Prepared for submission to JCAP

Primordial Gravitational Waves: Theory and Detection

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Abstract. It is believed that primordial tensor perturbations in the form of gravitational waves is one the consequences of inflationary paradigm. The primordial gravitational waves were produced during inflation because of the quantum fluctuations and were freely propagating until they reach us today, while interacting very little with other elements of the universe. Therefore, providing us a new window to look at the universe. In this paper, we discuss the emergent of primordial gravitational waves from first principles, and then talk about the detection possibility of them and challenges along the way.

Contents

1 Homogeneous Universe

From the observational standpoint, the universe is homogeneous and isotropic on large scales ($\sim$ 100Mpc). This is cosmological principle: there is no preferred direction or position in the 3-dimensional space. Comoving observers, are the frames with no peculiar motion with respect to Hubble flow. In fact, comoving frames, are free falling under Hubble flow of the universe. Geometry of an expanding homogeneous and isotropic universe in a comoving frame is given by the Friedman-Robertson-Walker (FRW) metric[5]:

$$ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2(\theta,\phi)\right), \tag{1.1}$$

where $a(t)$ is the scale factor, usually normalized such that $a(t_0) = 1$. The metric 1.1 is expressed in terms of *comoving* coordinates r, θ, ϕ that do not change with space or time. Parameter K is the curvature of the universe, which has three possible values: -1 for an open universe, 0 for a flat one and $+1$ for a closed universe. K can be precisely measured from the acoustic peaks of CMB power spectrum, which confirm that the universe if flat.

One of the immediate consequences of expansion of the universe, is the redshift effect. Simply, the photons need to overcome the expansion of the universe to travel from place to place. Therefore, they lose energy and thus, decreasing their frequency. For the null geodesics of 1.1 it is easy to show:

$$z \equiv \frac{\Delta\nu}{\nu_0} = \frac{1}{1+a}, \tag{1.2}$$

where ν_0 is the lab frequency of the photon's absorption line.

The universe is filled with uniform cosmic fluid such as CMB photons, dark matter, baryonic matter and dark energy. We assume that the cosmic fluid is of the form of perfect fluid, which means there is no dissipation. Also, the fluid is assumed to be rotationally invariant and inviscid. Therefore, the cosmic fluid is obeying the perfect fluid energy momentum tensor. At any given point, the fluid has energy density $\rho(t)$, pressure $p(t)$ and four velocity $u^\mu(t)$:

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu. \tag{1.3}$$

The dynamic of the universe is solely encoded in $a(t)$, since it is the only parameter that depends on time. We can find its dynamic with Einstein field equations: $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.[1] Because of the symmetries of the metric tensor, there are only two independant equations. Because of the cosmological principle, all the vector equations (i.e $G_{0i} \sim T_{0i}$) vanish. $G_{00} \sim T_{00}$ will yield the first Friedman equation and tensor sector $G_{ij} \sim T_{ij}$ will give us the second Friedman equation:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{1.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{p}{3}\right). \tag{1.5}$$

Here we have defined Hubble parameter as $H \equiv \frac{\dot{a}}{a}$. The Hubble parameter was first measured by Edwin Hubble in the late 1920s, and describes how fast the universe is expanding. Hubble found an experimental correlation between redshift of the galaxies and their distance. He wrote the relation as $z = H_0 d$, where H_0 is the Hubble parameter at present time, commonly known as Hubble constant.

H^{-1} would be the age of the universe if it has been expanding at a constant rate. Although the universe is not expanding with constant rate, H^{-1} is a time scale of the universe which we call *Hubble time*. We also can define the length scale of the universe as cH^{-1} which we call *Hubble Horizon*, or simply horizon. The notion of horizon plays and important role in the cosmological perturbation theory [2]. The comoving Hubble horizon is therefore simply $(aH)^{-1}$.

The solutions of Friedmann equations 1.4 and 1.5 depend on the composition of the universe. Today we know the universe consists of several components such as radiation (mostly photons, but neutrinos included as well.), dark matter (the form of matter that only interacts gravitationally), baryonic matter (starts, galaxies,...) and dark energy (which is responsible for the late accelerated expansion.). Each component has its own equaton of state, which is the relation between pressure and density of the fluid. The equations of state is written as:

$$p = \rho\omega. \tag{1.6}$$

1.6 Simply adds a new equation to the system of Friedmann equations. For matter component (DM+BM) we can confidently put $\omega = 0$ since the matter is cold and collisionless, therefore the pressure of the matter is zero. Assuming the black body distribution, we can find $\omega = \frac{1}{3}$ for radiation. In 1998 The () team found out that the universe is expanding with positive

[1] Details of the calculations can be found in many cosmology textbooks, so we do not bring them here.

[2] e.g. when we talk about the evolution of perturbations in the next sections.

acceleration using the supernova data. Looking at 1.5, we see the condition for accelerated expansion is:

$$\rho + 3p < 0 \longrightarrow \omega < -\frac{1}{3}, \tag{1.7}$$

which means that the dark energy must violate the strong energy condition. As far as the measurements show now, the dark energy is a cosmological constant with $\omega = -1$.

The Friedmann equation 1.4 can easily be solved as [7]:

$$a(t) = \begin{cases} t^{2/3(1+\omega)}, & \omega > -1 \\ e^{H_0 t}, & \omega = -1 \end{cases}. \tag{1.8}$$

It is usually more convenient to express the quantities in terms of dimensionless variables. We First define critical density, as the density of the universe when it is flat:

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{1.9}$$

Now we define the density parameter as the ratio of present density to present critical density,$\rho_c^0 = \frac{3H_0^2}{8\pi G}$:

$$\Omega = \frac{\rho_0}{\rho_c^0}. \tag{1.10}$$

We assign a density parameter to each component in the universe. 1.4 can be written as:

$$H^2 = H_0^2 \left(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_K a^{-2}\right) \tag{1.11}$$

From 1.10 it is clear that:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_K = 1, \tag{1.12}$$

where $\Omega_K = -\frac{K}{a_0^2 H_0^2}$.

Lastest measurements of cosmological parameters imply that the universe is mostly filled with dark energy in the form of cosmological constant ($\Omega_\Lambda \sim 0.7$). About 30% of energy density belongs to matter ($\Omega_m \sim 0.3$). The rest (under 1%) belongs to radiation sector and the universe is flat [11].

2 Slow roll inflation

2.1 Why Inflation?

In the standard cosmological paradigm, below the Planck energy scale, the universe underwent a rapid accelerated expansion, called inflation. the inflation era is necessary to explain some of the issues with hot Big Bang model, such as horizon problem, flatness problem and also, provides it with a natural process in which we can describe anisotropies in the cosmic microwave background (CMB) and structure formation of the large scale structure (LSS) of the universe.

