$1/r^2$ Potential

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Put $\hbar = 2m = 1$. The radial Schrödinger equation is

$$\left(-\frac{d^2}{dr^2} - \frac{\alpha^2 + 1/4}{r^2}\right) u(r) = Eu \tag{1}$$

The solutions are $\sqrt{r}J_{\pm i\alpha}(kr)$ with $k^2 = E$ and J are bessel functions of order $\pm i\alpha$. Introduce a cutoff radius ϵ below which the potential is a constant continuous with the above at $r = \epsilon$. Then the solution for $r < \epsilon$ is just $\sin(k'r)$, with $k'^2 = k^2 + (4\alpha^2 + 1)/4\epsilon^2$. Matching logarithmic derivatives of the wave functions at the cutoff gives

$$k'\epsilon \cot(k'\epsilon) = \frac{1}{2} + k\epsilon \frac{AJ'_{-i\alpha}(k\epsilon) + BJ'_{i\alpha}(k\epsilon)}{AJ_{-i\alpha}(k\epsilon) + BJ_{i\alpha}(k\epsilon)} \tag{2}$$

For energies such that $k\epsilon << 1$, the matching condition can be simplified. Note that

$$J_\nu(z) \sim \frac{1}{\Gamma(1+\nu)} \left(\frac{z}{2}\right)^\nu .$$

as $z \to 0$. For $\nu = i\alpha$, $z^{i\alpha}$ is periodic under $z \to ze^{2\pi/\alpha}$, so we can go to small cutoff by writing $\epsilon = ae^{-N_0\pi/\alpha}$ with a fixed and finite and $N_0 \to \infty$. The matching condition becomes independent of N_0:

$$\sqrt{\alpha^2 + 1/4} \cot \sqrt{\alpha^2 + 1/4} - 1/2 = -i\alpha \frac{A(ka/2)^{-i\alpha}e^{i\phi(\alpha)} - B(ka/2)^{i\alpha}e^{-i\phi(\alpha)}}{A(ka/2)^{-i\alpha}e^{i\phi(\alpha)} + B(ka/2)^{i\alpha}e^{-i\phi(\alpha)}} \tag{3}$$

where we have put $\Gamma(1 \pm i\alpha) = |\Gamma(1 + i\alpha)|e^{\pm i\phi(\alpha)}$. For $E > 0$, is k real and this equation determines the phase shift in terms of a. For $E < 0$, $k \equiv i\kappa$ is imaginary, and B/A is fixed by the condition that $u = AJ_{i\alpha}(i\kappa r) + BJ_{-i\alpha}(i\kappa r) \propto K_{i\alpha}(\kappa r)$ is exponentially damped at large r, so the matching condition gives the energy quantization condition.

$$\sqrt{\alpha^2 + 1/4} \cot \sqrt{\alpha^2 + 1/4} - 1/2 = \alpha \cot[\alpha \ln(\kappa a/2) - \phi(\alpha)].$$

We have immediately from the scaling periodicity that if κ_0 is a solution so is $\kappa_n = \kappa_0 e^{-n\pi/\alpha}$, where n is any positive or negative integer. In particular there are discrete energy levels extending to $-\infty$. Of course with N_0 finite the above solution breaks down for $n \sim -N_0$ and then one has to go back to the unsimplified matching condition to obtain the quantization condition.

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