Quantum Field Theory of the Standard Model

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Chapter 1

Introduction

1.1 Problems with relativistic quantum mechanics

One of the most basic facts about relativistic physics is the equivalence of mass and energy, as in Einstein’s famous relation $E = mc^2$. This fact has profound consequences for relativistic quantum mechanics, because of Heisenberg’s uncertainty principle

$$\Delta x \Delta p \geq \hbar. \quad (1.1)$$

In ordinary nonrelativistic quantum mechanics this follows from the fact that the position and momentum are represented by operators that do not commute: $[x, p] = i\hbar$. According to (1.1) an accurate position measurement can be made only at the expense of a compensatingly large inaccuracy in the momentum of the particle. This implies a correspondingly large expense of energy in the accurate measurement of position. In nonrelativistic quantum mechanics the rest energy of a massive particle is effectively infinite so one can subject a particle to arbitrarily large energy changes without compromising its integrity as a particle, and accurate position measurements are compatible with particle number conservation. The concept of an $N$ particle wave function as a function of the $N$ particle coordinates is sustainable. This is drastically changed in relativistic quantum mechanics. As soon as one attempts to measure the position of a particle to an accuracy much less than the particle’s Compton wavelength, $\frac{\hbar}{mc}$, the associated momentum uncertainty is $> mc$, corresponding to an energy change larger than $mc^2$. Thus accurate position measurements require the supply of energy sufficient to produce additional particles. If the particle possesses a conserved charge, additional such particles can be produced but only in association with particles of opposite charge. Thus we should expect any consistent version of relativistic quantum mechanics to require abandoning the concept of a wave function for a system with a definite number of particles. Further, in retrospect, it is not surprising that when there are conserved charges, relativistic quantum physics requires antiparticles.

The preceding is not meant to be a rigorous argument, but rather a simple explanation of why we must expect at least the complexities of quantum field theory when we attempt to extend quantum mechanics into the relativistic domain. The many-body aspects of relativistic quantum mechanics are universal, but the manner in which they make their appearance...
varies. In quantum electrodynamics it is immediate once one tries to interpret classical radiation in terms of photons. For massive particles, like the electron, it arises from the existence of negative energy solutions of the Dirac equation.

Applications of quantum field theory will focus primarily on quantum electrodynamics (QED), the theory of electrons and positrons interacting with the quantized electromagnetic field. But since the fields of QED have multiple components which mix under Lorentz transformations, we shall begin with a much simpler quantum field theory— that of a self-interacting scalar field $\phi(x)$. Then we shall show how Dirac’s theory of electrons and positrons can be converted to a quantum field theory through the device of “second quantization”. We shall study this Dirac quantum field theory first in the presence of external (classical) electromagnetic fields. Only then will we turn to the problem of quantizing the electromagnetic field and studying it in interaction with electrons and positrons.

The message of QFT is that, to develop quantum theory consistently with special relativity, we should abandon the idea of taking particle coordinates as the fundamental degrees of freedom, and we should instead use quantum fields.

1.2 Review of Lorentz Invariance, with $c = 1$

In elementary discussions of special relativity we learn that frames of reference with a constant relative velocity $V$ have their coordinates related by the Lorentz “boost”

$$
x' = \gamma(x + Vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t + Vx) \tag{1.2}
$$

where $\gamma = 1/\sqrt{1 - V^2}$. But more generally the boost could be in any direction specified by a three vector $\vec{V}$. In addition to boost invariance, we also require rotational invariance, a symmetry we are very familiar with in nonrelativistic mechanics. We define a general Lorentz transformation of space-time coordinates $x^\mu = (t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$ as a linear transformation

$$
x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu, \quad \text{Lorentz} \tag{1.3}
$$

where $\Lambda$ preserves Minkowski scalar products $v \cdot w = v^1w^1 + v^2w^2 + v^3w^3 - v^0w^0 \equiv v^\mu v^\nu \eta_{\mu\nu}$. ($\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = 1, \eta_{\mu\nu} = 0$ for $\mu \neq \nu$. Note that we use the convention that repeated indices are summed. This requirement implies the following constraints on $\Lambda$:

$$
\eta_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu = \eta_{\mu\nu}. \tag{1.4}
$$

Exercise: You should convince yourself that the special boost in the $x$ direction does indeed preserve Minkowski scalar products.

