Circular Motion and Simple Harmonic Motion

The projection of uniform circular motion along any axis (the \( x \)-axis here) is the same as simple harmonic motion. We use our understanding of uniform circular motion to arrive at the equations of simple harmonic motion.

The projection of the position along the \( x \)-axis gives

\[
x(t) = A \cos \omega t
\]

The acceleration is inward and (recall \( a_r = \omega^2 r \))

\[
a(t) = -\omega^2 A \cos \omega t
\]

The velocity can be shown to be

\[
v(t) = -\omega A \sin \omega t
\]
If the projection on the y-axis is used,

\[ y(t) = A \sin \omega t \]
\[ v(t) = A \omega \cos \omega t \]
\[ a(t) = -A \omega^2 \sin \omega t \]

The angular frequency is

\[ \omega = \sqrt{\frac{k}{m}} \]

For the ideal spring mass system

\[ f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad T = \frac{1}{f} = 2\pi \sqrt{\frac{m}{k}} \]
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How do you keep these straight? Memorize one and derive the others.

The Pendulum
For small oscillations, the simple pendulum executes simple harmonic motion. A simple pendulum is point mass suspended from a string.

Applying Newton’s second law

\[ \sum F_x = ma_x \]
\[ -T \sin \theta = ma_x \]
\[ -T \frac{x}{L} = ma_x \]

For small angles \( \sin \theta = \theta \). Motion along the \( y \)-axis is negligible since the angle is small.

\[ \sum F_y = ma_y \]
\[ T \cos \theta - mg \approx 0 \]
\[ T \approx mg \]

For small angles \( \cos \theta \approx 1 \). Substituting for \( T \)

\[ -T \frac{x}{L} = ma_x \]
\[ -mg \frac{x}{L} = ma_x \]
\[ a_x = -\frac{g}{L} x \]
This is the condition of simple harmonic motion \((a \propto x)\). We can pick off the angular frequency as

\[
\omega = \sqrt{\frac{g}{L}}
\]

The period of the pendulum

\[
T = 2\pi \sqrt{\frac{L}{g}}
\]

**Physical Pendulum.**
Instead of a point mass, the vibrating object has size.

The period is

\[
T = 2\pi \sqrt{\frac{I}{mgd}}
\]

where \(d\) is the distance from the rotation axis to the center of mass of the object and \(I\) is the rotational inertia about the rotation axis (not the center of mass).

**Damped Oscillations**
In reality, a vibrating system will stop vibrating. Energy is lost to the surroundings. Since it is similar to the drag experienced when moving through water, it is called damped motion.
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(b) is more strongly damped
(c) is overdamped (or perhaps critically damped) and there are no oscillations

**Forced Oscillations and Resonance**
To compensate for energy lost in damped systems, it is possible to drive the system at its resonant frequency. Pushing a swing is an example.

Forced oscillations can occur when a periodic driving force acts on a system that can vibrate. If the system is driven at its resonance frequency, vibrations can increase in amplitude until the system is destroyed.

**The Short Life and Tragic End of Gallopin’ Gertie**
http://en.wikipedia.org/wiki/Tacoma_Narrows_Bridge_%281940%29
http://www.youtube.com/watch?v=j-zczXSNxw

**Chapter 11 Waves**
Energy can be transported by particles or waves:

Lesson 14, page 5
A wave is characterized as some sort of disturbance that travels away from a source. The key difference between particles and waves is a wave can transmit energy from one point to another without transporting any matter between the two points.

Waves transport energy without transporting matter.

The **intensity** is the average power per unit area. It is measured in W/m$^2$.

\[ I = \frac{P}{A} \]

As you move away from the source, the intensity drops off

\[ I = \frac{P}{A} = \frac{P}{4\pi r^2} \]

**Two types of waves**
- **Transverse** – the motion of the particles in the medium are perpendicular to the direction of propagation of the wave. (wave in string, electromagnetic)
- **Longitudinal** – the motion of the particles in the medium are along the same line as the direction of the wave. (sound)
In a sound wave, there are

- **Compressions** – Regions where air is slightly more dense than usual.
- **Rarefactions** – Regions where the air is slightly less dense than usual.

Some waves can have both transverse and longitudinal behavior. Water waves for example.

