HW C Solutions

1. (21.1)

$$\sum_{k} \longrightarrow \left(\frac{L}{2\pi}\right)^D \int d^D k.$$

For $D=2$,

$$\left(\frac{L}{2\pi}\right)^2 \int d^2 k = \int \frac{A}{2\pi} k \, dk \quad \leftarrow g_{2D}(k).$$

$$\therefore Z_1 = \int_0^\infty g_{2D}(k)\, e^{-\frac{\hbar^2 k^2}{2m}\beta} dk.$$

$$= \frac{A}{2\pi}\int_0^\infty k e^{-\frac{\hbar^2 k^2}{2m}\beta} dk. \qquad \left(\begin{array}{l} \frac{\hbar^2\beta}{2m} k^2 \equiv y \\ \frac{\hbar^2\beta}{m} k\, dk = dy \end{array}\right)$$

$$= \frac{A}{2\pi}\frac{m}{\hbar^2\beta} \underbrace{\int_0^\infty e^{-y} dy}_{=1}$$

$$= \frac{A}{\lambda_{th}^2} \quad \text{where } \lambda_{th} = \frac{h}{\sqrt{2\pi m k_B T}}.$$

2. (21.4)

$$Z_{atom} = \sum_{i=1,2} e^{-\beta E_i} = \underbrace{e^{-0\cdot\beta} + e^{-0\cdot\beta} + \cdots + e^{-0\cdot\beta}}_{g_1} + \underbrace{e^{-\Delta\beta} + \cdots + e^{-\Delta\beta}}_{g_2}$$

$$= g_1 + g_2 e^{-\Delta\beta}.$$

$$U_{atom} = -\frac{\partial}{\partial\beta}\ln Z_{atom} = -\frac{-\Delta g_2 e^{-\beta\Delta}}{g_1 + g_2 e^{-\beta\Delta}} = \frac{\Delta g_2 e^{-\beta\Delta}}{g_1 + g_2 e^{-\beta\Delta}}.$$

$$C_V = \left(\frac{\partial U_{atom}}{\partial T}\right) = -\frac{1}{k_B T^2}\frac{\partial}{\partial\beta} U_{atom}$$

$$= \frac{g_1 g_2}{k_B T^2} \cdot \frac{\Delta^2 e^{-\beta\Delta}}{(g_1 + g_2 e^{-\beta\Delta})^2}$$

$$= k_B g_1 g_2 \Delta^2 \frac{\beta^2 e^{-\beta\Delta}}{(g_1 + g_2 e^{-\beta\Delta})^2}$$

$$Z = (Z_{atom})^N Z_N \quad \text{where} \quad Z_N = \left(\frac{1}{N!}\right)\left(\frac{V}{\lambda_{th}^3}\right)^N$$

$$U = -\frac{\partial}{\partial\beta}\ln Z = \underbrace{-N\left(\frac{\partial}{\partial\beta}\right)\ln Z_{atom}}_{U_{atom}} \underbrace{-\left(\frac{\partial}{\partial\beta}\right)\ln Z_N}_{U_N}.$$

$$= N U_{atom} + U_N.$$

$$\therefore\ C = \left(\frac{\partial U}{\partial T}\right) = NC_a + C_N. \quad \text{where} \quad C_N = \frac{3}{2}Nk_BT.$$

$$= N\left(\frac{3}{2}k_B + C_a\right).$$

3. (21.6)

Just shift the ground state energy

Then, $Z_1 \rightarrow \frac{V}{\lambda_{th}^3}e^{\beta R}$

4. (22.1)

$S = -k_B \sum_i P_i \ln P_i$: Maximize S with constraints

$$\sum_i P_i = 1$$

$$\sum_i P_i E_i = U$$

$$\sum_i P_i N_i = N$$

$$S = S(p_1, p_2, \cdots)$$

$$g_1(p_1, p_2 \ldots) = \sum_i P_i - 1 \quad ; \text{ Normalization}$$

$$g_2(p_1, p_2, \ldots) = \sum_i P_i E_i - U \quad ; \text{ Energy Cons.}$$

$$g_3(p_1, p_2, \ldots) = \sum_i P_i N_i - N.$$

$$\delta[S - \lambda_1 g_1 - \lambda_2 g_2 - \lambda_3 g_3] = 0.$$

$$\sum_i \frac{\partial}{\partial p_i}\left[-k_B P_i \ln P_i - \lambda_1 P_i - \lambda_2 P_i E_i - \lambda_3 P_i N_i\right] = 0$$

