+ \frac{\partial y}{\partial x^\mu \partial x'(\rho \partial x^\sigma \partial x^\nu)} \left[ \beta_2(F_2) \right] \\
+ \frac{\partial y}{\partial x^\mu \partial x'(\rho \partial x^\nu \partial x^\sigma \partial x^\nu)} \left[ \gamma_2(F_2) \right]
\]
\[ + \bar{g}_{\nu \rho} \frac{\partial y}{\partial x^\rho} H^4 \left[ \frac{\partial y}{\partial x^\nu} \right] \]
\[ + \left[ g_{\mu \nu} \frac{\partial y}{\partial x^\rho} \frac{\partial y}{\partial x^\sigma} + \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu} \bar{g}_{\rho \sigma} \right] H^2 \left[ \epsilon_2 (F_2) \right]. \quad (77) \]

Now we remind ourselves that the construction of the spin 2 operator using Weyl tensor is not unique. It can differ by constants. So the coefficients \( \alpha_2, \beta_2, \gamma_2, \delta_2, \epsilon_2 \) multiplied by a constant can also be a good set of the coefficients. One way to determine the constant is to compare with the flat space case which is given in section 4. It turns out that if we multiply by \( \frac{D-2}{16(D-3)} \), the leading term of the \( F_1 \) and \( F_2 \) are exactly agree with the corresponding ones in flat space. So by multiplying by \( \frac{D-2}{16(D-3)} \) we get,

\[ \alpha_2 (F_2) = \alpha_2 (F_2) + \alpha_2 (F_2) + \alpha_2 (F_2) + \alpha_2 (F_2), \quad (78) \]

and \( \beta_2 (F_2), \ldots \), \( \epsilon_2 (F_2). \) Here the coefficients \( \alpha_2, \ldots \) are quite lengthy so we give them in the Tables 3 - 7. 

Now in order to solve for \( F_1 \) and \( F_2 \) what we have to do is just to equate the corresponding coefficients:

\[ \alpha_A (A) = \alpha_1 (F_1) + \alpha_2 (F_2), \quad (79) \]
\[ \beta_A (A) = \beta_1 (F_1) + \beta_2 (F_2), \quad (80) \]
\[ \gamma_A (A) = \gamma_1 (F_1) + \gamma_2 (F_2), \quad (81) \]
\[ \delta_A (A) = \delta_1 (F_1) + \delta_2 (F_2), \quad (82) \]
\[ \epsilon_A (A) = \epsilon_1 (F_1) + \epsilon_2 (F_2). \quad (83) \]

We recall that \( \alpha_A, \ldots \epsilon_A \) are given as the functions of \( A(y) \) defined in (31). Because the infinite series which vanishes for \( D = 4 \), we only need to keep potentially divergent terms,

\[ A(y) = \frac{H^D - 2}{(4\pi)^2} \left\{ \frac{\Gamma(D)}{2} \left( \frac{y}{2} \right)^{D-1} + \frac{\Gamma(D+1)}{2} \left( \frac{y}{2} \right)^{D-2} + \frac{\Gamma(D+2)}{2(D-3)} \left( \frac{y}{2} \right)^{D-3} \right\} + \cdots. \quad (84) \]

We see this function \( A(y) \) is basically power series of \( \left( \frac{y}{2} \right)^{D-1} \). Keeping only the potentially divergent powers, we have

\[ \alpha_A = -\frac{K}{2^5} \left[ \left( \frac{y}{2} \right)^D + D \left( \frac{y}{2} \right)^{D-1} + \frac{D(D+1)}{2} \left( \frac{y}{2} \right)^{D-2} + \cdots \right]. \quad (85) \]
\[
\beta_A = \frac{K}{2^7} \left[ D \left( \frac{4}{y} \right)^{D+1} + (D-1)D \left( \frac{4}{y} \right)^{D} + \frac{(D-2)D(D+1)}{2} \left( \frac{4}{y} \right)^{D-1} + \cdots \right] ,
\]

(86)

\[
\gamma_A = -\frac{K}{2^{11}} \left[ D^2 \left( \frac{4}{y} \right)^{D+2} + (D-2)D^2 \left( \frac{4}{y} \right)^{D+1} \\
+ \frac{(D^2-3D-2)D^2}{2} \left( \frac{4}{y} \right)^{D} + \cdots \right] ,
\]

(87)

\[
\delta_A = -\frac{K}{2^5} \left[ (D^2-D-4) \left( \frac{4}{y} \right)^{D} + (D^3-5D^2+4D-4) \left( \frac{4}{y} \right)^{D-1} \\
+ \frac{D^4-8D^3+19D^2-28D+8}{2} \left( \frac{4}{y} \right)^{D-2} + \cdots \right] ,
\]

(88)

\[
\epsilon_A = \frac{K}{2^8} \left[ (D-2)D \left( \frac{4}{y} \right)^{D+1} + (D^3-5D^2+6D-4) \left( \frac{4}{y} \right)^{D} \\
+ \frac{D(D^3-7D^2+12D-12)}{2} \left( \frac{4}{y} \right)^{D-1} + \cdots \right] .
\]

(89)

where the constant \( K \) is,

\[
K \equiv \frac{\kappa^2 H^{2D-4} \Gamma^2 \left( \frac{D}{2} \right)}{(4\pi)^D} .
\]

(90)

Now to solve for \( F_1 \) and \( F_2 \) we can choose any two of the coefficient equations (79 - 83). For example, we can pick the “\( \alpha^- \)” and “\( \beta^- \)” coefficient equations,

\[
\alpha_A(A) = \alpha_1(F_1) + \alpha_2(F_2) ,
\]

\[
\beta_A(A) = \beta_1(F_1) + \beta_2(F_2) .
\]

These equations are not easy to solve in general, but considering \( \alpha_A \) and \( \beta_A \) given in terms of \( \left( \frac{4}{y} \right) \) we can assume that \( F_1(y) \) and \( F_2(y) \) have the series expansions of \( \left( \frac{4}{y} \right) \) as well. So let us set

\[
F_1(y) \equiv a_1 \left( \frac{4}{y} \right)^{D-2} + b_1 \left( \frac{4}{y} \right)^{D-3} + c_1 \left( \frac{4}{y} \right)^{D-4} + \cdots ,
\]

(91)

\[
F_2(y) \equiv a_2 \left( \frac{4}{y} \right)^{D-2} + b_2 \left( \frac{4}{y} \right)^{D-3} + c_2 \left( \frac{4}{y} \right)^{D-4} + \cdots .
\]

(92)

After arranging the two “\( \alpha^- \)” and “\( \beta^- \)” equations in powers of \( \left( \frac{4}{y} \right) \) we can determine the coefficients of \( F_1 \) and \( F_2 \) as,

\[
a_1 = -K \frac{1}{8(D-1)^2} ,
\]

(93)
\[ a_2 = -K \frac{1}{4(D-2)^2(D-1)(D+1)}, \]  
(94)  
\[ b_1 = -K \frac{D(D^2 - 5D + 2)}{8(D-4)(D-3)(D-1)^2}, \]  
(95)  
\[ b_2 = -K \frac{D^4 - 3D^3 - 8D^2 + 60D - 96}{4(D-4)(D-3)(D-2)^3(D-1)(D+1)}, \]  
(96)  
\[ c_1 = -K \frac{D^2(D^4 - 12D^3 + 39D^2 - 16D - 36)}{16(D-6)(D-4)^2(D-3)(D-1)^2}, \]  
(97)  
\[ c_2 = -K \frac{1}{8(D-6)(D-4)^2(D-3)(D-2)^4(D-1)(D+1)} \times [D^8 - 8D^7 - 13D^6 + 348D^5 - 1136D^4] \times -1024D^3 + 15056D^2 - 38208D + 34560]. \]  
(98)

First we note that the leading coefficients \( a_1 \) and \( a_2 \) agree with the flat space case which is summarized in section 4 for the correspondence limit. Second, we note that the subleading terms are ultraviolet finite in \( D = 4 \) but have divergent factors either \( \frac{1}{D-4} \) or \( \frac{1}{(D-4)^2} \). Third, the divergent factors for the same powers of \( \left(\frac{1}{y}\right) \) are the same: \( \frac{1}{D-4} \) for \( b_2 \) and \( c_2 \) and \( \frac{1}{(D-4)^2} \) for \( b_3 \) and \( c_3 \). In the next section we introduce the appropriate counterterms to absorb the divergences and fully renormalize this result.

4 Renormalization

In this section we fully renormalize the graviton self-energy. The ultraviolet divergent terms are all absorbed by BPHZ counterterms and the remained terms are completely finite. We begin with reviewing the BPHZ procedure of obtaining counterterms for our case. Next we summarize the flat space case to provide the correspondence limit as well as to illustrate the procedure of our computation briefly. By taking trace we renormalize the spin 0 part first. This enables us to subtract all the divergent factors \( \frac{1}{D-4} \) for subleading terms of the spin 2 parts. Finally we completely renormalize the spin 2 part.

4.1 BPHZ counterterms

The theorem of Bogoliubov, Parasiuk, Hepp and Zimmerman (BPHZ) tells us how to construct the local counterterms to absorb the ultraviolet divergences
of any quantum field theory at some fixed order in the loop expansion [23].
For our case, the 1PI 2-point function at one loop order, it implies we need
two counterterms

\begin{align}
\Delta L_{C_1} &= \alpha R^2 \sqrt{-g} , \\
\Delta L_{C_2} &= \beta C^{\rho\sigma\mu\nu} C_{\rho\sigma\mu\nu} \sqrt{-g} .
\end{align}

(99) (100)

for \(D = 4\) spacetime dimensions. The divergent parts of which were originally
obtained by ‘t Hooft and Veltman in 1974[21].