2.1.1 Horizon Problem

Let's start discussing the horizon problem by introducing conformal time. Conformal transformations, are the ones that transform the light cone angle back to 45°. In the case of FRW metric, it is easy to do such transformation: we just need to factorize the scale factor. Therefore, the metric 1.1 can be written as:

$$ds^2 = a^2(t)\left(-c^2 d\tau^2 + \left(d\chi^2 + S_k(\chi) d\Omega^2(\theta,\phi)\right)\right), \tag{2.1}$$

where $\chi = \int \frac{dr}{\sqrt{1-Kr^2}}$ and $\tau = \int dt/a$ is the conformal time. The particle horizon is defined as the maximum comoving distance light can travel between τ_1 and $\tau_2 > \tau_1$. From 2.1 is it clear that the particle horizon from $t_i = 0$ is:

$$\chi_{\mathrm{PH}}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)}. \tag{2.2}$$

To address the horizon problem, let's look at 1.

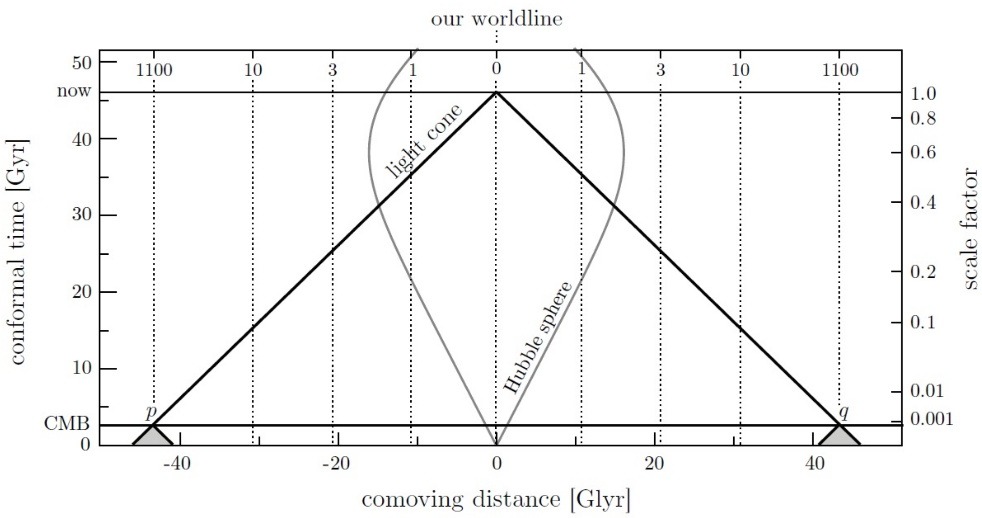

Figure 1. Horizon problem addresses the fact that we should not be able to observe the CMB isotropic unless there was a primordial fine-tuning.[7]

The vertical axis shows the conformal time and the horizontal one is comoving distance. Our past lightcone is shown with thick 45° lines. The last scattering surface of the CMB can be seen at the bottom of the figure. We are in fact looking at two opposite directions where p and q are the corresponding points on last scattering surface. As we can see, the particle horizon associated with p and q are shown as their past lightcones. It is clear that these two lightcones do not intersect, and therefore these two points were never in casual contact. How can they have the same temperature as we see in the CMB? It is not possible to accept that these two points were not in casual contact never before and yet have the same temperature unless we accept the existence of extreme fine-tuning at τ_i.

The solution to horizon problem is to extend τ_i back to infinity, meaning we push the singularity back to $-\infty$ which provides us with extra time for casual contact 2.

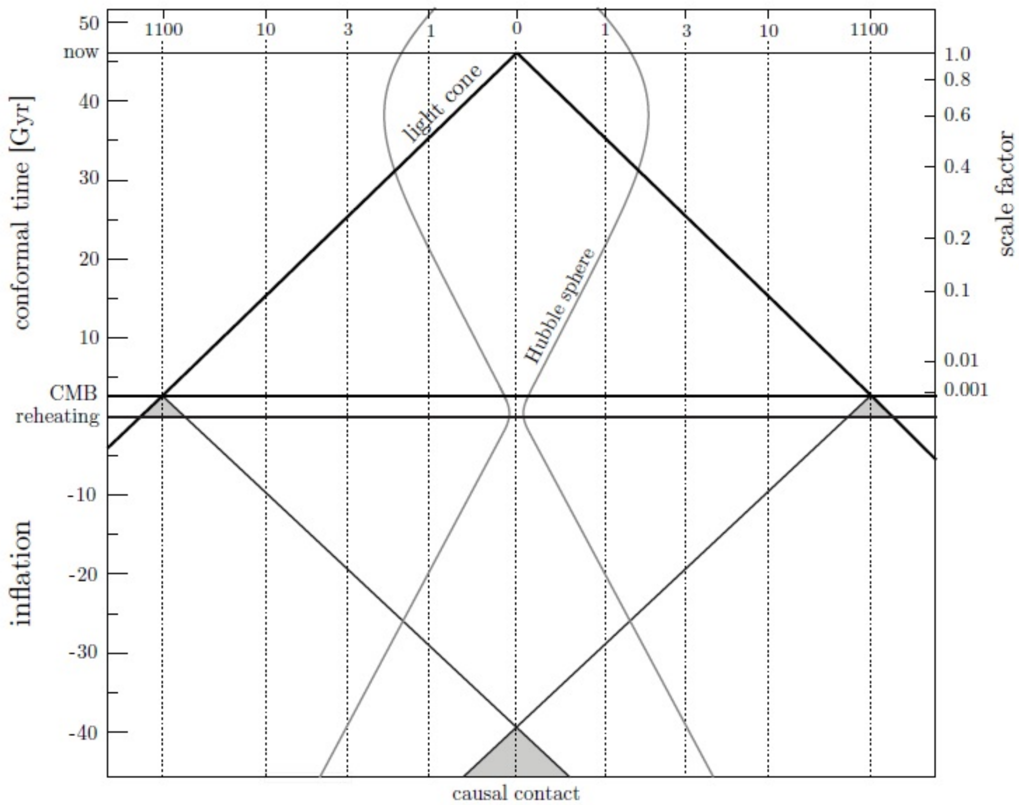

Figure 2. The horizon problem is solved by assuming a period of positive acceleration which violates the SEC condition.[7]

But how are we allowed to do this? Let's assume the universe is filled with a single fluid with equation of state ω. Now let's compute the particle horizon. It is easy to show[7]:

$$\chi_{\rm PH}(a) = \frac{2H_0^{-1}}{1+3\omega}\left[a^{\frac{1}{2}(1+3\omega)} - a_i^{\frac{1}{2}(1+3\omega)}\right]. \tag{2.3}$$

For a standard cosmology with $\omega > -\frac{1}{3}$, it is not possible to push the initial time back to $-\infty$. It is required to violate the SEC to solve the horizon problem. Therefore, solution to horizon problem requires a period of positive acceleration in the early times.