For a wave on a string, the speed of the wave is

\[ v = \sqrt{\frac{F}{\mu}} \]

\( F \) is the tension in the string and \( \mu \) is the linear density of the string. The linear density is the total mass of the string divided by its length,

\[ \mu = \frac{m}{L} \]

We will find similar expressions for the speed of a wave in other media. They always involve the square root of a fraction. The numerator of the fraction involves a restoring force and the denominator involves a measure of inertia. The particulars will be different for different types of waves or media.

\[ v = \sqrt{\frac{\text{Restoring Force}}{\text{Inertia}}} \]

The text summarizes the situation near the top of page 407:

**More restoring force makes faster waves; more inertia makes slower waves.**

The speed at which a wave propagates is not the same as the speed at which a particle in the medium moves. The speed of propagation of the wave \( v \) is the speed at which the pattern moves along the string in the \( x \)-direction. If the string is uniform, the speed \( v \) is constant. A point on the string vibrates up and down in the \( y \)-direction with a different speed that constantly changes.

**Wave Parameters**

- **Period** \( (T) \) – While staring at a point in the wave, how long does it take for the wave to repeat itself.
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- **Frequency** \( f \) – The number of times the wave repeats per unit time. The inverse of the period.
- **Wavelength** \( \lambda \) – While looking at a photograph of the wave, it is the distance along the wave where the pattern will repeat itself.
- **Amplitude** \( A \) – The furthest from equilibrium for the wave.

A hugely important relationship for waves is

\[
v = f\lambda
\]

Since the speed of the wave is determined by the properties of the medium, it is impossible to change both the frequency and wavelength independently. A high frequency wave must have short wavelengths and a long wavelength wave must have a low frequency.

In harmonic waves, the disturbance can be described by a sinusoidal function. For a harmonic wave on a string, every point on the string move in simple harmonic motion with the same amplitude and frequency, although different points reach their maximum displacements at different times. The maximum speed and maximum acceleration of a point along the wave are

\[
v_m = \omega A \quad \text{and} \quad a_m = \omega^2 A
\]

The larger the amplitude of the wave, the more energy it possesses. It can be shown that

**the intensity of a wave is proportional to the square of its amplitude.**

**Mathematical Description of a Wave**

For a harmonic (sinusoidal) wave traveling at speed \( v \) in the positive \( x \) direction (not pictured to the right!)

\[
y(x,t) = A\cos[\omega(t - x/v)]
\]


The equation can be rewritten as
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\[ y(x,t) = A\cos(\omega t - kx) \]

where the wavenumber (not the spring constant), \( k \) is

\[ k = \frac{2\pi}{\lambda} \]

and

\[ v = f\lambda = \frac{\omega}{k} \]

The argument of the cosine function, \((\omega t \pm kx)\), is called the phase of the wave. Phase is measured in radians. The phase of a wave at a given point and at a moment of time tells us how far along that point is in the repeating pattern of its motion. If the phase of two points along the wave have phases that differ by \(2\pi n\) radians, where \(n\) is an integer, they move in the same way since

\[ \cos(\theta + 2\pi n) = \cos\theta \]

Two points that satisfy this condition are said to be in phase.

The phase of the wave tells us which direction the wave is travelling.

\[ y(x,t) = A\cos(\omega t - kx) \]

describes a wave traveling in the +x direction and

\[ y(x,t) = A\cos(\omega t + kx) \]

describes a wave traveling in the −x direction.

The properties of a wave can be understood better by graphing the wave.

Problem 33 A sine wave is traveling to the right on a cord. The lighter line in the figure represents the shape of the cord at \(t = 0\); the darker line the shape of the cord at time \(t = 0.10\) s. (Note that the horizontal and vertical scales are different.) What are (a) the amplitude and (b) the wavelength of the wave? (c) What is the speed of the wave? What are (d) the frequency and (e) the period of the wave?
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(a) The amplitude corresponds to the largest (or smallest value of \( y \))

\[ A = 2.6 \text{cm} \]

(b) The wavelength is the distance it takes for the pattern to repeat itself. Looking at peak to peak distance in the lighter plot

\[ \lambda = 19.5 \text{m} - 5.5 \text{m} = 14 \text{m} \]

(c) The speed of the wave is the distance traveled divided by the time. Use the distance between the adjacent peaks on the lighter and darker plots,

\[ v = \frac{\Delta x}{\Delta t} = \frac{7.5 \text{m} - 5.5 \text{m}}{0.10 \text{s}} = 20 \text{m/s} \]