The procedure is summarized as follows: First we check the superficial
degree of divergence (SDD) of the 1PI function we consider at a certain order
for a particular dimension; in our case we consider the graviton self-energy
at one loop order for \(D = 4\) dimensions. Its SDD is four and this means
it is quartically divergent in \(D = 4\) dimensions. Then the BPHZ theorem
says that we need four or fewer derivatives of the field, in our case, \(g_{\mu
\nu}\).
So the counterterm vertex functions have to be in the form of four or fewer
derivatives acting on a delta function. Now recalling that the Lagrangian has
to be generally coordinate invariant, the only possibilities for the counterterm
Lagrangian are,

\[\sqrt{-g} \times \text{[constant, } R, \ R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}, \ R^{\mu\nu} R_{\mu\nu}, \ R^2] .\]

(101)

It turns out we do not need the constant and \(R\) terms [21]. Also we note that
there exists a scalar called Gauss-Bonnet Scalar, which is a total derivative

\[\left[R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4R^{\mu\nu} R_{\mu\nu} + R^2\right] \sqrt{-g} = \partial_\mu S^\mu \]

(102)
in \(D = 4\) dimensions. Because the total derivative makes no contribution
to the action the remained possibilities are two of the three 4th-order cur-
vature scalars. We choose the two linearly independent combinations, (99)
and (100). Further we make a slight change to (99) for computational con-
venience,

\[R^2 \to \left[R - D(D-1)H^2\right]^2 \]

(103)

With this change it is left with only quadratic dependence on \(h:\)

\[R^2 \sim 1 + h + h^2 , \]

\[\left[R - D(D-1)H^2\right]^2 \sim h^2 . \]

(104)
For the price of writing $R^2$ of the form $[R - D(D-1)H^2]^2$, it turns out we need two more counterterms to absorb the divergent factor $\frac{1}{D-1}$ in the subdominant terms:

$$
\Delta L_{C3} = \gamma \left[ R - (D-1)(D-2)H^2 \right] H^2 \sqrt{-g}, \quad (105)
$$

$$
\Delta L_{C4} = \delta H^4 \sqrt{-g}. \quad (106)
$$

In fact we can show that these two extra counterterms and the subtracted terms from $R^2$ add making only finite contributions:

$$
-\alpha \times 2D(D-1) + \gamma = \text{finite at } D = 4, \quad (107)
$$

$$
\alpha D^2(D-1)^2 - \gamma(D-1)(D-2) + \delta = \text{finite at } D = 4. \quad (108)
$$

This means we only need the two counterterms (99) and (100) to absorb all the divergences.

One problem for quantum general relativity concerns us is that these counterterms are not present in the original Lagrangian. We could include (99); it would add a massive, positive energy scalar particle which poses no essential problem for the theory. However, incorporating (100) on a nonperturbative level would add a negative energy, spin two particle whose presence would cause the universe to decay instantly. We must therefore treat the one loop counterterms perturbatively, and regard them as proxies for the still unknown ultraviolet completion of the theory.

### 4.2 Flat space limit

In this section we consider the flat space case to illustrate the procedure as well as to give the correspondence limit. In $D$ dimensional flat space the MMC scalar propagator is,

$$
i\Delta(x; x') = \frac{\Gamma(D/2 - 1)}{4\pi D/2} \left[ \frac{1}{\Delta x^2(x; x')} \right]^{D/2 - 1}, \quad (109)
$$

where $\Delta x^2(x; x') = ||\vec{x} - \vec{x}'||^2 - (|t - t'| - i\epsilon)^2$. All the vertices in flat space follow from those of de Sitter given in Tables 1 and 2 by taking $H \to 0$ and $\bar{g}_{\mu\nu} \to \eta_{\mu\nu}$. 

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Recalling that dimensional regularization works by choosing $D$ so that any $D$-dependent power of zero vanishes, we note that all the 4-point contributions,

$$\frac{1}{2} \sum_{l=1}^{4} F_{l}^\mu\nu\rho\sigma\alpha\beta(x) \times \partial_\alpha \partial'_\beta i \triangle(x; x') \times \delta^D(x - x') ,$$  \hspace{1cm} (110)

vanish in flat space as well as on de Sitter.

The 3-point one loop scalar contribution to the graviton self-energy in flat space is,

$$-i [\mu\nu\Sigma_{\rho\sigma}]_{3\text{-point}}(x; x')$$

$$= \frac{1}{2} \sum_{l=1}^{2} T_{\mu\nu\alpha\beta}^l(x) \sum_{J=1}^{2} T_{\rho\sigma\gamma\delta}^J(x') \times \partial^\alpha \partial'^\gamma i \triangle(x; x') \times \partial^\beta \partial'^\delta i \triangle(x; x')$$

$$= C \left\{ \eta_{\mu(\rho}|\eta_{\sigma)\nu} \left[ -\frac{2}{\Delta x^{2D}} \right] + \Delta x_{(\mu}(\eta_{\nu)(\rho} \Delta x_{\sigma)} \left[ \frac{4D}{\Delta x^{2D+2}} \right] \right.$$

$$+ \Delta x_{\mu} \Delta x_{\nu} \Delta x_{\rho} \Delta x_{\sigma} \left[ -\frac{2D^2}{\Delta x^{2D+4}} \right] + \eta_{\mu\nu} \eta_{\rho\sigma} \left[ -\frac{1}{2} (D^2 - D - 4) \right]$$

$$+ \left[ \eta_{\mu\nu} \Delta x_{\rho} \Delta x_{\sigma} + \Delta x_{\mu} \Delta x_{\nu} \eta_{\rho\sigma} \right] \left[ \frac{D(D - 2)}{\Delta x^{2D+2}} \right] \left\} . \hspace{1cm} (111)$$

Here $C \equiv \kappa^2 \Gamma^2(D) \frac{\pi^{D/2}}{\Gamma(D/2)}$ for simplicity. We decompose this into the spin 0 part and spin 2 part as well as on de Sitter:

$$-i [\mu\nu\Sigma_{\rho\sigma}]_{3\text{-point}}(x; x')$$

$$= \left[ \text{spin 0 operator} \right] F_1(\Delta x^2) + \left[ \text{spin 2 operator} \right] F_2(\Delta x^2)$$

$$= \left[ \eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right] \eta_{\rho\sigma} \partial^2 - \partial_\rho \partial_\sigma \right] F_1(\Delta x^2)$$

$$+ \left\{ \eta_{\mu(\rho} \eta_{\sigma)\nu} \partial^4 - 2 \partial_\mu (\eta_{\nu)}(\rho) \partial_\sigma) \partial^2 + \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \right.$$  

$$- \frac{1}{D - 1} \left[ \eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right] \eta_{\rho\sigma} \partial^2 - \partial_\rho \partial_\sigma \right\} F_2(\Delta x^2) . \hspace{1cm} (112)$$

Next we determine the structure functions $F_1$ and $F_2$ by comparing this eqn (112) with eqn (111). The procedure is the same as on de Sitter: First we act all the derivatives on $F_1$ and $F_2$ to arrange the spin 0 and spin 2 parts in terms of the five basis tensors. Then we compare the coefficient of each tensor to determine $F_1$ and $F_2$. Here the flat space version of the five basis tensors are from (111),

$$\alpha : \eta_{\mu(\rho}|\eta_{\sigma)\nu} \right.$$  

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\[ \begin{align*} 
\beta &: \Delta x_{(\mu} \eta_{\nu)} (\rho \Delta x_{\sigma)} , \\
\gamma &: \Delta x_{\mu} \Delta x_{\nu} \Delta x_{\rho} \Delta x_{\sigma} , \\
\delta &: \eta_{\mu \nu} \eta_{\rho \sigma} , \\
\epsilon &: \eta_{\mu \nu} \Delta x_{\rho} \Delta x_{\sigma} + \Delta x_{\mu} \Delta x_{\nu} \eta_{\rho \sigma} . 
\end{align*} \]

Acting all the derivatives on \( F_1 \) and \( F_2 \) eqn (112) becomes

\[ -i [\mu \nu \Sigma]_{\rho \sigma} |_{\text{point}}(x; x') = [\text{spin 0 operator}] F_1 (\Delta x^2) + [\text{spin 2 operator}] F_2 (\Delta x^2) = \eta_{(\mu} (\eta_{\nu)} (\rho \Delta x_{\sigma}) \left\{ \alpha_1 (F_1) + \alpha_2 (F_2) \right\} + \Delta x_{(\mu} \eta_{\nu)} (\rho \Delta x_{\sigma}) \left\{ \beta_1 (F_1) + \beta_2 (F_2) \right\} + \Delta x_{\mu} \Delta x_{\nu} \Delta x_{\rho} \Delta x_{\sigma} \left\{ \gamma_1 (F_1) + \gamma_2 (F_2) \right\} + \eta_{\mu \nu} \eta_{\rho \sigma} \left\{ \delta_1 (F_1) + \delta_2 (F_2) \right\} + [\eta_{\mu \nu} \Delta x_{\rho} \Delta x_{\sigma} + \Delta x_{\mu} \Delta x_{\nu} \eta_{\rho \sigma}] \left\{ \epsilon_1 (F_1) + \epsilon_2 (F_2) \right\} . \] (113)