It is worth to mention how the comoving Hubble horizon is behaving in this era. The comoving Hubble horizon can be easily obtained as:

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3\omega)}. \tag{2.4}$$

For a standard matter where $\omega > -\frac{1}{3}$, the Hubble horizon grows with time since $1+3\omega$ is positive. During inflation, however, since the SEC is violated, the Hubble horizon is shrinking. It is worth noting in the late times, when dark energy dominates the universe, the Hubble horizon is shrinking again.

2.1.2 CMB & LSS

Cosmic microwave background is an almost isotropic relic radiation with almost perfect black body spectrum that confirms the Thomson scattering from electrons in the early universe.

The existence of small fluctuations in the temperature map of CMB was first detected by COBE satellite, 1989 [1]. Inflation provides us with a natural process in which these fluctuations can emerge. It is strongly believed that the quantum fluctuations of the inflation field is responsible for producing the temperature fluctuations in the CMB map and density perturbations of the matter distribution on large scales. Explaining these fluctuations is one of the most significant successes of inflation[2].

2.2 Scalar Field Inflation

The inflation is commonly described by a single scalar field, coupled minimally to gravity with Lagrangian density[5]:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi). \tag{2.5}$$

The energy momentum tensor of such theory can be obtained by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial S}{\partial g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial_\lambda \phi \partial^\lambda \phi + V(\phi) \right). \tag{2.6}$$

Energy density and pressure of the energy momentum tensor can be easily found. By analogy of 1.3 we get:

$$\rho_\phi = \frac{1}{2}\dot{\phi} + V(\phi), \tag{2.7}$$

$$p_\phi = \frac{1}{2}\dot{\phi} - V(\phi). \tag{2.8}$$

We already have talked about the conditions for having an accelerated expansion is to have $\omega < -\frac{1}{3}$. This condition can be satisfied if we get $V(\phi) \gg \dot{\phi}$. The dynamic of the field can be found by using Lagrange-Euler theorem:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{2.9}$$

There are two criteria that need to be satisfied:

First, we need to get the accelerated expansion, which means $V(\phi) \gg \dot{\phi}$.

Second, we need to get enough inflation period to solve the horizon problem. This means the acceleration term in 2.9 needs to be much smaller than other terms, $\ddot{\phi} \ll 3H\dot{\phi}, V'(\phi)$.

These two conditions are called *Slow-roll* conditions. They can be expressed in terms of two dimensionless parameters ϵ and η[7]:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{M_{Pl}^2}{2} \left(\frac{V'}{V} \right)^2, \tag{2.10}$$

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = M_{Pl}^2 \left(\frac{V''}{V} \right). \tag{2.11}$$

Inflation conditions means $\epsilon \ll 1$ and $\eta \ll 1$.

The slow-roll condition imposes an important constrain on the shape of the inflationary potential, $V(\phi)$. This condition implies that inflation occurs when the potential is substantially flat. As ϕ increases, $V(\phi)$ should remain constant. Then, at the end of inflation, we need an oscillatory behaviour that guarantees the inflation field decays into standard model particles through a process we commonly call *Reheating*.

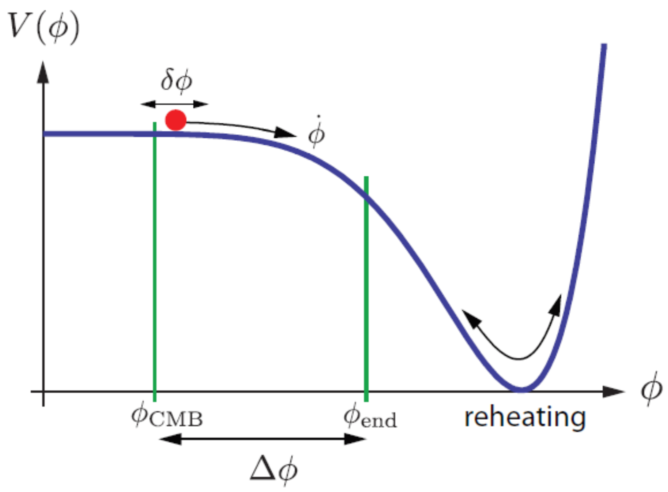

Figure 3. Inflationary potential example. Inflation occurs when the potential is flat. Inflation field decays into radiation during reheating.

3 Perturbation Theory of the Early Universe

3.1 Field Equation Perturbations and SVT Decomposition

The anisotropies in the CMB spectrum as well as density perturbations in the large scale structure are described by quantum fluctuations of the inflation field, during the inflation era. The fluctuations in the density, pressure and four velocity of the field causes the perturbations in the metric tensor through Einstein field equations. Let's suppose a small perturbation in the metric can be written as[3]:

$$g_{\mu\nu}(\vec{x},t) = \bar{g}_{\mu\nu}(t) + h_{\mu\nu}(\vec{x},t), \tag{3.1}$$

where $\bar{g}_{\mu\nu}$ is the unperturbed flat FRW metric with:

$$\bar{g}_{00} = -1,\ \bar{g}_{0i} = 0,\ \bar{g}_{ij} = a^2(t)\delta_{ij}. \tag{3.2}$$