The Poincaré group consists of Lorentz transformations together with translations

$$
x^\mu \rightarrow x^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \quad \text{Poincare.} \tag{1.5}
$$
According to special relativity the laws of physics should look the same in all frames related by Lorentz transformations. The systematic way to implement this requirement is to identify all physical quantities as the components of 4 tensors $A_{\mu\nu\cdots}$ which transform as

$$A'_{\mu\nu\cdots} = \Lambda^\mu_\rho A^\nu_\sigma \cdots A^\rho_\sigma$$

For example energy and momentum are the components of a four vector $p^\mu = (E, \mathbf{p})$ which transform just like the coordinates $x^\mu$.

$$p^\mu \rightarrow p'^\mu = \Lambda^\mu_\nu p^\nu$$

Tensors written with index superscripts are called contravariant tensors. It is also useful to introduce covariant tensors $A_{\mu\nu\cdots}$, written with index subscripts. A covariant index transforms like $\partial/\partial x^\mu$:

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho} (\Lambda^{-1})^\rho_\mu = \frac{\partial x^\rho}{\partial x^\mu} \eta^{\rho\kappa} \Lambda^\kappa_\sigma \eta_{\sigma\mu} \equiv \Lambda^\rho_\mu \frac{\partial}{\partial x^\rho}$$

A general tensor can have any number of upper and lower indices. the metric tensor is used to raise and lower indices.

The $\Lambda$’s can be divided into 4 disjoint sets according to the signs of $\det \Lambda$ and $\Lambda^0_0$. This is because it is easy to show from the above property that $(\det \Lambda)^2 = 1$ and $(\Lambda^0_0)^2 \geq 1$. Thus a continuous variation of $\Lambda$ always stays within one of these sets. In the following we restrict ourselves to the proper Lorentz Group, i.e. with $\det \Lambda = +1$ and $\Lambda^0_0 \geq +1$. The complete Lorentz group is then obtained by adjoining parity and time reversal.

### 1.2.1 Infinitesimal generators of Lorentz Group

Symmetries in quantum mechanics should be realized by unitary operators. For Lorentz symmetry, this means that there should be a unitary operator $U(\lambda)$ associated with each Lorentz transformation $\Lambda$, such that $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1 \Lambda_2)$. For the rotation subgroup this construction is familiar to all. Lorentz transformations with $\Lambda^0_k = \Lambda^k_0 = 0$ are simply rotations and form a subgroup. We know from basic quantum mechanics all the unitary irreducible representations of the Rotation group, namely those labeled by angular momentum $j = 0, 1/2, 1, 3/2, \ldots$. This conclusion is reached by considering infinitesimal rotations which are generated by the angular momentum $\vec{J}$, $U(R) = e^{-i\hat{u} \cdot \vec{J}} \approx 1 - i \hat{u} \cdot \vec{J}$. The commutators of $J^k$ are well-known:

$$[J^k, J^l] = i\hbar \epsilon_{klm} J^m$$

and the $|jm\rangle$ basis of eigenstates of $\vec{J}^2$, $J_3$ is constructed in the familiar way.

The unitary representations of the Lorentz group must be extensions of this construction of the rotation group, $U(\Lambda) = U(R)$ when $\Lambda^0_k = \Lambda^k_0 = 0$. To find the generators of boosts we first look at a single particle. For example, consider the boost in the $x$ direction of the momentum of a particle of mass $m$:

$$p'^1 = \gamma (p^1 + V\sqrt{p^2 + m^2}) \approx p^1 + V\sqrt{p^2 + m^2} + O(V^2)$$
for infinitesimal \( V \). Calling the infinitesimal generator for this transformation \( K^1 \), it should be identified with \( p^1 = iV[p^1, K^1] \). So we can infer

\[
K^1 = -(x^1 \sqrt{p^2 + m^2} + \sqrt{p^2 + m^2} x^1)/2 + F(p) \tag{1.11}
\]

The symmetrized product, which can be represented as an anticommutator \( \{A, B\} \equiv AB + BA \), in the first term is to keep \( K \) hermitian when the operators don’t commute. To find \( F \) we consider the transform of the coordinate

\[
x'^1(t') \approx x^1(t) + Vt \approx x^1(t') - V(x^1 \dot{x}^1(t) + \dot{x}^1(t) x^1)/2 + Vt \tag{1.12}
\]