Now we can get the equations for \( F_1 \) and \( F_2 \) by equating the coefficients of the five basis tensors from (113) and (111):

\[ \begin{align*} 
\alpha &= \alpha_1 (F_1) + \alpha_2 (F_2) , \\
\beta &= \beta_1 (F_1) + \beta_2 (F_2) , \\
\gamma &= \gamma_1 (F_1) + \gamma_2 (F_2) , \\
\delta &= \delta_1 (F_1) + \delta_2 (F_2) , \\
\epsilon &= \epsilon_1 (F_1) + \epsilon_2 (F_2) . 
\end{align*} \] (114)

Here the coefficients are different combinations of derivatives, up to four four derivatives, of \( F_1 \) and \( F_2 \). Now considering the origianl coefficients \( \alpha, \beta, \gamma, \delta \) and \( \epsilon \) are functions of \( \frac{1}{\Delta x^2} \) we can assume

\[ \begin{align*} 
F_1 &= \frac{K_1}{(\Delta x^2)^{D-2}} , \\
F_2 &= \frac{K_2}{(\Delta x^2)^{D-2}} . 
\end{align*} \] (115, 116)

To determine the two constants \( K_1 \) and \( K_2 \), we pick any two of the five equations (114). For example, if we choose \( \beta \) and \( \gamma \) coefficient equations, we
have
\[ \beta : \quad C \frac{4D}{\Delta x^{2D+2}} = \frac{16(D - 2)D}{\Delta x^{2D+2}} \left[ -2(D - 1)K_1 - (D - 2)(D + 1)K_2 \right], \quad (117) \]
\[ \gamma : \quad -C \frac{2D^2}{\Delta x^{2D+4}} = \frac{16(D - 2)D(D + 1)}{\Delta x^{2D+4}} \left[ (D - 1)K_1 + (D - 2)K_2 \right]. \quad (118) \]
Solving for \( K_1 \) and \( K_2 \) gives,
\[ K_1 = -\frac{\kappa^2 \Gamma^2(D)}{64\pi^D} \frac{1}{2(D - 1)^2}, \quad (119) \]
\[ K_2 = -\frac{\kappa^2 \Gamma^2(D)}{64\pi^D} \frac{1}{(D - 2)(D - 1)(D + 1)}. \quad (120) \]
So the flat space 3-point contributions become,
\[ -i [\mu \nu \Sigma \rho \sigma]_{3\text{-point}}(x; x') = -\frac{\kappa^2 \Gamma^2(D)}{64\pi^D} \left\{ \left[ \text{spin 0 operator} \right] \frac{1}{2(D - 1)^2} \frac{1}{\Delta x^{2D-4}} \right. \]
\[ + \left[ \text{spin 2 operator} \right] \frac{1}{(D - 2)(D - 1)(D + 1)} \frac{1}{\Delta x^{2D-4}} \right\}. \quad (121) \]
We can check that these results for \( F_1 \) and \( F_2 \) in flat space agree with the leading terms of \( F_1 \) and \( F_2 \) on de Sitter. We can compare them with the following correspondences:
\[ y \rightarrow H^2 \Delta x^2, \quad (122) \]
\[ \frac{\partial y}{\partial x^\mu} \rightarrow 2H^2 \Delta x_\mu, \quad (123) \]
\[ \frac{\partial y}{\partial x^\rho} \rightarrow -2H^2 \Delta x_\rho, \quad (124) \]
\[ \frac{\partial^2 y}{\partial x^\mu \partial x^\rho} \rightarrow -2H^2 \eta_{\mu \rho}, \quad (125) \]
\[ \frac{\partial}{\partial y} \rightarrow \frac{1}{H} \frac{\partial}{\partial \Delta x^2}. \quad (126) \]
This means we can read off the leading term of \( F_1 \) and \( F_2 \) on de Sitter by the correspondences,
\[ (F_1)_{\text{flat}} = -\frac{\kappa^2 \Gamma^2(D)}{64\pi^D} \frac{1}{2(D - 1)^2} \left( \frac{1}{\Delta x^2} \right)^{D-2}, \quad (127) \]
\[
\rightarrow (F_1)_{\text{de Sitter}} = -\kappa^2 \Gamma^2 \left(\frac{D}{2}\right) \frac{H^{2D-4}}{(4\pi)^D} \left[ \frac{1}{8(D-1)^2} \left(\frac{4}{y}\right)^{D-2} + \cdots \right], \quad (128)
\]

\[
(F_2)_{\text{flat}} = -\kappa^2 \Gamma^2 \left(\frac{D}{2}\right) \frac{1}{64\pi^D} \frac{1}{(D-2)^2(D-1)(D+1)} \left(\frac{1}{\Delta x^2}\right)^{D-2}, \quad (129)
\]

\[
\rightarrow (F_2)_{\text{de Sitter}} = -\kappa^2 \Gamma^2 \left(\frac{D}{2}\right) \frac{H^{2D-4}}{(4\pi)^D} \left[ \frac{1}{4(D-2)^2(D-1)(D+1)} \left(\frac{4}{y}\right)^{D-2} \right.
\]

\[
+ \cdots \right]. \quad (130)
\]

Now let us recall that we can use the graviton self-energy to get the quantum corrections to the linearized field equation,

\[
- \int d^D x' [\mu\nu\Sigma^{\rho\sigma}](x; x') h_{\rho\sigma}(x') \ . \quad (131)
\]

Basically we need to do the following integral,

\[
\int d^D x' \frac{1}{\Delta x^{2D-4}} h_{\rho\sigma}(x') \ . \quad (132)
\]

We note that this is logarithmically divergent in \( D = 4 \) dimensions. After one partial integration we can get a finite integral,

\[
\frac{\partial^2}{(D-4)(2D-6)} \int d^D x' \frac{1}{\Delta x^{2D-4}} h_{\rho\sigma}(x') \ , \quad (133)
\]

however this integral is multiplied by a divergent factor, \( \frac{1}{\Delta x^{D-4}} \) for \( D = 4 \) dimension. Thus we need to cancel this divergence using appropriate counterterms. Recalling that counterterms should be local, we need to isolate the divergence in the form of delta function. Our strategy for doing that is adding zero in the form,

\[
\partial^2 \frac{1}{\Delta x^{2D-4}} - \frac{i4\pi^{D/2}}{\Gamma(D/2 - 1)} \delta^D(x - x') = 0 \ . \quad (134)
\]

Using this equation (134) we add zero to \( 1/\Delta x^{2D-4} \) in the form of a delta function,

\[
\frac{1}{\Delta x^{2D-4}} = \frac{\partial^2}{(D-4)(2D-6)} \frac{1}{\Delta x^{2D-6}} - \frac{\mu \partial^2}{(D-4)(2D-6)} \left[ \partial^2 \frac{1}{\Delta x^{D-2}} - \frac{i4\pi^{D/2}}{\Gamma(D/2 - 1)} \delta^D(x - x') \right] \ . \quad (135)
\]
Here a dimensional factor $\mu^{D-4}$ is added to make the dimensions consistent and $\mu$ is called the dimensional regularization mass scale. Now we can write $1/\Delta x^{2D-4}$ as a nonlocal finite term plus a local divergent term,

$$
\frac{1}{\Delta x^{2D-4}} = - \frac{\partial^2}{4} \left( \frac{\ln(\mu^2 \Delta x^2)}{\Delta x^2} \right) + \frac{4\pi^{D/2} \mu^{D-4}}{\Gamma(D/2 - 1)} \frac{i\delta^D(x - x')}{2(D - 4)(D - 3)} .
$$

(136)

Here for the finite part the $D = 4$ limit is taken. Then the 3-point contribution (121) can be written as

$$
-i\left(\mu^\rho\Sigma^\sigma\right)_{3\text{-point}}(x; x') = -\frac{\kappa^2 \Gamma^2(D^2)}{64\pi^D} \left\{ \frac{1}{2(D - 1)^2} \Pi_{\mu\nu}(x)\Pi_{\rho\sigma}(x') \left[ \frac{1}{D - 1} \Pi_{\rho\sigma} \right] + \frac{1}{D - 1} \Pi_{\mu\nu} \Pi_{\rho\sigma} \right\} \frac{1}{\Delta x^{2D-4}} ,
$$

$$
= -\frac{\kappa^2 \Gamma^2(D^2)}{64\pi^D} \left\{ \frac{1}{2(D - 1)^2} \Pi_{\mu\nu}(x)\Pi_{\rho\sigma}(x') \left[ \frac{1}{D - 1} \Pi_{\rho\sigma} \right] + \frac{1}{D - 1} \Pi_{\mu\nu} \Pi_{\rho\sigma} \right\} \frac{1}{\Delta x^{2D-4}} ,
$$

$$
+ \frac{\kappa^2 \Gamma^2(D^2)}{64\pi^D} \left\{ \frac{10}{9} \frac{1}{D - 1} \Pi_{\mu\nu} \Pi_{\rho\sigma} + \frac{1}{3} \left[ \Pi_{\mu(\rho \Pi_{\sigma})\nu} - \frac{1}{3} \Pi_{\mu\nu} \Pi_{\rho\sigma} \right] \right\} \frac{1}{\Delta x^{2D-4}} ,
$$

$$
- \frac{\kappa^2 \Gamma^2(D^2)}{64\pi^D} \left\{ \frac{1}{2(D - 1)^2} \Pi_{\mu\nu} \Pi_{\rho\sigma} \right\} \frac{1}{\Delta x^{2D-4}} ,
$$

$$
+ \frac{\kappa^2 \Gamma^2(D^2)}{64\pi^D} \left\{ \frac{1}{2(D - 1)^2} \Pi_{\mu\nu} \Pi_{\rho\sigma} \right\} \frac{1}{\Delta x^{2D-4}} ,
$$