Working out the perturbations in the Ricci tensor we find:

$$\delta R_{00} = \frac{1}{2a^2}\nabla^2 h_{00} + \frac{3\dot{a}}{2a}h_{00} - \frac{1}{a^2}\partial_i h_{i0} + \frac{1}{2a^2}\left[\ddot{h}_{ii} - \frac{2\dot{a}}{a}\dot{h}_{ii} + 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)h_{ii}\right], \tag{3.3}$$

$$\delta R_{0i} = \frac{\dot{a}}{a}\partial_j h_{00} + \frac{1}{2a^2}\left(\nabla^2 h_{i0} - \partial_i\partial_j h_{j0}\right) - \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)h_{i0} + \frac{1}{2}\partial_t\left[\frac{1}{a^2}(\partial_i h_{jj} - \partial_j h_{ij})\right], \tag{3.4}$$

$$\begin{aligned}\delta R_{ij} =& -\frac{1}{2}\partial_i\partial_j h_{00} - (2\dot{a}^2 + a\ddot{a})\delta_{ij}h_{00} - \frac{1}{2}a\,\dot{a}\,\delta_{ij}\dot{h}_{00} - \frac{1}{2}\ddot{h}_{ij} + \frac{\dot{a}}{2a}(\dot{h}_{ij} - \delta_{ij}\dot{h}_{kk}) \\ &+ \frac{1}{2a^2}\left(\nabla^2 h_{ij} - \partial_k\partial_i h_{kj} - \partial_k\partial_j h_{ki} + \partial_i\partial_j h_{kk}\right) + \frac{\dot{a}^2}{a^2}(-2h_{ij} + \delta_{ij}h_{kk}) \\ &+ \frac{\dot{a}}{a}\delta_{ij}\partial_k h_{k0} + \frac{1}{2}(\partial_i\dot{h}_{j0} + \partial_j\dot{h}_{i0}) + \frac{\dot{a}}{2a}(\partial_i h_{j0} + \partial_j h_{i0}).\end{aligned} \tag{3.5}$$

[3]We follow the treatment of [5] in this section

The perturbations in the Einstein tensor can be simplified because of the symmetries of the unperturbed FRW metric into scalars, vectors and tensor perturbations. Such decomposition is called *SVT decomposition* and simplifies the future calculations extremely easier. We write:

$$h_{00} = -E, \tag{3.6}$$
$$h_{0i} = a\left(\partial_i F + G_i\right), \tag{3.7}$$
$$h_{ij} = a^2\left(A\delta_{ij} + \partial_i C_j + \partial_j C_i + D_{ij}\right), \tag{3.8}$$

where the perturbations $A, B, C_i, D_{ij} = D_{ji}, E, F, G_i$ are functions of $\vec{x}$ and t with:

$$\partial_i C_i = \partial_i G_i = 0, \tag{3.9}$$
$$\partial_i D_{ij} = 0,\ D_{ii} = 0. \tag{3.10}$$

D_{ij} is a symmetric tensor with 6 DOF initially. We have already killed 4 of them with 3.10 which means it has only two independant, dynamical DOF which are the gravitation radiation polarization. Similar treatment can be applied to energy momentum tensor. For the sake of brevity, we do not bring them here though[4]. Einstein equations for tensor perturbations in the absence of anisotropic stress is:

$$\nabla^2 D_{ij}(\vec{x},t) - a^2\ddot{D}_{ij}(\vec{x},t) - 3a\dot{a}\dot{D}_{ij}(\vec{x},t) = 0. \tag{3.11}$$

Equation 3.11 is wave equation and implies the existence of primordial gravitational waves. The solution to wave equation can be obtained by Fourier transformation. Fourier analysis is extremely helpful here because it separates different Fourier modes. The Fourier transform is:

$$D_{ij}(\vec{x},t) = \int \frac{d^3\vec{k}}{2\pi^3} e^{i\vec{k}.\vec{x}}, \tilde{D}_{ij}(\vec{k},t), \tag{3.12}$$

and also the inverse FT:

$$\tilde{D}_{ij}(\vec{k},t) = \int d^3\vec{x} e^{-i\vec{k}.\vec{x}} D_{ij}(\vec{x},t). \tag{3.13}$$

Fourier transforming 3.11 will give:

$$\left(\frac{k^2}{a^2}\right)\tilde{D}_{ij}(\vec{k},t) + \ddot{\tilde{D}}_{ij}(\vec{k},t) + 3H\dot{\tilde{D}}_{ij}(\vec{k},t) = 0. \tag{3.14}$$

Now, we take the solution to 3.14 to be of the form $D_{ij}(\vec{k},t) = d(t)e_{ij}(\vec{k})$, with e_{ij} satisfying $e_{ii} = 0$, $k_i e_{ij} = 0$ and $e_{ij} = e_{ji}$. We find:

$$\ddot{d} + 3H\dot{d} + (\frac{k^2}{a^2})d = 0. \tag{3.15}$$

The solution to 3.15 is generally not easy to find. Fortunately though, we can make huge simplifications by expressing it in terms of conformal time, τ. We get:

$$d^{''} + 2\mathcal{H}d^{'} + k^2 d = 0. \tag{3.16}$$

Now I replace d with a new variable v defined as $v = ad$. We find[5],[12]:

$$v^{''} + \left(k^2 - \frac{a^{''}}{a}\right)v = 0. \tag{3.17}$$

We will solve this equation in the next section.

[4]For details see (Weinberg)

3.2 Conservation Outside Horizon

At first glance, 3.17 is not easy to solve. In general, v and a both have nontrivial dependence on conformal time which makes it impossible to solve it in full generality. Fortunately though, we can solve this equation in two different regimes: sub-horizon scales, where k is much larger than comoving Hubble horizon $(aH)^{-1}$, and super-horizon, where k is much smaller than horizon. For simplicity, I assume slow roll approximation is valid, therefore, 3.17 can be written as[12],[3]:

$$v'' + \left(k^2 - 2a^2H^2\right) v = 0. \tag{3.18}$$

3.2.1 Sub-Horizon

In this regime, $k \gg (aH)^{-1}$, corresponding to the scales which are deep inside the horizon. Neglecting the second term in the parentheses of 3.17 in favor of first one we obtain:

$$v'' + k^2 v = 0, \tag{3.19}$$

which is a free harmonic oscillator equation of motion. The solution of gravitational wave amplitude, D_{ij} is simply:

$$D_{ij} \propto \frac{e^{ik\tau}}{a}, \tag{3.20}$$

which implies the oscillatory damping of the tensor fluctuations due to the expansion of the universe.

3.2.2 Super-Horizon

The super-horizon regime is the one with $k \ll (aH)^{-1}$, corresponding to the scales that are larger than the comoving Hubble horizon. In this case, 3.17 can be expressed as:

$$v'' - 2a^2H^2 v = 0, \tag{3.21}$$

with two solutions of the form:

$$v_1 \propto a, \text{ and } v_2 \propto 1/a^2. \tag{3.22}$$

The linear solution v_1 implies that the amplitude of gravitational wave perturbations is a constant outside horizon. This is a very important result because it shows that the modes freeze when they reach the horizon scale. Remember that the comoving Hubble horizon $(aH)^{-1}$ is shrinking during the inflation but the mode wavelength is constant, meaning at early times, when the mode is still inside the horizon, it oscillates. After horizon exit, it freezes whilst preserving it's original information (we will clarify what we mean by information shortly). In later times, e.g. during radiation or matter domination era, when then mode re-enters the horizon, it has kept it's original information. The fact that the modes are simply constant outside of the horizon allows us to study both the scalar and tensor perturbations.