But

\[
-iV[x^1, K^1] = -\frac{V}{2} \left( x^1 \frac{p^1}{\sqrt{p^2 + m^2}} + \frac{p^1}{\sqrt{p^2 + m^2}} x^1 \right) + V \frac{\partial F}{\partial p^1}
\]

\[
= -\frac{1}{2} \left( x^1 \dot{x}^1 + \dot{x}^1 x^1 \right) + \frac{\partial F}{\partial p^1} \tag{1.13}
\]

Thus we are led to

\[
K^1 = -(x^1 \sqrt{p^2 + m^2} + \sqrt{p^2 + m^2} x^1)/2 + p^1 t \tag{1.14}
\]

We easily see that \( K^1 \) is constant in time. Boosts in all three directions are generated by

\[
K^k = -(x^k \sqrt{p^2 + m^2} + \sqrt{p^2 + m^2} x^k)/2 + p^k t \tag{1.15}
\]

Notice that according to our 4-vector notation \( x^0 = t \) and \( p^0 = E = \sqrt{p^2 + m^2} \) so we can write \( K^k = x^0 p^k - \{x^k, p^0\}/2 \). We can also replace the angular momentum \( J \) with a two index tensor \( M^{kl} = \epsilon_{klm} J^m = x^k p^l - x^l p^k \), so that the Lorentz generators can be combined into the two index tensor

\[
M^{\mu\nu} = \frac{1}{2} \{x^\mu, p^\nu\} - \frac{1}{2} \{x^\nu, p^\mu\} . \tag{1.16}
\]

Again the point of writing the anticommutator for the product of operators on the right is to ensure that \( M^{\mu\nu} \) is Hermitian. In the classical limit where operators commute we could write more simply \( M^{\mu\nu} \rightarrow x^\mu p^\nu - x^\nu p^\mu \). It is now a straightforward

**Exercise**: to complete the commutator algebra of Lorentz generators:

\[
[J^k, J^l] = i\hbar \epsilon_{klm} J^m, \quad [K^k, J^l] = i\hbar \epsilon_{klm} K^m, \quad [K^k, K^l] = -i\hbar \epsilon_{klm} J^m
\]

An alternative more covariant presentation of the Lorentz algebra is to use \( M^{ij} = \epsilon_{ijk} J^k \), \( M^{0i} = K^i \), and find

\[
[M_{\mu\nu}, M_{\rho\sigma}] = i \left( \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho} \right) . \tag{1.17}
\]

where now and henceforth we will choose units so that \( \hbar = c = 1 \).

We have derived the Lorentz commutator algebra for a particularly simple system: a single free particle. But the universality of Lorentz invariance means that the same commutator algebra must hold for any physical system. In another system \( M^{\mu\nu} \) is a function of the dynamical variables of the other system, and that function can be much more complicated. But the commutator algebra must be the same. In the next section we shall see how this works for quantum fields.
1.2.2 Lorentz transforming fields

In classical electromagnetic theory we learn that Maxwell’s equations are covariant under Lorentz transformations of the electromagnetic field $\Phi_{\mu\nu}$ ($F_{0k} \equiv -E^k$ and $F_{kl} \equiv \epsilon_{klm}B^m$):

$$F'_{\mu\nu}(x') = \Lambda_{\mu}^{\rho} \Lambda_{\nu}^{\sigma} F_{\rho\sigma}(x) = \Lambda_{\mu}^{\rho} \Lambda_{\nu}^{\sigma} F_{\rho\sigma}(\Lambda^{-1}x')$$  \hspace{1cm} (1.18)

Since $F_{\mu\nu}(x)$ is a field with a different value at each point, notice that two changes are going on simultaneously: the components of $F$ are mixed and the old and new fields are compared at different space-time coordinates $x'$, $x$ related by $x' = \Lambda x$. We are going to consider at the beginning a scalar field $\phi(x)$ with only one component. The scalar field then has the much simpler Lorentz transformation

$$\phi'(x') = \phi(\Lambda^{-1}x')$$  \hspace{1cm} (1.19)

The simplest Lorentz covariant scalar field equation is the Klein-Gordon equation

$$-\eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \phi + \mu^2 \phi = 0$$  \hspace{1cm} (1.20)

Exercise: To show Lorentz covariance we want to prove

$$-\eta^{\mu\nu} \frac{\partial^2}{\partial x'^\mu \partial x'^\nu} \phi(\Lambda^{-1}x') + \mu^2 \phi(\Lambda^{-1}x') = 0.$$  \hspace{1cm} (1.21)

This follows by using the the chain rule

$$\frac{\partial}{\partial x'^\mu} = (\Lambda^{-1})^\rho_\mu \frac{\partial}{\partial x^\rho}$$  \hspace{1cm} (1.22)

and the defining property of Lorentz transformation.

