Ex I Solutions

1.

(a) N: total # of rods.

N_f: # of rods heading forward

N_b: # of rods heading backward

$\Big\}\ N = N_f + N_b.$

$$x = (N_f - N_b)a = (2N_f - N)a$$

$$\therefore\quad N_f = \frac{1}{2}\left(N + \frac{x}{a}\right) \quad \text{and} \quad N_b = \frac{1}{2}\left(N - \frac{x}{a}\right).$$

# of microstates with the end-to-end length x : $\Omega(x)$

$$\Omega(x) = \frac{N!}{N_f!\,(N-N_f)!} = \frac{N!}{\left[\frac{1}{2}(N+\frac{x}{a})\right]!\,\left[\frac{1}{2}(N-\frac{x}{a})\right]!}$$

$$S(x) = k_B \ln \Omega(x)$$

$$= k_B \ln \frac{N!}{\left[\frac{1}{2}(N+\frac{x}{a})\right]!\,\left[\frac{1}{2}(N-\frac{x}{a})\right]!}$$

$$\simeq k_B \left\{ \ln N! - \ln\left[\frac{1}{2}\left(N+\frac{x}{a}\right)\right]! - \ln\left[\frac{1}{2}\left(N-\frac{x}{a}\right)\right]! \right\}$$

$$\simeq k_B \Big\{ N\ln N - \cancel{N} - \frac{1}{2}\left(N+\frac{x}{a}\right)\ln\left(N+\frac{x}{a}\right) - \frac{1}{2}\left(N+\cancel{\frac{x}{a}}\right)\ln\left(\frac{1}{2}\right) + \frac{1}{2}\left(\cancel{N}+\frac{x}{a}\right)$$
$$- \frac{1}{2}\left(N-\frac{x}{a}\right)\ln\left(N-\frac{x}{a}\right) - \frac{1}{2}\left(N-\cancel{\frac{x}{a}}\right)\ln\left(\frac{1}{2}\right) + \frac{1}{2}\left(\cancel{N}-\frac{x}{a}\right) \Big\}$$

$$= k_B \Big\{ \cancel{N\ln N} + N\ln 2 - \frac{1}{2}\left(\cancel{N}+\cancel{\frac{x}{a}}\right)\ln N - \frac{N}{2}\left(1+\frac{x}{Na}\right)\ln\left(1+\frac{x}{Na}\right)$$
$$- \frac{1}{2}\left(\cancel{N}-\cancel{\frac{x}{a}}\right)\ln N - \frac{N}{2}\left(1-\frac{x}{Na}\right)\ln\left(1-\frac{x}{Na}\right) \Big\}$$

$$= N k_B \left\{ \ln 2 - \frac{1}{2}\left(1+\frac{x}{Na}\right)\ln\left(1+\frac{x}{Na}\right) - \frac{1}{2}\left(1-\frac{x}{Na}\right)\ln\left(1-\frac{x}{Na}\right) \right\}$$

(b) $dU = TdS + fdx$,

$dT=0 \rightarrow dU=0 \quad \therefore \; f = -T\left(\frac{\partial S}{\partial x}\right)_T$.

$$f = -Nk_BT\left(\frac{\partial}{\partial x}\right)\left[\ln 2 - \frac{1}{2}\left(1+\frac{x}{Na}\right)\ln\left(1+\frac{x}{Na}\right) - \frac{1}{2}\left(1+\frac{x}{Na}\right)\ln\left(1-\frac{x}{Na}\right)\right]$$

$$= Nk_BT\left\{\frac{1}{2Na}\ln\left(1+\frac{x}{Na}\right) + \cancel{\frac{1}{2Na}} - \frac{1}{2Na}\ln\left(1-\frac{x}{Na}\right) - \cancel{\frac{1}{2Na}}\right\}$$

$f \approx \frac{k_BT}{Na^2}x \quad : \; \ln\left(1\pm\frac{x}{Na}\right) \approx \pm\frac{x}{Na}$ for $\frac{x}{Na} \ll 1$.

Hooke's law

(c) Isothermal $dU=0$.