3.3 Power Spectrum of Fluctuations

The perturbations caused by the quantum fluctuations of a single scalar field during inflation grow to become the structures as we see today in the shape of galaxies and CMB. These perturbations are actually a field which we can schematically denote by $\delta(\vec{x}, t)$. The parameter δ can be either matter density contrast, CMB temperature fluctuation or basically any other quantity of interest defined as[4]:

$$\delta(\vec{x}, t) = \frac{\rho(\vec{x}, t) - \bar{\rho}}{\bar{\rho}}, \tag{3.23}$$

where $\bar{\rho}$ is the background observable. Typically, we are not interested in the fluctuation field itself, but it's statistical properties. Since the fluctuation distribution is completely random with mean $\langle \delta(\vec{x}) \rangle = 0$, it's statistical properties can be characterised as n-point correlation functions of the field. The n-point correlation function of a random field is defined as:

$$C_n(\vec{x}_1, \vec{x}_2, ..., \vec{x}_n) = \langle \delta(\vec{x}_1)\delta(\vec{x}_2)...\delta(\vec{x}_n) \rangle. \tag{3.24}$$

The observations of CMB and LSS imply that the fluctuations are Gaussian, meaning at each point $\vec{x}$, the probability of measuring the fluctuations to be a certain number, follows a Gaussian distribution function. With this regard and use of Wick's theorem, we can show that all the n-point correlation functions will vanish if n is odd. Even correlations can be expressed in terms of two point moments:

$$C_n = \begin{cases} 0, \text{ if n is odd} \\ \sum\limits_{\text{all pair associations}} \prod\limits_{\text{p pairs}} \langle \delta(\vec{x}_i)\delta(\vec{x}_j) \rangle. \end{cases} \tag{3.25}$$

Therefore, the observable that contains all the information regarding the field is the two point correlation function (simply correlation function from now on) $\xi(r)$:

$$\xi(r) = \langle \delta(\vec{x})\delta(\vec{x} + \vec{r}) \rangle. \tag{3.26}$$

We can compute the correlation in the Fourier space:

$$\langle \delta(\vec{k}_1)\delta(\vec{k}_2) \rangle = \int d^3\vec{x}\ d^3\vec{r}\ \langle \delta(\vec{x})\delta(\vec{x} + \vec{r}) \rangle\ e^{-i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}-i\vec{k}_2\cdot\vec{r}}. \tag{3.27}$$

Integrating over $\vec{x}$ we obtain:

$$\langle \delta(\vec{k}_1)\delta(\vec{k}_2) \rangle = \delta_D(\vec{k}_1 + \vec{k}_2) \int d^3\vec{r}\ \xi(r)\ e^{-i\vec{k}\cdot\vec{r}} \tag{3.28}$$

The quantity in the integral is the Fourier transform of correlation function which we call *power spectrum*, $P(k)$:

$$P(k) = \int d^3\vec{r}\ \xi(r)\ e^{-i\vec{k}\cdot\vec{r}} = \langle \delta(\vec{k}_1)\delta(\vec{k}_2) \rangle\ \delta_D(\vec{k}_1 + \vec{k}_2). \tag{3.29}$$

The power spectrum of fluctuations is the most important observable in cosmology and contains all the information about the perturbations. Another quantity we can construct out of power spectrum is the *dimensionless power spectrum* defined as:

$$\Delta^2(k) = \frac{k^3}{2\pi^3} P(k). \tag{3.30}$$

3.3.1 Power Spectrum of Tensor Perturbations

Equation 3.17 has an exact solution in the de Sitter space. The solutions are[12]:

$$v(k,\tau) = \sqrt{-\tau}\left(C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right), \tag{3.31}$$

where $H_\nu^{(1)}(-k\tau)$ and $H_\nu^{(2)}(-k\tau)$ are the Hankel functions of the first and second kind with $\nu = 3/2 + \epsilon$. C_1 and C_2 are the integration functions and τ lies between $-\infty < \tau < 0$. Using the asymptotic form of the Hankel functions on super-horizon scales ($k\tau \ll 1$) we are able to calculate the power spectrum of perturbations as:

$$\Delta_t^2(k) = \frac{8}{M_{Pl}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{2\epsilon}. \tag{3.32}$$

As we can see, the power spectrum is nearly scale invariant, meaning there is no characteristic scale that the power spectrum depends on.

The other observable that is very important and quantifies the energy scale of inflation, is *tensor to scalar ratio*:

$$r = \frac{\Delta_t^2}{\Delta_s^2}. \tag{3.33}$$

Large values of tensor to scalar ratio $r > 0.01$ corresponds to inflation occurring at GUT energy scales. Usually, the value of Δ_s^2 is fixed by temperature anisotropy measurements, leaving r as a free parameter. Therefore, the amplitude of the tensor fluctuations is not known initially and should be measured with other measurements (such as B-mode polarization power spectrum).

The power spectrum of scalar and tensor perturbations are usually parametrized as[10]:

$$P_t(k) = A_t\left(\frac{k}{k_*}\right)^{n_t}, \tag{3.34}$$

$$P_s(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1}. \tag{3.35}$$

where A_t and A_s are the amplitude of tensor and scalar perturbations, respectively. n_t and n_s are called the spectral index for the tensor and scalar perturbations and k_* is a pivot scale. The tensor to scalar ratio is then $r = P_t(k_*)/P_s(k_*)$. 4 shows the constrains of r and n_s for common inflation models measured by Planck satellite.

A famous consistency relation holds between tensor to scalar ratio and tensor spectral index. It it shown that for a single field inflation model in the slow roll approximation we have $r = -8n_t$. this relation will allow us to check the conditions inflation occurred.