(137)

Here we symbolically write the spin 0 and spin 2 operators as

$$
\text{spin 0 operator} = \Pi_{\mu\nu} \Pi_{\rho\sigma} ,
$$

(138)

$$
\text{spin 2 operator} = \Pi_{\mu(\rho \Pi_{\sigma})\nu} - \frac{1}{D - 1} \Pi_{\mu\nu} \Pi_{\rho\sigma} .
$$

(139)

To absorb the divergences we add the two invariant one loop counterterms (99) and (100) with $H \to 0$ and $g_{\mu\nu} \to \eta_{\mu\nu}$ to the original Lagrangian,

$$
\Delta L_1 = \frac{\alpha R^2}{\kappa^2 \Gamma^2(D^2)} ,
$$

(140)

$$
\Delta L_2 = \beta C_{\rho\sigma\mu\nu} C_{\rho\sigma\mu\nu} .
$$

(141)

By expanding them, we get two counterterms

$$
\frac{i\delta^2 S_1}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \bigg|_{h_{\alpha\beta} = 0} = 2\alpha \frac{\kappa^2 \Gamma^2(D^2)}{64\pi^D} \frac{1}{\Delta x^{2D-4}} ,
$$

(142)

$$
\frac{i\delta^2 S_2}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \bigg|_{h_{\alpha\beta} = 0} = 2\beta \frac{\kappa^2 \Gamma^2(D^2)}{64\pi^D} \left\{ \Pi_{\mu(\rho \Pi_{\sigma})\nu} - \frac{1}{D - 1} \Pi_{\mu\nu} \Pi_{\rho\sigma} \right\} \frac{1}{\Delta x^{2D-4}} ,
$$

(143)
To cancel the divergences requires

\[
\alpha = \frac{\Gamma(D/2)\mu^{D-4}D}{64\pi^{3/2}} \frac{D-2}{4(D-4)(D-3)(D-1)^2} + \text{finite}, \quad \text{(144)}
\]

\[
\beta = \frac{\Gamma(D/2)\mu^{D-4}2}{64\pi^{3/2}} \frac{1}{2(D-4)(D-3)^2(D-1)(D+1)} + \text{finite}. \quad \text{(145)}
\]

These results were originally obtained in momentum space by ‘t Hooft and Veltman in 1974[21]. In the next sections we compare these results with the de Sitter case.

Note the agreement of the flat space limit of our final result in section four with [24].

4.3 Spin 0 Part: Fully Renormalize $F_1$

In the previous section we separated the graviton self-energy in two pieces: We could find the $F_1(y)$ and $F_2(y)$ by comparing (44) with (70). Let us recall that the subdominant terms of $F_1$ and $F_2$ have the same divergent structure: $1/(D - 4)$ and $1/(D - 4)^2$ otherwise finite powers of $\frac{4}{y}$. Thus we expect the counterterms from $L_{C3}$ and $L_{C4}$ will take care of the divergent factors for both $F_1$ and $F_2$. (Better reasoning?) Thus recalling that the spin 2 part is traceless, taking trace on (44) and (70) gives an equation of $F_1$. To solve this equation for $F_1$ and fully renormalize it is the task of this section.

4.3.1 Determine $F_1$ alternatively by taking trace

Because the spin 2 part is traceless, taking trace on both sides of this equation (70) gives an equation including only $F_1$:

\[
[\Box + DH^2][\Box' + DH^2]F_1(y) = -\frac{1}{8}(D - 2)^2\kappa^2H^4\left\{[(2 - y)A' - k]^2 + \frac{4}{D - 1}(A')^2\right\}. \quad \text{(146)}
\]

Noting

\[
\Box F(y) = H^2[(4y - y^2)F'' + D(2 - y)F'] = \Box' F(y), \quad \text{(147)}
\]

we can write the equation (146) as

\[
[\Box + DH^2]^2F_1(y) = -\frac{1}{8}(D - 2)^2\kappa^2H^4\left\{[(2 - y)A' - k]^2 + \frac{4}{D - 1}(A')^2\right\}. \quad \text{(148)}
\]
Again we note that $A(y)$ is essentially a series expansion of $\frac{1}{y}$, and we only need to keep the powers which become divergent in $D = 4$ dimensions because the infinite series of $A(y)$ vanishes for $D = 4$. Thus keeping only potentially divergent terms, the right hand side of the equation (148) can be written as

$$\begin{align*}
-\frac{1}{8}(D-2)^2\kappa^2 H^4 \left\{ [(2-y)A' - k]^2 + \frac{4}{D-1}(A')^2 \right\} \\
= -\frac{1}{8}(D-2)^2\kappa^2 H^4 \frac{H^{2D-4}\Gamma^2(D/2)}{(4\pi)^D} \left\{ \frac{D}{4(D-1)} \left( \frac{4}{y} \right)^D + \frac{(D-2)^2}{4(D-1)} \left( \frac{4}{y} \right)^{D-1} \\
+ \frac{D^3 - 7D^2 + 16D - 8}{8(D-1)} \left( \frac{4}{y} \right)^{D-2} \right\}.
\end{align*}$$

Now we have the differential equation

$$[\Box + DH^2] F_1(y) = A_1 \left( \frac{4}{y} \right)^D + B_1 \left( \frac{4}{y} \right)^{D-1} + C_1 \left( \frac{4}{y} \right)^{D-2} + \cdots. \quad (150)$$

where the constants $A_1, B_1$ and $C_1$ are

$$A_1 = -H^4 \times \frac{\kappa^2 \Gamma^2(D/2) H^{2D-4}}{(4\pi)^D} \times \frac{D(D-2)^2}{32(D-1)}, \quad (151)$$

$$B_1 = -H^4 \times \frac{\kappa^2 \Gamma^2(D/2) H^{2D-4}}{(4\pi)^D} \times \frac{(D-2)^4}{32(D-1)}, \quad (152)$$

$$C_1 = -H^4 \times \frac{\kappa^2 \Gamma^2(D/2) H^{2D-4}}{(4\pi)^D} \times \frac{(D^3 - 7D^2 + 16D - 8)(D-2)^2}{64(D-1)} \quad (153)$$

and want to solve for $F_1(y)$. Again we set $F_1(y)$ as a series expansion of $\frac{4}{y}$.

$$F_1(y) \equiv a_1 \left( \frac{4}{y} \right)^{D-2} + b_1 \left( \frac{4}{y} \right)^{D-3} + c_1 \left( \frac{4}{y} \right)^{D-4} + \cdots \quad (154)$$

Plugging (154) into (150) and comparing both sides we can get the coefficients $a_1, b_1$ and $c_1$:

$$a_1 = -\frac{\kappa^2 \Gamma^2(D/2) H^{2D-4}}{(4\pi)^D} \times \frac{1}{8(D-1)^2} \quad , \quad (155)$$

$$b_1 = -\frac{\kappa^2 \Gamma^2(D/2) H^{2D-4}}{(4\pi)^D} \times \frac{D(D^2 - 5D + 2)}{8(D-4)(D-3)(D-1)^2} \quad , \quad (156)$$

$$c_1 = -\frac{\kappa^2 \Gamma^2(D/2) H^{2D-4}}{(4\pi)^D} \times \frac{D^2(D^4 - 12D^3 + 39D^2 - 16D - 36)}{16(D-6)(D-4)(D-3)(D-1)^2} \quad . \quad (157)$$

Of course these results agree with the coefficients for $F_1(y)$ we have found in the previous section. In the subsequent section we fully renormalize $F_1$. 28
4.3.2 Renormalization of Spin 0 part

Here we add the counterterms to absorb the divergences occurring in \( F_1(y) \). The structure function \( F_1(y) \) is given as,

\[
F_1(y) = -K \left\{ a_1(D) \left( \frac{4}{y} \right)^{D-2} + \frac{b_1(D)}{(D-4)} \left( \frac{4}{y} \right)^{D-3} + \frac{c_1(D)}{(D-4)^2} \left( \frac{4}{y} \right)^{D-4} + \cdots \right\}. \tag{158}
\]

Here we factored the divergent coefficient out and redefined \( b_1 \) and \( c_1 \) as follows:

\[
a_1(D) = \frac{1}{8(D-1)^2}, \tag{159}
\]

\[
b_1(D) = \frac{D(D^2 - 5D + 2)}{8(D - 3)(D - 1)^2}, \tag{160}
\]

\[
c_1(D) = \frac{D^2(D^4 - 12D^3 + 39D^2 - 16D - 36)}{16(D - 6)(D - 3)(D - 1)^2}. \tag{161}
\]

We see that the first term is logarithmically divergent in \( D = 4 \) dimensions as well as the flat space case. The new problem on de Sitter is that the second and third terms which will give finite integrals have the divergent factors, \( \frac{1}{D-4} \) and \( \frac{1}{(D-4)^2} \), respectively. Our procedure is again to segregate the divergence on the local term (in the form of a delta function). For doing that the key identity we can use is,

\[
\frac{\Box}{H^2} \left( \frac{4}{y} \right)^{D/2-1} = \frac{(4\pi)^{D/2}}{\Gamma(D/2 - 1)} \frac{i\delta^D(x - x')}{H^D \sqrt{-g}} + \frac{D(D - 2)}{4} \left( \frac{4}{y} \right)^{D/2-1}. \tag{162}
\]

Let us first note that in \( D = 4 \) dimensions

\[
\left( \frac{4}{y} \right)^{D/2-1} = \left( \frac{4}{y} \right)^{D/2-4}, \tag{163}
\]

\[
\left( \frac{4}{y} \right)^{D/2-2} = 1. \tag{164}
\]