Therefore, each term should be zero,

$$\frac{\partial}{\partial p_j}\left[-k_B p_j \ln p_j - \lambda_1 p_j - \lambda_2 p_j E_j - \lambda_3 p_j N_j\right] = 0$$

$$-\ln p_j - 1 - \lambda_1 - \lambda_2 E_j - \lambda_3 N_j = 0$$

$$\therefore\ p_j = \frac{e^{-\lambda_2 E_j} e^{-\lambda_3 N_j}}{e^{1+\lambda_1}}$$

By identifying $e^{1+\lambda_1} = Z$, $\lambda_2 = \beta$, and $\lambda_3 = \mu\beta$.

$$p_j = \frac{1}{Z} e^{-\beta(E_j - \mu N_j)}.$$

5. (22.5)

$$F = -k_B T \ln Z_N \approx -k_B T N \ln Z_1 + k_B T (N \ln N - N)$$

$$dF = -SdT - pdV + \mu dN$$

$$\therefore\ \mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -k_B T \ln\left(\frac{Z_1}{N}\right).$$

6. $H \rightleftharpoons p^+ + e^-$

(a) Read Text 22.8.

$$\mu_H = \mu_p + \mu_e \quad \text{———} (*)$$

$$Z_H = \frac{V}{(\lambda_{th}^{H})^3} e^{\beta R}$$: You know this from the previous prob.

$$\therefore\ \mu_H = -k_B T \ln\left(\frac{Z_H}{N_H}\right)$$

$$\mu_p = -k_B T \ln\left(\frac{Z_p}{N_p}\right) \quad ; \quad Z_p = \frac{V}{(\lambda_{th}^{p})^3}$$

$$\mu_e = -k_B T \ln\left(\frac{Z_e}{N_e}\right) \quad ; \quad Z_e = \frac{V}{(\lambda_{th}^{e})^3}$$

where $$\lambda_{th}^{\square} = \frac{h}{\sqrt{2\pi m_\square k_B T}}.$$

Following Eq. (*).

$$-\cancel{k_B T} \ln \frac{Z_p^*}{N_p} = \cancel{k_B T} \ln \frac{Z_e^*}{N_e} = \cancel{k_B T} \ln \frac{Z_H^*}{N_H}.$$

Knowing $Z_H = Z_1^H e^{\beta R}$.

$$e^{-\beta R} = \frac{Z_1^H}{N_H} \frac{N_e N_p}{Z_p Z_e}.$$

$$= \left(\frac{V}{N_H}\right)\left(\frac{N_e}{V}\right)\left(\frac{N_p}{V}\right)\left(\frac{\cancel{m_H}}{\cancel{m_p} m_e}\right)^{3/2} h^3 \frac{1}{(2\pi k_B T)^{3/2}}$$

$$m_H \approx m_p.$$

$$\therefore e^{-\beta R} \cdot \frac{(2\pi m_e k_B T)^{3/2}}{h^3} = \frac{n_e n_p}{n_H} \quad ; \quad \left(\frac{N_a}{V}\right) = n_a$$

(b) $y = n_p/n_H$: degree of ionization.

$$n = n_H + n_p. \longrightarrow \underset{= n_e}{n_p = yn}, \quad n_H = (1-y)n.$$

Plug these in Eq (22.96).

$$\frac{n_e n_p}{n_H} = \frac{y^2 n^2}{(1-y)n} = \left(\frac{y^2}{1-y}\right) \cdot n = \frac{(2\pi k_B T m_e)^{3/2}}{h^3} e^{-\beta R}.$$

$$\therefore \frac{y^2}{1-y} = \frac{1}{n} \frac{e^{-\beta R}}{(\lambda_{th}^e)^3}$$

At $T \approx 1000\,K$, $\lambda_{th}^e = \frac{h}{\sqrt{2\pi m_e k_B T}} \approx 2.4 \times 10^{-19}$ m.

$$\therefore \frac{y^2}{1-y} = \frac{1}{n} e^{-13.6/0.025 \times 3.3} \approx 2 \times 10^{-63}$$

for $n \approx 10^{20}\ m^{-3}$.

$$y \ll 1, \quad y^2 \approx 2 \times 10^{-63}$$

and $y \approx 5 \times 10^{-31}$

7. (23.1).

Total radiation power from the Sun : R_S

$R_S = 4\pi r_S^2 \sigma T_S^4$ where r_S and T_S are the radius and temperature of the Sun.

Then the earth absorbs

$\alpha_e = (1-A)\frac{\pi r_e^2}{4\pi D^2}\cdot R_S$. where D is the distance between S & E.