4 Detection

4.1 Direct detection

Since the gravitational waves are decoupled from other elements in the universe below Planck energy scale, they are free to propagate. Unlike CMB photons who are bound to free electrons through Thomson scattering before recombination, the universe is always transparent to gravitational waves. As a result, detection of gravitational waves has significant implications

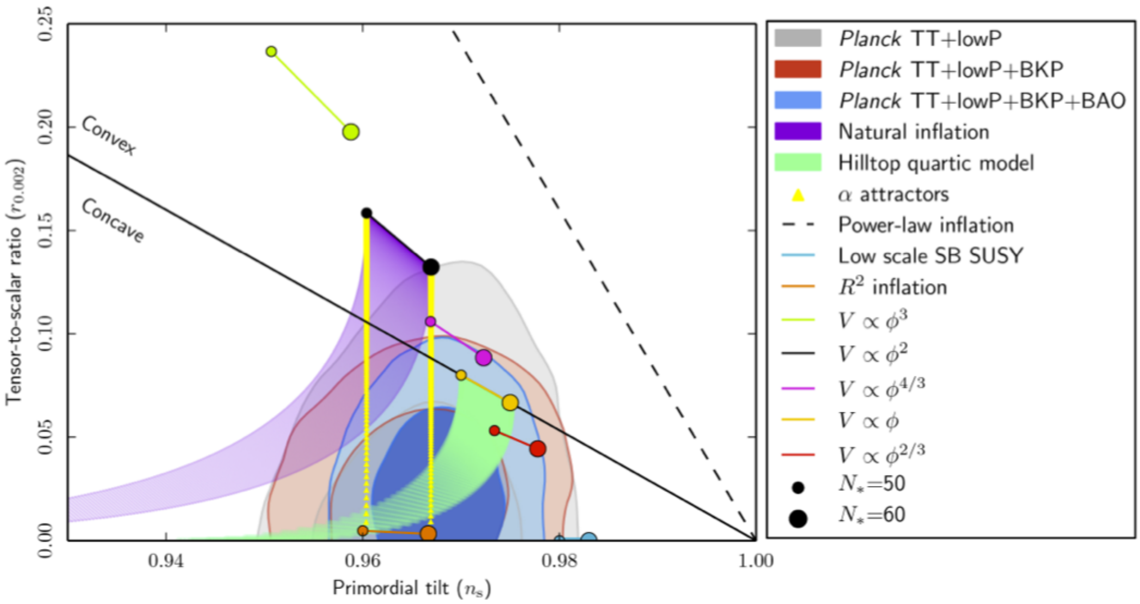

Figure 4. Tensor to scalar ratio r vs primordial tilt (spectral index) n_s. Popular models of inflation are tested and many of them were ruled out by this measurement[10]

on the early universe and inflation. Basically, there are two detection approaches: direct and indirect. The direct detection is performed by experiments like LIGO and LISA where we use interferometer arms to directly detect the perturbation of space, either on the ground or in the space with satellites. Indirect measurements are done by looking at the effects gravitational waves might have on the other probes like CMB and LSS.[13]

The current generation of gravitational wave detectors do not have the required sensitivity to detect the primordial gravitational waves. Since the amplitude of the gravitational waves is too small, space-based interferometer with at least 3-4 order of magnitude enhancement in sensitivity is needed. On the other hand, even if such interferometer existed, because of the foreground effect such as astrophysical process, it is unlikely we could detect the gravitational waves unless we could erase the contamination first[6].

4.2 Cosmic Microwave Background

After inflation, during reheating era, the inflation field decays into standard model particles such as baryons, photons and etc. Although the details of such mechanism is not known completely, it is necessary to heat the universe with reheating. After reheating, the universe is a plasma filled with photons, electrons, protons and fraction of other particles like neutrinos[5] and Helium nucleus. The most efficient interaction at this temperature, is the Thomson scattering of photons off of electrons. As the universe expands, the temperature of the the plasma decays as $T = T_0/a$. This temperatures eventually falls below the decoupling temperature of the photons from electrons, releasing the photons. Once the photons are allowed to free stream, neutral Hydrogen atoms form and the universe which once was opaque, becomes transparent. The moment which photons release from electrons and hydrogen atom forms, is called recombination. Recombination, is in fact a 2D snapshot of the universe which

[5]The neutrinos decouple from early plasma at $T_\nu \sim 1$MeV

we can study by observing the relic photons that reach us today. This relic radiation, is called *cosmic microwave background* or CMB. The mean temprature of the CMB today is about $T_0 \sim 2.725 \pm 0.001K$ which implies that the redshift of recombination should be about $z_{Rec} = 1100$.

Apart from the fact that discovery of CMB in 1968 by Penzias & Wilson was a strong evidence in establishing the Big Bang hypothesis, measurements of CMB anisotropy power spectrum has been the most reliant source of gathering cosmological data. As we have discussed so far, the primordial perturbations in during the inflation will seed the later perturbations, allowing us to look for information about the early universe in the CMB spectrum.

Since the CMB is seen as a 2D sphere at $z_{Rec} = 1100$, it is best to expand the temperature anisotropies in the terms of spherical harmonics $Y_{lm}(\theta, \phi)$[11]:

$$\frac{\Delta T(\theta, \phi)}{\bar{T}} = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi), \tag{4.1}$$

where (θ, ϕ) are the coordinates on the last scattering surface and a_{lm}s are the harmonic coefficients. The power spectrum of temperature fluctuations is given by:

$$C_l = \frac{1}{2l+1} \sum_{m=-l}^{m=+l} \langle |a_{lm}|^2 \rangle. \tag{4.2}$$

The power spectrum of temperature fluctuations measured by Planck satellite is shown in 6. Notice the angular seperation corresponds to multipole moment l with $\Delta\theta \sim 2\pi/l$.

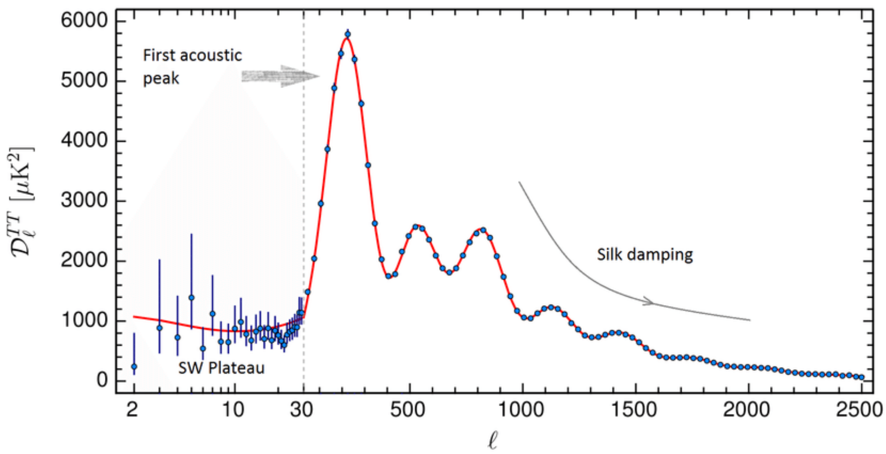

Figure 5. CMB temperature power spectrum ($D_l^{TT} = l(l+1)C_l^{TT}/2\pi$). The location of the first peak reveals that the universe if flat. Other peaks contain information on the composition of the universe. Notice how the low l moments have substantially higher variances. This is because of the *cosmic variance* and occurs only because we have one realization of the universe, limiting the precision at low ls. Silk damping refers to the smoothing of the power at high l because of the scattering between photons and electrons[11]

The CMB temperature power spectrum contains many information about the both the geometry of the universe and it's composition. The location of the first peak of corresponds to the sound horizon of plasma perturbations at the recombination, which makes it a perfect standard ruler. By exact measurements of the first peak, we can infer that the universe is flat. The next peaks of the power have information on the composition of the universe like Ω_m and dark energy.