For simplicity, let’s change the variable \( X \equiv y/4 \). For the second and third term of eqn (158) we may add

\[
\frac{b_1(D)}{(D-4)} \left[ - \left( \frac{1}{X} \right)^{D/2-1} \right] \quad \text{to} \quad \frac{b_1(D)}{(D-4)} \left( \frac{1}{X} \right)^{D-3}, \tag{155}
\]

\[
\frac{c_1(D)}{(D-4)^2} \left[ - 2 \left( \frac{1}{X} \right)^{D/2-2} + 1 \right] \quad \text{to} \quad \frac{c_1(D)}{(D-4)^2} \left( \frac{1}{X} \right)^{D-4}. \tag{166}
\]
Then by taking $D = 4$ limit, we have

$$
\frac{b_1(D)}{(D-4)} \left[ \left( \frac{1}{X} \right)^{D-3} - \left( \frac{1}{X} \right)^{D/2-1} \right] \rightarrow -\frac{b_1(4)}{2} \frac{1}{X} \ln(X) , \quad (167)
$$

$$
\frac{c_1(D)}{(D-4)^2} \left[ \left( \frac{1}{X} \right)^{D-4} - 2 \left( \frac{1}{X} \right)^{D/2-2} + 1 \right] \rightarrow \frac{c_1(4)}{4} \ln^2(X) , \quad (168)
$$

which are finite in $D = 4$. We expect that the divergences from these, which will occur acted by the differential operator $\left[ \frac{\Box}{M^2} + D \right]^2$, will be canceled by the $\gamma$- and $\delta$- counterterms.

Now let us review what the possible counterterms are. From the counterterm Lagrangians we discussed in the previous section, we get the corresponding counterterms, by varying the Lagrangians with respect to the graviton field $h_{\mu\nu}$:

$$
\Delta L_{C_1} = \alpha \left[ R - D(D-1)H^2 \right]^2 \sqrt{-g} ,
$$

$$
\Rightarrow \frac{i \delta^2 \Delta S_1}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \bigg|_{h_{\alpha\beta} = 0} = 2\alpha \kappa^2 \sqrt{-g} \left[ \Box g^{\mu\rho} - D^\mu D^\rho + (D-1)H^2 g^{\mu\rho} \right] \times \sqrt{-g} \left[ \Box g^{\rho\sigma} - D^\rho D^\sigma + (D-1)H^2 g^{\rho\sigma} \right] \delta^D(x-x') \sqrt{-g} , \quad (169)
$$

$$
\Delta L_{C_2} = \beta C^{\rho\sigma\mu\nu} C_{\rho\sigma\mu\nu} \sqrt{-g} ,
$$

$$
\Rightarrow \frac{i \delta^2 \Delta S_2}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \bigg|_{h_{\alpha\beta} = 0} = 2\beta \kappa^2 \sqrt{-g} \left[ \Box g^{\mu\rho} - D^\mu D^\rho + (D-1)H^2 g^{\mu\rho} \right] \times \sqrt{-g} \left[ \Box g^{\rho\sigma} - D^\rho D^\sigma + (D-1)H^2 g^{\rho\sigma} \right] \frac{i \delta^D(x-x')} \sqrt{-g} , \quad (170)
$$

$$
\Delta L_{C_3} = \gamma \left[ R - (D-1)(D-2)H^2 \right] H^2 \sqrt{-g} ,
$$

$$
\Rightarrow \frac{i \delta^2 \Delta S_3}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \bigg|_{h_{\alpha\beta} = 0} = \gamma \kappa^2 \sqrt{-g} \left[ 2\Box g^{\mu\rho} D^\rho D^\sigma - \Box g^{\mu\rho} g^{\rho\sigma} \right] \times \sqrt{-g} \left[ 2\Box g^{\mu\rho} D^\rho D^\sigma + \Box g^{\mu\rho} g^{\rho\sigma} \right] \delta^D(x-x') , \quad (171)
$$

$$
\Delta L_{C_4} = \delta H^1 \sqrt{-g} ,
$$

$$
\Rightarrow \frac{i \delta^2 \Delta S_4}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \bigg|_{h_{\alpha\beta} = 0} = \frac{\delta}{4} \kappa^2 H^1 \sqrt{-g} \left[ g^{\mu\nu} g^{\rho\sigma} - 2\Box g^{\mu\rho} g^{\nu\sigma} \right] \delta^D(x-x') . \quad (172)
$$
Taking trace on the primitive terms (70) and the counterterms (169 - 172) gives,

\[
(D - 1)^2 \sqrt{-g} \sqrt{-g} [\Box + DH^2] [\Box' + DH^2] F_1(y) + 0 \\
+ 2\alpha \kappa^2 (D - 1)^2 \sqrt{-g} [\Box + DH^2] [\Box' + DH^2] i\delta^D (x - x') + 0 \\
+ \frac{\gamma}{2} \kappa^2 H^2 (D - 1)(D - 2) \sqrt{-g} [\Box + DH^2] i\delta^D (x - x') \\
+ \frac{\delta}{4} \kappa^2 H^4 D(D - 2) \sqrt{-g} i\delta^D (x - x').
\]  

(173)

We note that the added terms on the subleading terms of \( F_1 \) produce the delta function and therefore can be absorbed by the \( \gamma^- \) and \( \delta^- \) counterterms:

Note that

\[
[\frac{\Box}{H^2} + D]^2 \left( \frac{1}{X} \right)^{D/2-1} = \left[ \frac{\Box}{H^2} + D \right] \frac{(4\pi)^{D/2} i\delta^D (x - x')}{\Gamma \left( \frac{D}{2} - 1 \right) \frac{D^D \sqrt{-g}}{2}} + \frac{D(D + 2)(4\pi)^{D/2} i\delta^D (x - x')}{4 \Gamma \left( \frac{D}{2} - 1 \right) \frac{D^D \sqrt{-g}}{2}} + \frac{D^2(D + 2)^2}{16} \left( \frac{1}{X} \right)^{D/2-1}.
\]  

(174)

\[
[\frac{\Box}{H^2} + D]^2 \left( \frac{1}{X} \right)^{D/2-2} = - \frac{D - 4(4\pi)^{D/2} i\delta^D (x - x')}{2 \Gamma \left( \frac{D}{2} - 1 \right) \frac{D^D \sqrt{-g}}{2}} - \frac{(D - 4)(D^2 + 2D - 4)}{4} \left( \frac{1}{X} \right)^{D/2-1} + \frac{(D - 2)^2(D + 4)^2}{16} \left( \frac{1}{X} \right)^{D/2-2}.
\]  

(175)

We write \( F_1 = \mathcal{F}_1 + \Delta F_1 \) where \( \mathcal{F}_1 \) is,

\[
\mathcal{F}_1 \equiv -K \left\{ a_1(D) \left( \frac{1}{X} \right)^{D-2} + b_1(D) \left[ \frac{1}{X} \right]^{D-3} - \left( \frac{1}{X} \right)^{D/2-1} \right\} \\
+ \frac{c_1(D)}{(D - 4)^2} \left[ \left( \frac{1}{X} \right)^{D-4} - 2 \left( \frac{1}{X} \right)^{D/2-2} + 1 \right].
\]  

(176)

\[
\rightarrow -K \left\{ a_1(D) \left( \frac{1}{X} \right)^{D-2} - \frac{b_1(4)}{2} \frac{1}{X} \ln(X) + \frac{c_1(4)}{4} \ln^2(X) \right\}.
\]  

(177)

Then we have,

\[
[\frac{\Box}{H^2} + D]^2 \mathcal{F}_1
\]
\[
\begin{align*}
&= \left[ \frac{\Box}{\mathcal{H}_D^2} + D \right]^2 \left[ a_1(D) \left( \frac{1}{X} \right)^{D-2} - \frac{b_1(4)}{2} \frac{1}{X} \ln(X) + \frac{c_1(4)}{4} \ln^2(X) \right], \\
&= -K \left\{ n_1(D) \left( \frac{1}{X} \right)^D + n_2(D) \left( \frac{1}{X} \right)^{D-1} + n_3(D) \left( \frac{1}{X} \right)^{D-2} \\
&\quad - \frac{b_1(D)}{D-4} \left[ \frac{\Box}{\mathcal{H}_D^2} + D \right] \left( \frac{4\pi D/2}{\Gamma(\frac{D}{2} - 1)} \right) i\delta^D(x-x') \right. \\
&\quad + \left[ - \frac{b_1(D) D(D+2)}{D-4} - 2 \frac{c_1(D)}{(D-4)^2} \times - \frac{D-4}{2} \right] \left( \frac{4\pi D/2}{\Gamma(\frac{D}{2} - 1)} \right) \mathcal{H}_D \sqrt{-g} \\
&\quad \left. - \frac{4}{3} \frac{1}{X} \ln(X) - \frac{8}{3} \ln^2(X) \right\}. \\
\end{align*}
\]

Here the constants \( n_1(D), n_2(D) \) and \( n_3(D) \) are
\[
\begin{align*}
n_1(D) &= \frac{D(D-2)^2}{32(D-1)}, \\
n_2(D) &= \frac{(D-2)^4}{32(D-1)}, \\
n_3(D) &= \frac{(D^3 - 7D^2 + 16D - 8)(D-2)^2}{64(D-1)}. \\
\end{align*}
\]

These \( n_1(D), n_2(D) \) and \( n_3(D) \) agree with the original expressions (151, 152, 153) as they should be. We see that the localized divergent terms of the eqn (179) in the form of the \( \gamma^- \) and \( \delta^- \) counterterms so that can be canceled by them. The left-over term,
\[
- \frac{4}{3} \frac{1}{X} \ln(X) - \frac{8}{3} \ln^2(X),
\]
can be canceled by adding \( \Delta F_1 \) which satisfies
\[
\left[ \frac{\Box}{\mathcal{H}_D^2} + D \right] \Delta F_1 = \frac{4}{3} \frac{1}{X} \ln(X) + \frac{8}{3} \ln^2(X). \\
\]
The explicit function of \( \Delta F_1 \) is given in the next subsection.