1.3 The Free Scalar Field

The field concept goes all the way back to Faraday in the first part of the nineteenth century. It was of course indispensable to a proper understanding of electromagnetism which he pioneered. This first field theory is actually quite complex involving three components for each of the electric and magnetic fields. For our first look at quantum field theory we consider the much simpler case of the single component scalar field.

A field is a dynamical variable $\phi(x, t)$ assigned to each point of space. One is therefore positing from the beginning an infinite number of degrees of freedom. We first assume that $\phi$ satisfies the Klein-Gordon equation (henceforth we shall use units in which the speed of light $c = 1$):

$$\left( \frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2 \right) \phi \equiv (-\partial^2 + \mu^2)\phi = 0.$$  \hspace{1cm} (1.23)
The parameter $\mu$ clearly has dimensions of $1/\text{Length}$, and represents an inverse wavelength (or wave number) rather than a mass. After quantization, we shall see that the particles associated with the quantum field have mass $\hbar \mu$. We are of course very familiar with the nature of the solutions to these equations: general superpositions of plane waves, called wave packets:

$$\phi(x, t) = \int d^3k f(k) e^{ik \cdot x - i\omega(k) t}$$ (1.24)

$$\approx \int d^3k f(k) e^{ik \cdot x - i\omega(k_0) t - (k - k_0) \cdot \nabla \omega |_{k_0}}$$ (1.25)

$$\approx e^{-i\omega(k_0) t + ik_0 \cdot v_g t} \phi(x - v_g t, 0)$$ (1.26)

where $\omega(k) = \sqrt{k^2 + \mu^2}$, moving with group velocity $v_g = k / \sqrt{k^2 + \mu^2}$. The approximations on the second and third lines are valid if the packet $f(k)$ is very narrowly peaked about some mean wave number $k_0$.

We have noticed that the wave equation is invariant in form under Lorentz transformations $\phi'(x') = \phi(x)$ where $x'' = \Lambda_{\nu}^{\mu} x^\nu$ and $\Lambda$ is a Lorentz transformation.

### 1.3.1 The quantum mechanics of $\phi$

To discuss the quantum mechanics of this field we must identify the canonical variables of the system. Since the equation of motion is of second order we can easily guess that the momentum conjugate to $\phi(x)$ is just $\pi(x) = \dot{\phi}$. A more systematic approach is to find an action principle which implies the field equation. Recall that for dynamical systems with discretely labeled degrees of freedom $q_i(t)$ one forms the action from the Lagrangian $L(q(t), \dot{q}(t), t)$ via $S = \int _{t_1}^{t_2} dt L$. Hamilton’s principle states that the equations of motion follow from the requirement that $S$ is stationary under variations $\delta q_i(t)$ with $\delta q_i(t_1) = \delta q_i(t_2) = 0$. By definition the momentum conjugate to $q_i$ is $p_i \equiv \partial L / \partial \dot{q}_i$. For our field the spatial coordinates $x$ play the role of the index $i$, and the Lagrangian will be an integral over spatial coordinates. It is easy to see that Hamilton’s principle applied to the action

$$S = \int_{t_1}^{t_2} dt \int d^3x \frac{1}{2} (\dot{\phi}^2 - (\nabla \phi)^2 - \mu^2 \phi^2) \equiv \int d^4x \mathcal{L}$$ (1.27)

$$\mathcal{L} = -\frac{1}{2} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mu^2 \phi^2]$$ (1.28)

implies the Klein-Gordon equation. The second form shows that the Lagrange density $\mathcal{L}$ is a Lorentz covariant scalar field, so its integral is a Lorentz invariant.