4.2.1 CMB Polarization

The most important interaction between photons and electrons in the times before recombination, is Thomson scattering. The scattering of anisotropic photons off of electrons produces a polarized radiation field at 10% level. The polarized radiation is described by a rank 2 tensor $I_{ij}(\theta, \phi)$. The *Stokes parameters* Q and U are defined as[12]:

$$Q = (I_{11} - I_{22})/4, \tag{4.3}$$
$$U = I_{12}/2. \tag{4.4}$$

The combination $Q \pm iU$ of the stokes parameters are then expanded into spin 2 spherical harmonics, since this combination possess definite transformation properties under rotation. Therefore:

$$Q \pm iU = \sum_{l,m} a_{lm}^{(\pm 2)} Y_{lm}^{(\pm 2)}(\theta, \phi). \tag{4.5}$$

E and B mode amplitudes are then defined as:

$$a_{lm}^{E} = -\frac{1}{2}\left(a_{lm}^{(2)} + a_{lm}^{(-2)}\right), \tag{4.6}$$
$$a_{lm}^{B} = -\frac{1}{2i}\left(a_{lm}^{(2)} - a_{lm}^{(-2)}\right). \tag{4.7}$$

Since the E mode is invariant under parity, it is curl free. B mode is odd under parity and it is divergence free. The power spectrum of polarization fluctuations are then:

$$C_l^{EE} = \frac{1}{2l+1}\sum_m \langle a_{lm}^{E} a_{lm}^{E} \rangle, \tag{4.8}$$
$$C_l^{BB} = \frac{1}{2l+1}\sum_m \langle a_{lm}^{B} a_{lm}^{B} \rangle. \tag{4.9}$$

The main source of temperature anisotropies are the scalar perturbations. Although tensor perturbations can impact the temperature power spectrum, their imprint can be seen in low multipoles, $l < 60$ but their amplitude is much lower than the scalar perturbation's amplitude. Also, cosmic variance limits the precision at low ls. Therefore, temperature power spectrum alone is not enough to constrain the tensor perturbation amplitudes.

The polarization, on the other hand, has a different story. Scalar perturbations can only produce E modes and therefore, any measured B mode, corresponds to existence of background tensor perturbations at the recombination. Detection of B modes is the most significant goal of the next generation of CMB polarization surveys, but it is extremely difficult since the amplitude of such modes are very small. Since gravitational waves at the time of recombination were on the super-horizon scales, their contribution to the BB power spectrum comes from low multipoles, $l < 150$. The problem with their detection is that the foreground is highly contaminated with dust, galactic magnetic field and astrophysical process which should be cleaned up. The existence of such environments can produce a fake B mode signal, that can be mistaken with primordial gravitational wave's signal (e.g. the BICEP2 claim, 2015[8],[9]). The other important obstacle, is the gravitational lensing. Gravitational lensing of CMB photons by galaxies and clusters of galaxies to the last scattering surface, can potentially transform E modes patterns to B modes, producing another fake effect which can be degenerate with gravitational waves signal. It is shown in 6 that the power spectrum of gravitational waves is much smaller than TT and EE power spectrums and it is highly degenerate with the lensing effect. Therefore, cleaning contamination from the data is absolutely necessary for the detection of gravitational wave B modes[6],[12],[5].

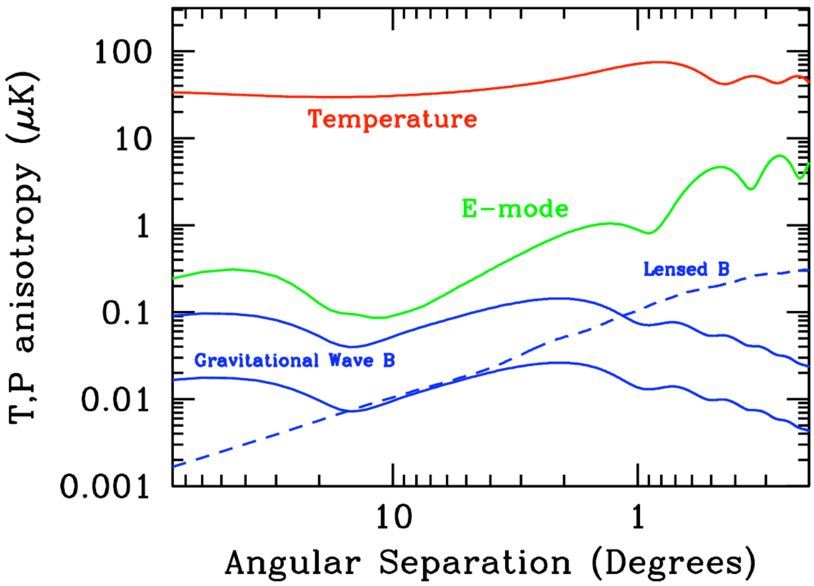

Figure 6. Amplitude of temperature, E mode and B mode power spectrums of the CMB. Notice that the B mode is substantially lesser than E mode and therefore, much harder to detect[6]

4.3 Large Scale Structure

On large scales, the clustering of matter is described by a dimensionless parameter known as density contrast:

$$\delta(\vec{x},t) = \frac{\rho(\vec{x},t) - \bar{\rho}}{\bar{\rho}}, \tag{4.10}$$

where $\bar{\rho}$ is the density of background universe. As we discussed earlier, the power spectrum of matter perturbations is:

$$P(k) = \langle \delta(\vec{k}_1)\delta(\vec{k}_2)\rangle\, \delta_D(\vec{k}_1 + \vec{k}_2). \tag{4.11}$$

Since fluctuations are homogeneous and isotropic on large scales on average, the power spectrum and correlation function $\xi(r)$ only depend on magnitude of $\vec{k}$ and $\vec{r}$, respectively.