Now we require that the localized divergences of (179) cancel with \( \gamma^- \) and \( \delta^- \) counterterms
\[
(D-1)^2 \frac{b_1(D)}{D-4} \times K \times \frac{(4\pi D/2)}{\Gamma(\frac{D}{2} - 1) \mathcal{H}_D} + \frac{\gamma^2}{2} \kappa^2 (D-1)(D-2) = 0,
\]

32
\[(D - 1)^2 \left[ - \frac{b_1(D)}{D - 4} \frac{D(D + 2)}{4} + \frac{c_1(D)}{D - 4} \right] \times (-K) \times \frac{(4\pi)^{D/2}}{\Gamma(D/2 - 1)H^D} \]
\[+ \delta^2 \kappa^2 D(D - 2) = 0 . \quad (186)\]

Thus we determine \(\gamma\) and \(\delta\) as,
\[
\gamma = \frac{D(D^2 - 5D + 2)}{8(D - 4)(D - 3)(D - 1)} \times \frac{\Gamma(D/2)H^{D-4}}{(4\pi)^{D/2}}, \quad (187)
\]
\[
\delta = \frac{D(D^3 - 11D^2 + 24D + 12)}{16(D - 6)(D - 3)} \times \frac{\Gamma(D/2)H^{D-4}}{(4\pi)^{D/2}}. \quad (188)
\]

Now to take care of the leading divergence \((\frac{1}{X})^{D-2}\) we extract a covariant d’Alembertian from it as we did for the flat space case:
\[
\left(\frac{1}{X}\right)^{D-2} = \frac{2}{(D - 3)(D - 4)} \frac{\Box}{H^2} \left(\frac{1}{X}\right)^{D-3} - \frac{4}{D - 4} \left(\frac{1}{X}\right)^{D-3} . \quad (189)
\]

Using the key identity (162), we add zero to \((\frac{1}{X})^{D-2}\):
\[
\left(\frac{1}{X}\right)^{D-2} = \frac{2}{(D - 3)(D - 4)} \frac{\Box}{H^2} \left[ \left(\frac{1}{X}\right)^{D-3} - \left(\frac{1}{X}\right)^{D/2-1} \right]
- \frac{4}{D - 4} \left(\frac{1}{X}\right)^{D-3} + \frac{2}{(D - 3)(D - 4)} \frac{D(D - 2)}{4} \left(\frac{1}{X}\right)^{D/2-1}
+ \frac{2}{(D - 3)(D - 4)} \frac{(4\pi)^{D/2}}{\Gamma(D/2 - 1)} \frac{i\delta^D(x - x')}{H^D \sqrt{-g}} . \quad (190)
\]

Now taking \(D = 4\) it becomes,
\[
\left(\frac{1}{X}\right)^{D-2} \rightarrow \frac{\Box}{H^2} \left[ - \frac{\ln(X)}{X} \right] + \frac{2}{X} \ln(X) - \frac{1}{X}
+ \frac{2}{(D - 3)(D - 4)} \frac{(4\pi)^{D/2}}{\Gamma(D/2 - 1)} \frac{i\delta^D(x - x')}{H^D \sqrt{-g}} . \quad (191)
\]

Then \(F_1\) becomes
\[
F_1 \rightarrow -K \left\{ a_1(D) \frac{2}{(D - 3)(D - 4)} \frac{(4\pi)^{D/2}}{\Gamma(D/2 - 1)} \frac{i\delta^D(x - x')}{H^D \sqrt{-g}} \right\}
+ a_1(4) \left[ \frac{\Box}{H^2} \left( - \frac{\ln(X)}{X} \right) + \frac{2}{X} \ln(X) - \frac{1}{X} \right]
- \frac{b_1(4)}{2} \frac{1}{X} \ln(X) + \frac{c_1(4)}{4} \ln^2(X) \right\} . \quad (192)
\]
Requiring that this leading divergence canceled by the $\alpha$-counterterm we can determine the coefficient $\alpha$:

$$(D - 1)^2 a_1(D) \frac{2}{(D - 4)(D - 3)} \times (-K) \times \frac{(4\pi)^{D/2}}{\Gamma(D/2 - 1) H^{D/2}} + 2\alpha \kappa^2 (D - 1)^2$$

$$= 0. \quad (193)$$

This gives,

$$\alpha = + \frac{(D - 2)}{16(D - 4)(D - 3)(D - 1)^2} \times \frac{\Gamma(D/2)H^{D-4}}{(4\pi)^{D/2}}. \quad (194)$$

We note that this agrees with the flat space case (144).

### 4.3.3 Spin 0 part: Determine $\Delta F_1$

To completely determine $F_1$ we need to find $\Delta F_1$ which satisfies in $D=4$ dimension,

$$\left[ \frac{\Box}{H^2} + 4 \right]^2 \Delta F_1 = + \frac{4}{3} \frac{1}{X} \ln(X) + \frac{8}{3} \ln^2(X) \quad (195)$$

We can invert one differential operator first:

$$\left[ \frac{\Box}{H^2} + 4 \right] \Delta F_1 \equiv \Delta G \quad (196)$$

By omitting the numerical factor $\frac{4}{3}$ which we can recover later easily, we want to solve the differential equation for $G(x)$:

$$\left[ \frac{\Box}{H^2} + 4 \right] \Delta G(X) = [X(1 - X)\left(\frac{d}{dX}\right)^2 + (2 - 4X)\frac{d}{dX} + 4] \Delta G(X) \quad (197)$$

$$= \frac{1}{X} \ln(X) + 2 \ln^2(X). \quad (198)$$

First we can find the Green’s function for the operator,

$$X(1 - X)\left(\frac{d}{dX}\right)^2 + (2 - 4X)\frac{d}{dX} + 4, \quad (199)$$

as

$$G(X; Y) = (2 - 4X)(2 - 4Y)Y(1 - Y) \times \left\{ \left[ - \frac{1}{4X} + \frac{1}{4(1 - X)} + \frac{4}{2 - 4X} + \frac{3}{2} \ln(X) - \frac{3}{2} \ln(1 - X) \right] \right. \right.$$ 

$$\left. - \left[ - \frac{1}{4Y} + \frac{1}{4(1 - Y)} + \frac{4}{2 - 4Y} + \frac{3}{2} \ln(Y) - \frac{3}{2} \ln(1 - Y) \right] \right\}. \quad (200)$$
Then basically we can find \( \Delta F_1(X) \) using the Green’s function by integrating twice over the Green’s function,

\[
\Delta F_1(X) = \int^X dX' G(X; X') \int^{X'} dX'' G(X'; X'') f(X'') ,
\]

where \( f(X) = \frac{1}{X} \ln(X) + 2 \ln^2(X) \). This gives after some simplification, \( \Delta F_1(X) \) as

\[
\Delta F_1(X) = -\frac{1}{54000(1 - X)X} \left\{ 62468X^3 - 95547X^2 + 28829X + 4340 \\
- 60(1 - X)(1418X^2 - 1129X + 70) \ln(1 - X) \\
- 120X(884X^2 - 1041X + 217) \ln(X) \\
- 43200(1 - X)X(1 - 2X) \ln(1 - X) \ln(X) \\
+ 18000(1 - X)X \ln^2(X) \\
- 68400(1 - X)X(1 - 2X) \text{Li}_2(X) \right\} .
\]

(202)

Here the \( \text{Li}_2(X) \) is the dilogarithm function.

### 4.4 Spin 2 Part: Renormalize \( F_2 \)

In the previous section we have found the \( F_1 = \overline{F}_1 + \Delta F_1 \) and fully renormalized it. Here we can use \( F_1 \) to find \( F_2 \) and renormalize it. The way we determine the coefficients of \( F_2 \) is again to compare the powers of \( (4/y) \) using the five coefficient equations (79 - 83). Now with the \( F_1 \) completely determined, we need only one of the five equations (79 - 83). Recalling the form of \( F_1 \),

\[
F_1 = \overline{F}_1 + \Delta F_1 = -K \left\{ a(D) \left( \frac{1}{X} \right)^{D-2} + \frac{b(D)}{(D-4)} \left[ \left( \frac{1}{X} \right)^{D-3} - \left( \frac{1}{X} \right)^{D/2-1} \right] \\
+ \frac{c(D)}{(D-4)^2} \left[ \left( \frac{1}{X} \right)^{D-4} - 2 \left( \frac{1}{X} \right)^{D/2-2} + 1 \right] \right\} + \Delta F_1,
\]

we suppose that \( F_2 \) has the same form,

\[
F_2 = -K \left\{ a(D) \left( \frac{1}{X} \right)^{D-2} + \frac{b(D)}{(D-4)} \left[ \left( \frac{1}{X} \right)^{D-3} - \left( \frac{1}{X} \right)^{D/2-1} \right] \\
+ \frac{c(D)}{(D-4)^2} \left[ \left( \frac{1}{X} \right)^{D-4} - 2 \left( \frac{1}{X} \right)^{D/2-2} + 1 \right] \right\} + \cdots.
\]