From this action, it is evident that the above guess for $\pi \equiv \partial \mathcal{L} / \partial \dot{\phi} = \dot{\phi}$ is correct. The canonical Hamiltonian is

$$H \equiv \int d^3x [\dot{\phi} \pi - \mathcal{L}] = \int d^3x \frac{1}{2} (\dot{\phi}^2 + (\nabla \phi)^2 + \mu^2 \phi^2).$$ (1.29)
One can easily check that Hamilton’s equations obtained from this Hamiltonian give that same old wave equation. To quantize canonically, we promote $\phi, \pi$ to operators with the canonical commutation relations

$$[\phi(x), \pi(y)] = i\hbar \delta(x - y).$$  \hfill (1.30)

In the language of Hamiltonian mechanics with canonical variables $q_i, p_i$, the field $\phi(x)$ is analogous to $q_i$ and $\pi(x)$ to $p_i$. The spatial coordinate plays the role of the index $i$ that distinguishes independent degrees of freedom.

One can interpret $L$ as a huge system of coupled oscillators. The $\mu^2$ term is like a spring constant, and the $\nabla^2 \phi$ term couples neighboring oscillators to each other. In the next optional section we make this analogy absolutely concrete by replacing space with a discrete rectangular lattice.

### 1.3.2 Space as a Discrete Lattice (Optional)

To find the energy eigenstates, we note the close resemblance of our Hamiltonian to that of a system of coupled harmonic oscillators. The only thing a little strange is that the oscillator coordinates are labeled by a continuous index. This can be remedied by replacing space by a lattice $an$ of spacing $a$ which we send to zero after finding the eigenstates. Thus we replace $\phi(an)$ with $\phi_n$ and $\pi(an)$ by $\pi_n/a^3$. We divide by $a^3$ so that $[\phi_n, \pi_m] = i\hbar \delta_{n,m}$

Doing this the Hamiltonian becomes

$$H_{\text{lattice}} = \frac{a^3}{2} \sum_n \left( \frac{1}{a^6} \pi_n^2 + \mu^2 \phi_n^2 + \frac{1}{a^2} \sum_i (\phi_{n+i} - \phi_n)^2 \right).$$  \hfill (1.31)

On the lattice our system is a system of coupled oscillators. To solve it, we just have to find the normal modes. This is done by a change of variables which renders diagonal the coupling matrix

$$V_{n,m} = (\mu^2 + \frac{2d}{a^2})\delta_{n,m} - \frac{1}{a^2} \sum_i (\delta_{n+i,m} + \delta_{n,m+i}),$$  \hfill (1.32)

i.e. we want to find the eigenvectors and eigenvalues of $V$. It is easy to see that the eigenvectors are $u_m = e^{iK_m}$, $-\pi < K_i < \pi$, belonging to eigenvalue

$$\lambda \equiv \omega^2 = \mu^2 + \frac{2}{a^2} (d - \sum_i \cos K \cdot \hat{i}).$$  \hfill (1.33)

Thus the transformation to normal modes is given by

$$\phi_n = \int_{-\pi}^{\pi} \frac{d^3 K}{(2\pi)^{3/2}} e^{iK_n} Q(K)$$  \hfill (1.34)

$$\pi_n = \int_{-\pi}^{\pi} \frac{d^3 K}{(2\pi)^{3/2}} e^{iK_n} P(K)$$  \hfill (1.35)
or inversely by
\[ Q(K) = \sum_n \frac{1}{(2\pi)^3/2} e^{-iK\cdot n} \phi_n \]  \hfill (1.36)
\[ P(K) = \sum_n \frac{1}{(2\pi)^3/2} e^{-iK\cdot n} \pi_n. \]  \hfill (1.37)

The inverse equations directly imply the commutation relations
\[ [Q(K), P(L)] = i\hbar \delta(K + L). \]  \hfill (1.38)

Note also that if \( \phi \) and \( \pi \) are hermitian, as we assume, then we have \( Q(K)^\dagger = Q(-K) \) and \( P(K)^\dagger = P(-K) \).