The power spectrum of matter can be seen in 7. At low ks, the power spectrum is linear which corresponds to very large scales that are set by inflation. Similar to acoustic peaks of the CMB, there are some wiggles around $k_s \sim 0.01\,\mathrm{h/Mpc}$. These wiggles correspond to the sound horizon at the time of recombination and set the scale of baryon acoustic oscillations

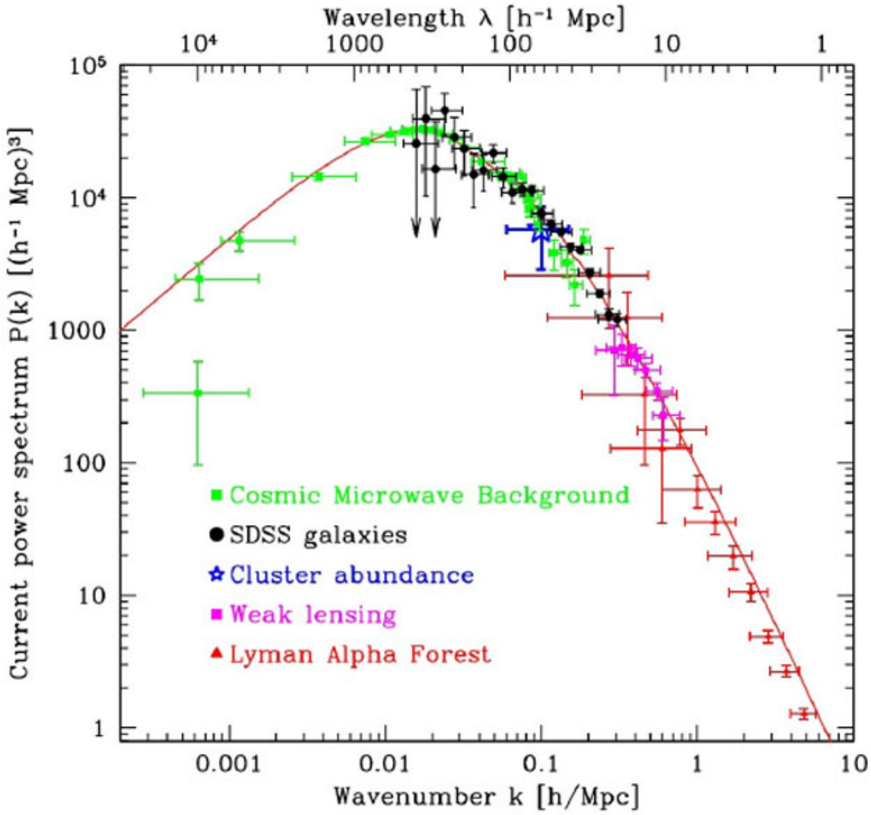

Figure 7. Matter power spectrum

(BAO) which is one of the most important observable in galaxy surveys. BAO scale appears as a bump in the correlation function, 8. The BAO scale is a standard ruler with comoving length of $r_s \sim 100$Mpc/h, allowing us to extract so many crucial information out of galaxy surveys on this scale. BAO scale is the length scale at which primordial fluctuations were allowed to travel in a form of a wave, before the time of recombination. After recombination, since photons decoupled, the fluctuations cannot travel in space, therefore they freeze on scale of r_s, producing a bump in the correlation function of galaxies.[5]

Gravitational waves leave observable imprint on the galaxy power spectrum and correlation function. They modify the power spectrum with the tidal effects during structure formation in presence of a long wavelength tensor mode. They produce a non-trivial elliptic galaxies correlation and the projection effect on galaxy distributions.[12]

Detecting such effects is not easy as well. One of the most important challenges along the way is the non-linear nature of galaxy clustering. Briefly, on smaller scales ($k \sim 0.1$h/Mpc, as the structures grow, they become more and more non-linear, hence, the linear perturbation theory we discussed earlier does not apply. Understanding the behavior of small scales is essential in order to detect early effects such as primordial non-Gaussianity and gravitational waves. Other important challenge is the bias. In a galaxy survey, we observe the position of galaxies and their redshift while we really want, is the underlying dark matter distribution. Baryons do not trace dark matter completely because they are involved in so many other interactions like supernovae backreaction, AGN feedback and other astrophysical processes. Also, the baryons were coupled to photons before recombination while dark matter was not

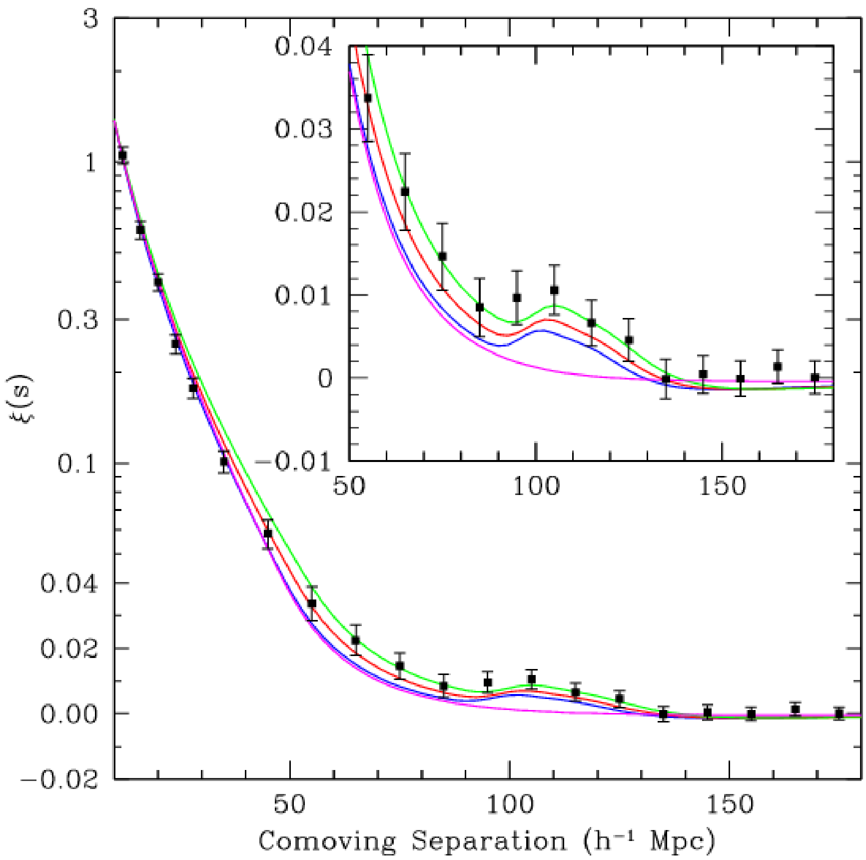

Figure 8. Correlation function of galaxies from SDSS and the BAO scale.

coupled to them. Therefore, what we measure in a galaxy survey, is not statistics of the dark matter density field, but rather a tracer of it. Understanding bias is significant for a clean detection of the primordial effects.

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