(204)
Then we just need to determine the coefficients $a_2(D)$, $b_2(D)$ and $c_2(D)$. The procedure is the same as before. We compare the coefficient of each power of $(4/y)$ then we get

\begin{align*}
a_2(D) &= -K \frac{1}{4(D-2)^2(D-1)(D+1)}, & (205) \\
b_2(D) &= -K \frac{D^4 - 3D^3 - 8D^2 + 60D - 96}{4(D-3)(D-2)^3(D-1)(D+1)}, & (206) \\
c_2(D) &= -K \frac{1}{8(D-6)(D-3)(D-2)^3(D-1)(D+1)} \\
&\quad \times \left[D^8 - 8D^7 - 13D^6 + 348D^5 - 1136D^4 - 1024D^3 \\
&\quad + 15056D^2 - 38208D + 34560\right]. & (207)
\end{align*}

Of course this is the same as the solutions we have in the Section 3. But now with this form we can take $D = 4$ limit. Then we have

\[ F_2 \to -K \left\{ a_2(D) \left( \frac{1}{X} \right)^{D-2} - \frac{b_2(4)}{2} \frac{1}{X} \ln(X) + \frac{c_2(4)}{4} \ln^2(X) \right\}. \]  

(208)

Now to make $F_2$ integrable in $D = 4$ we do one more partial integration using the identity (191),

\[ \left( \frac{1}{X} \right)^{D-2} \to \frac{\Box}{\mathcal{H}^2} \left[ - \ln(X) \right] + \frac{2}{X} \ln(X) - \frac{1}{X} \]

\[ + \frac{2}{(D-3)(D-4)} \frac{1}{\Gamma(\frac{D}{2} - 1)} \frac{1}{\mathcal{H}^D \sqrt{-\eta}} \]

as $D \to 4 - \epsilon$

Then $F_2$ becomes

\[ F_2 \to -K \left\{ a_2(D) \left( \frac{1}{X} \right)^{D-2} - \frac{b_2(4)}{2} \frac{1}{X} \ln(X) + \frac{c_2(4)}{4} \ln^2(X) \right\} . \]

(209)

Now we can cancel the local divergent part with the $\beta$-counterterm by requiring

\[ 2\beta \kappa^2 + a_2(D) \frac{2}{(D-3)(D-4)} \frac{1}{\Gamma(\frac{D}{2} - 1)} \frac{1}{\mathcal{H}^D} = 0. \]

(210)
And we can determine $\beta$ as

$$\beta = \frac{1}{8(D - 4)(D - 3)(D - 2)(D - 1)(D + 1)} \frac{\Gamma(\frac{D}{2})H^{D-4}}{(4\pi)^{D/2}}. \quad (211)$$

This agrees with the flat space case (145).

Finally the finite part of the graviton self-energy becomes

$$-i[\mu\nu\Sigma_{\rho\sigma}]_{3\text{-point}}(x; x')$$

$$= \sqrt{-g} \sqrt{-g'} \left\{ \text{Spin 0 Operator} \right\} \left\{ a_1(4) \left[ \frac{\Box}{H^2} \left( - \frac{\ln(X)}{X} \right) + 2 \frac{\ln(X)}{X} - \frac{1}{X} \right] - \frac{b_1(4)}{2} \frac{1}{X} \ln(X) + \frac{c_1(4)}{4} \ln^2(X) + \Delta F_1 \right\} + \left\{ \text{Spin 2 Operator} \right\} \left\{ a_2(4) \left[ \frac{\Box}{H^2} \left( - \frac{\ln(X)}{X} \right) + 2 \frac{\ln(X)}{X} - \frac{1}{X} \right] - \frac{b_2(4)}{2} \frac{1}{X} \ln(X) + \frac{c_2(4)}{4} \ln^2(X) + \Delta F_1 \right\}. \quad (212)$$

Here $a_1(4) = -K \times \frac{1}{72}$, $b_1(4) = -K \times -\frac{1}{9}$, $c_1(4) = -K \times -\frac{2}{9}$, $a_2(4) = -K \times \frac{1}{240}$, $b_2(4) = -K \times \frac{1}{5}$, $c_2(4) = -K \times -1$.

Note the agreement of the flat space limit of our final result in section four with [24].

5 Discussion

- Summarize results.
- Note that the result is de Sitter invariant.
- Note agreement of the form reached in section 3 with the stress tensor correlator of Perez-Nadal, Roura and Verdaguer [20].
- Note agreement of our counterterms with those found for this model by ‘t Hooft and Veltman [21].
- Note the agreement of the flat space limit of our final result in section four with [24].
• Discuss how one uses our result to quantum correct the linearized Einstein equations. In particular, give the prescription for getting the Schwinger-Keldysh effective field equations [25, 26, 27, 28, 29, 30, 31].

• Discuss what we expect from solving the equation: nothing much for dynamical gravitons (because scalars have no spin) but a potential fractional enhancement of the gravitational potential of the form,

\[
\frac{\Delta \Phi}{\Phi} \sim -\frac{\hbar G H^2}{c^5} \ln[a(t)].
\]  

(213)

We have computed the one loop contribution to the graviton self-energy

\[-i[\mu^\nu\Sigma^\rho\sigma](x; x')\] from a massless, minimally coupled (MMC) scalar on a locally de Sitter background. We used dimensional regularization and renormalized by absorbing the divergences with the BPHZ counterterms. The finite result is given in (212).

This result can be used to quantum correct to quantum correct the linearized Einstein equations.

\[D^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) - \int d^D x' [\mu^\nu\Sigma^\rho\sigma](x; x') h_{\rho\sigma}(x') = \frac{1}{2} \kappa T^{\mu\nu}.\]  

(214)

Here \(D^{\mu\nu\rho\sigma} h_{\rho\sigma}(x') = \frac{1}{2} \kappa T^{\mu\nu}\) is \((\sqrt{-g}/c)^\times\) the linearized classical Einstein equation. In de Sitter background the differential operator \(D^{\mu\nu\rho\sigma}\) is

\[D^{\mu\nu\rho\sigma} = \frac{1}{2} \sqrt{-g} \{ \tilde{g}^{\mu\rho} D^\sigma D^{\nu'} + \tilde{g}^{\nu'\rho} D^\sigma D^\mu - \tilde{g}^{\mu\nu'} \tilde{g}^{\rho\sigma} D^2 - \tilde{g}^{\rho\sigma} D^\mu D^{\nu'} - \tilde{g}^{\nu'\rho} D^\sigma D^\mu + \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma} D^2 - 2(D-1)H^2 \tilde{g}^{\mu\nu'} \tilde{g}^{\rho\sigma} + (D-1)H^2 \tilde{g}^{\rho\sigma} \tilde{g}^{\mu\nu'} \}.\]  

(215)

Here \(D^\mu\) is the covariant derivative operator with respect to the de Sitter background. For a stationary point mass the source is,

\[T_{\mu\nu} = \frac{m \delta^3(x) \tilde{g}_{\mu\nu} \tilde{g}_{\rho\sigma}}{\sqrt{-\tilde{g}_{00}}} .\]  

(216)

The usual, in-out formalism has two features which make it undesirable for applications in cosmology: quantum corrections can derive from the future of the observation point and quantum corrections to Hermitian operators need not be real. Neither of these features means the in-out formalism
is wrong. In fact it is the right answer to questions about matrix elements between scattering states if the system begins in free vacuum in the asymptotic past and ends up the same way in the far future. However, that question isn’t typically very relevant for cosmology in which the universe begins with an initial singularity and no one knows how it ends. The question of greater relevance for cosmology is, “what happens when the universe is released at some finite time in a prepared state?” The Schwinger-Keldysh formalism[25, 26, 27, 28, 29, 30, 31] provides a way of answering this more relevant question which is almost as simple to use as the Feynman diagrams of the in-out formalism.

We can get the four propagators of the Schwinger-Keldysh formalism from the Feynman propagator once that is known. The rule is just to replace \( \Delta x^2(x; x') \) in the Feynman propagator with \( \Delta x^2_{\pm\pm}(x; x') \) where

\[
\Delta x^2_{++}(x; x') \equiv \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| - i\epsilon)^2,
\]

\[
\Delta x^2_{+-}(x; x') \equiv \| \vec{x} - \vec{x}' \|^2 - (\eta - \eta' + i\epsilon)^2,
\]

\[
\Delta x^2_{-+}(x; x') \equiv \| \vec{x} - \vec{x}' \|^2 - (\eta - \eta' - i\epsilon)^2,
\]

\[
\Delta x^2_{--}(x; x') \equiv \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| + i\epsilon)^2.
\]

The “graviton self-energy” which belongs in equation (214) in the S-K formalism becomes

\[
\left[ \mu^\nu \Sigma_{\rho\sigma} \right](x; x') \rightarrow \left[ \mu^\nu \Sigma_{\rho\sigma} \right]_{++}(x; x') + \left[ \mu^\nu \Sigma_{\rho\sigma} \right]_{+-}(x; x').
\]

The linearized gravitational field equations tell us two things:

- How dynamical gravitons propagate through the background geometry;
- What the force of gravity is in the background geometry.

The quantum correction term in equation (214) describes how both of these things are affected by virtual particles, in our case virtual scalars. If there are not many virtual particles or they interact with gravitons only weakly then there will not be much effect once the field is renormalized. That is what happens in flat space.
In a inflationary background, for example, the de Sitter background we consider, we can show that the number of particles with wave number $k$ grows in time as the square of the scale factor,

$$N(k, t) = \left( \frac{H a(t)}{2k} \right)^2$$  \hspace{1cm} (222)

Note from expression (222) that there is not much excitation above the instantaneous 0-point energy of $k/2a(t)$ for “ultraviolet” wavelengths with $k \gg H a(t)$. It is only for cosmological scale wavelengths with $k \sim H a(t)$ that the occupation number $N(k, t)$ becomes of order unity.