Expressed in terms of normal modes the lattice Hamiltonian becomes
\[ H_{\text{lattice}} = \int_{-\pi}^{\pi} d^3K \left( \frac{1}{2\alpha^3} P(K)P(-K) + \frac{\alpha^3}{2} \omega^2(K)Q(K)Q(-K) \right). \]  \hfill (1.39)

In the standard fashion raising and lowering operators can be constructed in terms of which
\[ Q(K) = \sqrt{\frac{\hbar}{2\omega(K)\alpha^3}} (A(K) + A^\dagger(-K)) \]  \hfill (1.40)
\[ P(K) = -i\sqrt{\frac{\hbar\omega(K)\alpha^3}{2}} (A(K) - A^\dagger(-K)) \]  \hfill (1.41)

with \([A(K), A^\dagger(L)] = \delta(K - L)\), and the Hamiltonian becomes the familiar
\[ H_{\text{lattice}} = \int_{-\pi}^{\pi} d^3K \hbar \omega(K) \frac{1}{2} (A(K)A^\dagger(K) + A^\dagger(K)A(K)). \]  \hfill (1.42)

\( A^\dagger \) and \( A \) are of course eigenoperators of \( H_{\text{lattice}} \) with eigenvalues \( \pm \omega(K) \). Now we can describe the exact energy eigenstates of the system. The ground state (the vacuum) \( |0\rangle \) is annihilated by all the \( A \)'s: \( A(K)|0\rangle = 0 \), and its energy \( E_0 = (\hbar/2)\delta(0) \int d^3K \omega(K) \). We shall measure all energies relative to \( E_0 \), \( i.e. \) from now on we take \( E_0 = 0 \). This means our energy operator is redefined to be
\[ H - E_0 = \int_{-\pi}^{\pi} d^3K \hbar \omega(K) A^\dagger(K)A(K) \]  \hfill (1.43)

The excited states are obtained by applying any number of \( A^\dagger \)'s to the vacuum. The simplest one is \( A^\dagger(K)|0\rangle \) with energy
\[ E(K) = \hbar \sqrt{\mu^2 + \frac{2}{a^2} (d - \sum_i \cos K \cdot \hat{\imath})}. \]  \hfill (1.44)

At this point we can consider the continuum limit \( a \to 0 \). All excited states with finite \( K \) will have infinite energy and be dynamically irrelevant. However since the components of \( K \) are continuous, we may consider states with \( K = ak \). The energy of these states becomes \( \hbar \sqrt{\mu^2 + k^2} \) independent of \( a \) in the continuum limit. We identify this energy as that of a relativistic particle of mass \( h\mu \) and momentum \( \hbar k \).
1.3.3 Continuous space quantization

We can pass to the continuum limit in all of our results by changing variables from $K$ to $k$ whose components range from $-\pi/a$ to $\pi/a$ which become $-\infty$ to $\infty$ as $a \to 0$. At the same time we replace $A(K) \equiv a(k)/a^{3/2}$ so that $[a(k), a^\dagger(q)] = \delta(k - q)$. After these changes we find as $a \to 0$, $\phi_n(t) \to \phi(x, t)$ and $\pi_n/a^3 \to \pi(x)$:

\begin{equation}
\phi(x, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega(k)}} (a(k)e^{ik \cdot x - i\omega(k)t} + a^\dagger(k)e^{-ik \cdot x + i\omega(k)t}) \tag{1.45}
\end{equation}

\begin{equation}
\pi(x, t) = -i \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar \omega(k)}{2}} (a(k)e^{ik \cdot x - i\omega(k)t} - a^\dagger(k)e^{-ik \cdot x + i\omega(k)t}) \tag{1.46}
\end{equation}

\begin{equation}
H_\phi(t) - E_0 = \int d^3k \hbar \omega(k)a^\dagger(k)a(k) \tag{1.47}
\end{equation}

where now $\omega(k) = \sqrt{\mu^2 + k^2}$, and we have given the time dependence of the fields in Heisenberg picture.

It should be clear that we really never needed the lattice in obtaining these results: we simply had to substitute the above expansions for $\phi, \pi$ into $H_\phi$. The lattice only served to make absolutely concrete the assertion that the field system was a set of coupled oscillators. **Exercise**: Use the expansion of the $\phi$ and $\pi$ in terms of $a(p)$ to prove the canonical commutation relations

\begin{align*}
[\phi(x, t), \phi(y, t)] &= [\pi(x, t), \pi(y, t)] = 0 \\
[\phi(x, t), \pi(y, t)] &= i\hbar \delta(x - y) \tag{1.48}
\end{align*}

We should also note that the momentum operator of the continuum field theory, identified as the generator of translations, is

\begin{equation}
P = -\int d^3x \pi(x) \nabla \phi(x) = \int d^3k \hbar k a^\dagger(k)a(k) \tag{1.49}
\end{equation}

confirming the interpretation of $\hbar k$ as the momentum of the one particle state $a^\dagger(k)|0\rangle$.