$dQ_{rev} = T_0 dS = -f dx$.

$$\therefore \Delta Q = -\int_{x_0}^{2x_0}\frac{k_BT_0}{Na^2}x\,dx$$

$$= -\frac{k_BT_0}{2Na^2}x^2\Big|_{x_0}^{2x_0} = -\frac{3}{2}\frac{k_BT_0}{Na^2}x_0^2 < 0.$$

2. $\varepsilon_n = \left(\frac{1}{2}+n\right)\hbar\omega$, $n = 0, 1, 2, \cdots$

(a) $$Z_1 = \sum_{n=0}^{\infty} e^{-\beta\left(\frac{1}{2}\hbar\omega + n\hbar\omega\right)} = e^{-\frac{\beta}{2}\hbar\omega} \cdot \frac{1}{1-e^{-\beta\hbar\omega}}$$

$$= \frac{e^{\frac{1}{2}\hbar\omega\beta}}{e^{\beta\hbar\omega}-1}$$

$$Z_t = Z_1^N = \left(\frac{e^{-\frac{1}{2}\hbar\omega\beta}}{-e^{-\beta\hbar\omega}+1}\right)^N$$

$\uparrow$

distinguishable and non-interacting

(b) $$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z_t$$

$$= -N\left(\frac{\partial}{\partial\beta}\right)\left\{-\frac{1}{2}\hbar\omega\beta - \ln\left(-e^{-\beta\hbar\omega}+1\right)\right\}$$

$$= \frac{N}{2}\hbar\omega + \frac{N\hbar\omega\, e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$= N\left(\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right).$$

(c) $$C = \frac{\partial\langle E\rangle}{\partial T} = -k_B\beta^2\left(\frac{\partial}{\partial\beta}\right)\langle E\rangle$$

$$= -k_B\beta^2 N\hbar\omega\left(\frac{\partial}{\partial\beta}\right)\frac{1}{e^{\beta\hbar\omega}-1}$$

$$= k_B\beta^2 N\hbar\omega\,\frac{\hbar\omega\, e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}$$

$$= N\frac{\hbar^2\omega^2}{k_B T^2}\,\frac{e^{\hbar\omega/k_BT}}{\left(e^{\hbar\omega/k_BT}-1\right)^2}$$

(d) Each 1-D H.O. $\varepsilon = \underbrace{\frac{1}{2}kx^2 + \frac{p^2}{2m}}_{\text{2 Degrees of freedom.}}$

$$\therefore \langle \varepsilon \rangle = 2 \cdot \frac{1}{2} k_B T = k_B T.$$

$$\therefore \langle E \rangle = N k_B T \rightarrow C = \frac{\partial \langle E \rangle}{\partial T} = \underline{\underline{N k_B}}$$

indep. of T.

From (c)

$$C = N \frac{\hbar^2 \omega^2}{k_B T^2} \frac{e^{\hbar\omega/k_B T}}{(e^{\hbar\omega/k_B T} - 1)^2}.$$

In the classical limit, $\hbar\omega/k_B T \ll 1$.

$$\therefore C \approx N \frac{\hbar^2 \omega^2}{k_B T^2} \cdot \frac{1}{(\cancel{1} + \frac{\hbar\omega}{k_B T} - \cancel{1})^2} = N k_B.$$

3. $\langle \varepsilon_H \rangle = 3 \times \frac{1}{2} k_B T = \frac{3}{2} k_B T = 1.5 \text{ eV}.$

$\therefore k_B T = 1 \text{ eV}.$ $P(n) \propto e^{-13.6/n^2}$

$\varepsilon_3 = -\frac{13.6}{3^2} = -1.51$

$\varepsilon_1 = -13.6.$

$$\frac{P(n=3)}{P(n=1)} = \frac{e^{-\varepsilon_3/1.0}}{e^{-\varepsilon_1/1.0}}$$

$$= e^{-(-1.51+13.6)}$$

$$\approx 5.6 \times 10^{-6}.$$

Boltzmann Distribution.

4. Only (II)