(d) The frequency can be found from the speed and the wavelength,

\[ f = \frac{v}{\lambda} = \frac{20 \text{m/s}}{14 \text{m}} = 1.43 \text{Hz} \]

(e) The period is the inverse of the frequency

\[ T = \frac{1}{f} = \frac{1}{1.43 \text{Hz}} = 0.70 \text{s} \]

The traveling wave can also be expressed as

\[ y(x,t) = A \sin(\omega(t \pm x/v)) = A \sin(\alpha t \pm kx) \]

Again, the \(-\) corresponds to waves traveling in the +\( x \) direction and the + corresponds to waves traveling in the \(-x \) direction.

**Superposition of Waves**

Suppose that two wave of the same type pass through the same region of space. Do they affect each other? Let’s try a simple test, everyone start talking. Do other waves affect your wave?

If the displacements in the wave are not too great, the disturbances do not effect each other and the waves pass through each other.

**Principle of Superposition**

When two or more waves overlap, the net disturbance at any time is the sum of the individual disturbances due to each wave.
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Reflection
At an abrupt boundary between one medium and another, **reflection** occurs.

- If the end of the string is a fixed point, the reflected wave is inverted.
- If the speed of the wave decreases (a light string is tied to a heavy string) the reflected wave is inverted.
- If the speed of the wave increases (a heavy string is tied to a light string) the reflected wave has the same orientation as the incident wave.

Mathematically complicated but with good animations:

Below is an illustration of the first bullet point.

When the medium changes, the speed of the wave will change. Since
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\[ \nu = f \lambda \]

what changes, the frequency of the wave, its wavelength or both? The frequency is a measure of the up and down motion of the wave. The up and down motion in one media causes the up and down motion in the other media. Hence, the frequency is the same from one media into another. Therefore, the wavelength must change when the wave’s speed changes as it passes into another medium.

**Refraction**
When the wave travels from one medium into another, the direction of the wave will change. In the picture below, the direction of the wave is given by the black arrow. Going from higher speed to lower speed, the wave bends towards the normal to the boundary. The speed of the wave is implied by the spacing between the wave crests.

**Interference**
Interference is a consequence of the principle of superposition. The waves need to be **coherent** – same frequency and maintain a constant phase relationship. (For incoherent waves the phase relation varies randomly.)

- **Constructive interference** occurs when the waves are in phase with each other. The amplitude of the resulting wave is the sum of the amplitudes of the two waves, \( |A_1 + A_2| \)

- **Destructive interference** occurs when the waves are \( 180^\circ \) out of phase with each other. The amplitude of the resulting wave is the difference of the amplitudes of the two waves, \( |A_1 - A_2| \)
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- Otherwise the wave has amplitude between $|A_1 - A_2|$ and $(A_1 + A_2)$.

The two rods vibrate and down in phase and produce circular water waves. If the waves travel the same distance to a point they arrive in phase with each other and interfere constructively. At other points, the phase difference is proportional to the path difference. Since one wavelength of path difference corresponds to a phase difference of $2\pi$ radians,

$$\frac{d_1 - d_2}{\lambda} = \frac{\text{phasedifference}}{2\pi \text{rad}}$$

If the path difference $d_1 - d_2 = n\lambda$ ($n$ is any integer) the phase difference is $2\pi n$ rad and constructive interference occur at $P$. If the path difference $d_1 - d_2 = \lambda/2, 3\lambda/2, 5\lambda/2,$ etc., the phase difference is $\pi, 3\pi, 5\pi$, etc. and destructive interference occurs.

When coherent waves interfere, the amplitudes add for constructive interference and subtract for destructive interference. Since intensity is proportional to the square of the amplitude, you cannot simply add or subtract the intensities of the coherent waves when they interfere. For incoherent waves, there is no fixed phase relation. The total intensity is the sum of the intensities of the individual waves.

**Diffraction**

Diffraction is the spreading of waves around an obstacle. The obstacle must be similar in size to the wavelength of the wave for the effect to be noticeable.

Many animations are on the web. [http://www.youtube.com/watch?v=uPQM12q_vPQ](http://www.youtube.com/watch?v=uPQM12q_vPQ)