The earth radiates.

$R_e = 4\pi r_e^2 \sigma T_e^4$.

In steady state, $\alpha_e = R_e$.

$(1-A)\frac{r_e^2 r_s^2}{4D^2}(4\pi)\sigma T_s^4 = 4\pi r_e^2 \sigma T_e^4$

$\therefore\ T_e = T_s(1-A)^{1/4}\sqrt{\frac{r_s}{2D}}$.

8. (23.5).

(a) $dU = TdS - pdV$.

$U \propto V$.

$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p$.

$\therefore\ s = \frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{p}{T} = \frac{u}{T} + \frac{u}{3T} = \frac{4}{3}\frac{u}{T} = 4\frac{p}{T}$.

(b) $G = U + pV - TS$

$= uV + \frac{u}{3}V - 4pV = 0$. $\therefore\ \mu = 0$

(c) $\cancel{s/=}$

$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial}{\partial T}\right)_V(3Vp) = 3V\left(\frac{\partial p}{\partial T}\right)_V = 3V\left(\frac{\partial S}{\partial V}\right)_T = 3V s$

Maxwell's relation.

$\therefore\ C_v = \frac{C_V}{V} = 3s$ → Entropy density

(d) Since $G = 0$, $dG = 0$.

$\therefore SdT = Vdp$

Therefore, if $dp = 0$ then $dT = 0$.

9. (23.3)

(a) $T_u = 2.73\ K$

$$u = \frac{4\sigma}{c} T^4 = \frac{4 \times 5.67 \times 10^{-8}}{3 \times 10^8} (2.73)^4 \approx 4 \times 10^{-14}\ J/m^3$$

Do (c) first!

(c) $P = \sigma T^4 = 3.1\ \mu W/m^2$

Area of your palm $10 \times 10\ cm^2 = 0.01\ m^2$

Michael Jordan probably has $\sim 0.03\ m^2$.

$\therefore 3.1 \times 10^{-2}\ \mu W = 31\ nJ/s$ on your palm.

(b) Average Energy/photon $= 2.7\ k_B T \approx 10^{-22}\ J$.

# of photons on your palm $= 31 \times 10^{-9} / 10^{-22}$

$\approx 3 \times 10^{14}$ photons.

(d) $p = \frac{u}{3} = \frac{4}{3} \times 10^{-14}\ J/m^3 \approx 10^{-14}\ Pa = 10^{-19}$ bar.

10. (23.6)

$$Z_\omega = \sum_{n=0}^{\infty} e^{-\beta n \hbar \omega} = \frac{1}{1 - e^{-\beta\hbar\omega}} \text{ for a single mode.}$$

$$\mathcal{Z} = \prod_\omega Z_\omega$$

continuum approx.

$$\therefore \ln \mathcal{Z} = \sum_\omega \ln Z_\omega \xrightarrow{} \int_0^\infty d\omega\, \mathcal{D}(\omega) \ln\left[\frac{1}{1 - e^{-\beta\hbar\omega}}\right]$$

$$\mathcal{D}(\omega)\, d\omega = \frac{V\omega^2}{\pi^2 c^3}\, d\omega$$

$$\therefore \ln Z = -\frac{V}{\pi^2 c^3} \underbrace{\int_0^\infty d\omega\, \omega^2 \ln\left[1 - e^{-\beta\hbar\omega}\right]}_{I}$$

$$I = \left[\frac{\omega^3}{3} \underbrace{\ln(1-e^{-\beta\hbar\omega})}_{\to 0}\right]_0^\infty - \int_0^\infty \frac{\omega^3}{3} \frac{\beta\hbar e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}\, d\omega$$

$$= \frac{1}{3(\beta\hbar)^3} \boxed{\int_0^\infty \frac{x^3\, dx}{1-e^{-x}}} = \frac{\pi^4}{15} \text{ as we did in class}$$

$$\therefore \ln Z = \frac{V\pi^2 (k_B T)^3}{45\hbar^3 c^3}$$

$$F = -k_B T \ln Z = -\frac{4\sigma V}{3c} T^4 \quad \left(\sigma \equiv \frac{\pi^2 k_B^4}{60 c^2 \hbar^3}\right)$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{16\sigma V}{3c} T^3$$

$$U = F + ST = 4\frac{\sigma V}{3c} T^4$$

$$p = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{4\sigma}{3c} T^4$$

$$\therefore \quad U = -3F \quad \text{and} \quad \underline{pV = \frac{U}{3}}$$