At this point it is useful to compare my thesis problem with those of two recent UF graduates, Shun-Pei Miao [34] and Emre Kahya [35]. Whereas they studied the effect of inflationary gravitons (which are produced copiously for the same reason as MMC scalars) on different types of matter particles, I am studying the effect of inflationary scalars on gravitons. Dr. Miao studied the effect of inflationary gravitons on fermions and she found that one loop corrections increase the fermion field strength by a factor of $G H^2 \ln(a)$ [36]. Dr. Kahya studied the effect of infrared gravitons on scalars and he found no similar growth [37]. The reason why they got different results is spin: gravitons and fermions both have spin and this leads to an extra interaction, in addition to the one due to their kinetic energies. This extra interaction is highly significant because it persists even for highly infrared particles, whose kinetic energies redshift to zero. In a subsequent paper, Dr. Miao showed that the secular growth factors of $G H^2 \ln(a)$ derive entirely from the vertices of the spin-spin interaction [38]. The absence of a secular enhancement in Dr. Kahya’s thesis computation derives from the fact that scalars have zero spin.

Now recall that I am really making two studies:

1. How inflationary scalars affect dynamical gravitons; and
2. How inflationary scalars affect the force of gravity.

The first of these studies is a sort of mirror-image of Dr. Kahya’s; whereas he studied the effect of inflationary gravitons on scalars, I am studying the effect of inflationary scalars on dynamical gravitons. I expect the result of this first study to be the absence of any secular enhancement for the same reason that Dr. Kahya found none: dynamical gravitons have spin two, and scalars have spin zero. On the other hand, the force of gravity is partly
carried by the non-dynamical (constrained) spin zero sector of the graviton field. This might well experience a secular enhancement.

To find the force of gravity one first puts down a point mass and computes the classical response to that, then this classical potential is integrated against the one loop graviton self-energy to serve as a source for the one loop correction to the potential. When Donoghue and collaborators worked out the same thing for flat space background they got \[39\],

\[
\Phi = -\frac{GM}{r} \left\{ 1 + \frac{41}{10\pi} \frac{G}{r^2} + O(G^2) \right\}.
\]

(223)

There are two ways to view this: from the perspective of dimensional analysis, and from the perspective of how virtual particles interact with the point mass. In the context of dimensional analysis, any one loop correction in quantum gravity must carry a factor of \(G\), which goes like a length-squared. The only other dimensionful parameters in this problem are the length \(r\) and the particle mass \(m\). If we focus on the linearized response to \(m\) then any one loop effect must give a fractional correction of \(G/r^2\).

The physical interpretation of (223) is that the classical potential \(GM/r\) interacts with the energy density of virtual particles to reduce the measured mass of the source, in the same way that the combined mass of the Earth-Moon system is lower than the sum of their masses by their gravitational interaction energy. The effect is infinite because the energy density of ultraviolet virtual particles diverges, but if we renormalize things so that \(M\) is the measured mass as \(r\) goes to infinity, then the residual potential must grow as one approaches the mass because the radius \(r\) encloses less and less of the negative interaction energy between the mass and the virtual particles. The reason it grows like \(1/r^2\) is that the energy of a virtual particle of wavelength \(r\) goes like \(1/r\).

Now consider my problem in de Sitter background. From the perspective of dimensional analysis one sees first that there is a new parameter with the dimensions of an inverse length: the Hubble constant \(H\). That means one can get a fractional correction of the form \(GH^2\). The physical interpretation is the same as in Donoghue’s case: the one loop correction to the source’s potential comes from the gravitational interaction between it and the density of virtual particles. However, because inflation rips more and more virtual scalars out of the vacuum, as is evident in expression (222), the effect ought to get bigger with time. Of course the flat space result (223) must be present.
as well, but I expect an additional contribution of the form,

\[ \Phi \sim -\frac{Gm}{r} \left\{ 1 - \text{const} \times GH^2 \ln(a) + \ldots \right\}. \]  \tag{224}

This intrinsic inflation effect would manifest as a secular decrease in the effective Newton constant,

\[ G \rightarrow G\left\{ 1 - \text{const} \times GH^2 \ln(a) + \ldots \right\}. \]  \tag{225}

That is all pure theory, but it is worth asking how big the effect would be for the phase of primordial inflation which is believed to have actually occurred. Based on the measured value of the scalar amplitude of cosmic microwave anisotropies, and the current upper limit on the tensor-to-scalar ratio [33], the Hubble parameter during primordial inflation can have been no greater than about \(10^{13}\) GeV. Because Newton’s constant is the inverse square of about \(10^{19}\) GeV, the dimensionless loop counting parameter is very small,

\[ GH^2 \sim \left( \frac{10^{13}}{10^{19}} \right)^2 \sim 10^{-12}. \]  \tag{226}

On the other hand, the secular factor of \(\ln(a)\) that I expect (and that Dr. Miao actually found for her problem) grows as long as inflation persists, which is eternity for de Sitter space. So I expect that the secular reduction of Newton’s constant could be significant for a very long period of inflation.

Acknowledgements

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References


[32] Gradshteyn and Ryzhik


Table 3: Coefficient of $F_2$: each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

<table>
<thead>
<tr>
<th></th>
<th>Coefficient of $F_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{21}$</td>
<td>$-(D-3)D^2(D+1)^2[-4(D-2) + (D-1)(4y-y^2)]$</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>$2(D-3)(D-1)D^2(D+1)^2(2-y)$</td>
</tr>
<tr>
<td>$\gamma_{21}$</td>
<td>$(D-3)(D-1)D^2(D+1)^2$</td>
</tr>
<tr>
<td>$\delta_{21}$</td>
<td>$4(D-3)D(D+1)^2[-4(D-2) + D(4y-y^2)]$</td>
</tr>
<tr>
<td>$\epsilon_{21}$</td>
<td>$-4(D-3)D^2(D+1)^2$</td>
</tr>
</tbody>
</table>

Table 4: Coefficient of $F'_2$: each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

<table>
<thead>
<tr>
<th></th>
<th>Coefficient of $F'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{22}$</td>
<td>$4(D-3)(D+1)^2(2-y)[-2(D-2)D + (D-1)(D+1)(4y-y^2)]$</td>
</tr>
<tr>
<td>$\beta_{22}$</td>
<td>$8(D-3)(D+1)^2[-3D^2 + (D-1)(D+1)(4y-y^2)]$</td>
</tr>
<tr>
<td>$\gamma_{22}$</td>
<td>$-4(D-3)(D-1)(D+1)^3(2-y)$</td>
</tr>
<tr>
<td>$\delta_{22}$</td>
<td>$-16(D-3)(D+1)^2(2-y)[-2(D-2) + (D+1)(4y-y^2)]$</td>
</tr>
<tr>
<td>$\epsilon_{22}$</td>
<td>$16(D-3)(D+1)^3(2-y)$</td>
</tr>
</tbody>
</table>
\[ \alpha_{23} = 2 \left[ 8(D-2)^2 D(D+1) - 4(D+1)(3D^3 - 8D^2 - 6D + 12)(4y - y^2) \\
+ (D-3)(D-1)(3D^2 + 9D + 7)(4y - y^2)^2 \right] \]
\[ \beta_{23} = -4(2-y) \left[ -2D(D+1)(3D^2 - 5D - 10) \\
+ (D-3)(D-1)(3D^2 + 9D + 7)(4y - y^2)^2 \right] \]
\[ \gamma_{23} = \frac{1}{2} \left[ -12(D^4 - D^3 - 7D^2 + D + 10) \\
+ (D-3)(D-1)(3D^2 + 9D + 72)(4y - y^2)^2 \right] \]
\[ \delta_{23} = -8 \left[ 8(D-2)^2(D+1) - 2(D+1)(6D^2 - 11D - 18)(4y - y^2) \\
+ (D-3)(3D^2 + 9D + 7)(4y - y^2)^2 \right] \]
\[ \epsilon_{23} = 8 \left[ -2(D+1)(5D^2 - 6D - 24) + (D-3)(3D^2 + 9D + 7)(4y - y^2) \right] \]

Table 5: Coefficient of \( F''_2 \): each term is multiplied by \( \frac{1}{16(D-2)(D-1)} \)

| \( \alpha_{24} \) | Coefficient of \( F''''_2 \)
|------------------|--------------------------------------------------|
| \( \beta_{24} \) | \[ -4(D-1)(2-y)(4y - y^2) \left[ -2(D-2)(D+1) \\
+ (D-3)(D+2)(4y - y^2)^2 \right] \] |
| \( \gamma_{24} \) | \[ \frac{4(y - y^2)}{(D-2)D(D-1)(D+2)} \left[ -4(D-2)(D^2 - 5) + (D-3)(D-1)(D+2)(4y - y^2) \right] \] |
| \( \delta_{24} \) | \[ 16(2-y)(4y - y^2) \left[ -2(D-2)(D+1) + (D-3)(D+2)(4y - y^2) \right] \] |
| \( \epsilon_{24} \) | \[ -16(2-y) \left[ -2(D-2)(D+1) + (D-3)(D+2)(4y - y^2) \right] \] |

Table 6: Coefficient of \( F''''_2 \): each term is multiplied by \( \frac{1}{16(D-2)(D-1)} \)
Table 7: Coefficient of $F_2'''$: each term is multiplied by $\frac{1}{16(D-2)(D-1)}$.