1.3.4 Energy Momentum Tensor

As we have seen the energy and momentum of a quantum field system are integrals over all space of corresponding densities. This is a little like expressing charge as the integral of a charge density. As we know the charge density is the time component of a locally conserved 4-vector current density field $J^\mu(x)$, $\partial_\mu J^\mu = 0$. Similarly the densities of energy momentum are time components of a locally conserved 4-tensor field $T^{\mu\nu}(x)$, $\partial_\nu T^{\mu\nu} = 0$:

\begin{align*}
P^\mu &= \int d^3x T^{\mu0}(x, t) \\
T^{00} &= -\partial_0 \phi \nabla^2 \phi, \quad T^{00} = \frac{1}{2} \left( \dot{\phi}^2 + (\nabla \phi)^2 + \mu^2 \phi^2 \right) \tag{1.50}
\end{align*}
You can confirm by inspection that these and the remaining components of $T^{\mu\nu}$ are given by

$$T^{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \eta_{\mu\nu} \mathcal{L} \tag{1.51}$$

where $\mathcal{L}$ is the Lagrange density

$$\mathcal{L} = -\frac{1}{2}((\partial \phi)^2 + \mu^2 \phi^2) \tag{1.52}$$

You can think of this as a covariant analogue of the Legendre transformation which defines the Hamiltonian in terms of the Lagrangian.

**Exercise**: Show that $\partial_\nu T^{\mu\nu} = 0$, by virtue of the Klein-Gordon field equations.

From the energy momentum tensor we can construct a conserved third rank tensor,

$$M^{\mu\nu\rho} = x^{\mu} T^{\nu\rho} - x^{\nu} T^{\mu\rho}$$

since $T^{\mu\nu} = T^{\nu\mu}$ and $T^{\mu\nu}$ is locally conserved. Then $M^{\mu\nu} \equiv \int d^3x M^{\mu\nu\rho}$ are conserved quantities which can be identified with the generators of the Lorentz group:

$$\partial_\rho M^{\mu\nu\rho} = T^{\nu\mu} - T^{\mu\nu} + x^{\mu} \partial_\rho T^{\nu\rho} - x^{\nu} \partial_\rho T^{\mu\rho} = 0 \tag{1.53}$$

1.3.5 Multi-particle States

If we apply two $a^\dagger$'s to the vacuum we get a two particle state

$$a^\dagger(k_1) a^\dagger(k_2) |0\rangle \tag{1.54}$$

with total momentum $\hbar(k_1 + k_2)$ and total energy $\hbar(\omega(k_1) + \omega(k_2))$. Clearly this energetics is that of noninteracting particles. It is highly significant that all multiparticle states are completely symmetric under interchange of the labels of any pair of particles: the scalar field theory predicts Bose statistics for the associated particles!

The example of the scalar field illustrates the main physical aspects of quantum field theory

1. It predicts multi-particle states together with their statistics (Bose or Fermi).

2. It incorporates the requirements of Special Relativity (Poincaré invariance).

3. One can consider two classical limits ($\hbar \to 0$):

   With $\mu$ fixed the limit gives a classical field theory. In this case the Compton wavelength $1/\mu$ is finite in the classical limit.

   With $m = \hbar \mu$ fixed it gives a classical theory of point particles (the Compton wavelength vanishes).

   Thus quantum field theory unites the particle and field concepts.
4. Quantum field theories with linear field equations predict noninteracting (free) particles. Interactions will arise if the field equations have nonlinear terms. Such terms are associated with terms in the action or Hamiltonian with three or more powers of fields. The presence of such terms implies that particle number is not conserved.

5. The scalar field describes spinless particles; more general fields e.g. spinor, vector, tensor describe particles of higher spin.

Finally, let us consider how an approximate quantum particle interpretation can be retrieved from this quantum field theory. Remember the observables are quantum fields, not particle coordinates! To discover properties of a single particle wave packet,

\[ |f\rangle \equiv \int d^3k f(k)a^\dagger(k)|0\rangle, \tag{1.55} \]

we must consider what we can get from measurements of the quantum field

\[ \phi(x, t) = \int \frac{d^3k}{(2\pi)^3/2} \sqrt{\frac{\hbar}{2\omega(k)}} (a(k)e^{ik\cdot x - i\omega t} + a^\dagger(k)e^{-ik\cdot x + i\omega t}). \tag{1.56} \]

First it is easily shown that \( \langle f|\phi(x, t)|f\rangle = 0 \). That is, if we make many measurements of \( \phi \), we find zero on average. This means nothing more than \( \phi \) is negative as often as it is positive. A more sophisticated measurement is to measure \( \phi(x, t)\phi(y, t) \), i.e. the product of the results of simultaneous measurement of the fields at \( x \) and at \( y \). If no particle is present, the average of many repeats of this measurement is

\[ \langle 0|\phi(x, t)\phi(y, t)|0\rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3/2} \frac{\hbar}{\omega(k)} e^{ik\cdot(x-y)}. \tag{1.57} \]

Note that due to the \( 1/\omega(k) \), this is not 0 even when the fields are not measured at the same point. This implies a correlation between the measurements at separate points, but does not contradict causality: \( [\phi(x, t), \phi(y, t)] = 0! \) The vacuum is an energy eigenstate which requires an infinite time to set up. Thus there is plenty of time to set up correlations at distant points.

The zero equal time commutator \( [\phi(x, t), \phi(y, t)] \) is simply one of the canonical commutation relations: fields at different spatial points but at the same time are independent dynamical variables, and hence have zero Poisson brackets. But in a relativistic QFT two fields at different times should continue to commute as long as the spacetime interval \( (x - y)^2 = (x - y)^2 - (x^0 - y^0)^2 > 0 \). We can check this by a simple

**Exercise:**

\[ [\phi(x), \phi(0)] = \int \frac{d^3p}{(2\pi)^32\omega(p)} \left[ e^{ix\cdot p - i\omega(p)t} - e^{-ix\cdot p + i\omega(p)t} \right]. \tag{1.58} \]

This expression is zero if \( t = 0 \) (i.e. the fields are at equal times) because one can do the variable change \( p \to -p \) so the two terms cancel. But the expression is also Lorentz invariant.
under \( x \to \Lambda x \), so if \( x^2 > 0 \) one can go to a Lorentz frame for which \( t = 0 \), proving the commutator is zero. This is important for relativistic causality because otherwise one would have the ability to send signals faster than the speed of light.

Now, suppose a particle is present in the wave packet \( f \), and the same quantity is measured.

\[
\langle f | \phi(x, t) \phi(y, t) | f \rangle = \langle 0 | \phi(x, t) \phi(y, t) | 0 \rangle + \psi^*(x, t) \psi(y, t) + \psi^*(y, t) \psi(x, t).
\]

The presence of the particle causes the change in average results given by the last 2 terms on the r.h.s. which are modulated by the function

\[
\psi(x, t) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\omega(k)}} e^{ik \cdot x - i\omega t} f(k) \quad (1.60)
\]

\[
\approx \psi(x - v_g t, 0) \quad (1.61)
\]

\[
\approx \sqrt{\frac{\hbar}{2\omega(k)}} \psi_S(x - v_g t, 0). \quad (1.62)
\]

The last two forms hold if the packet function \( f \) is narrowly peaked about \( k_0 \). \( \psi_S \) is the one particle Schrödinger wave function corresponding to momentum wave function \( f \). \( v_g \equiv \partial \omega / \partial k|_{k = k_0} \) is the usual group velocity. The disturbance is nonzero only when the point is in the support of the wave function.

One can also measure functions of \( \phi, \partial_\mu \phi \) such as the energy momentum tensor \( T_{\mu\nu}(x, t) \) to learn about the energy and momentum of a particle. One will find that departures of these quantities from their vacuum values are also modulated by the wave function \( \psi_S \). In other words the physics of the free particles predicted by linear quantum field theory is contained in the wave function corresponding to \( f(p) \).

### 1.4 Interacting Scalar Field Theory

So far we have only considered linear field equations, whose associated Lagrangians are quadratic functions of the field. As soon as the field equations become non-linear, the particles associated with the quantum fields will interact with each other. In particular they can scatter.

The simplest way to introduce non-linear terms in the field equations, while preserving Lorentz invariance, is to add Lorentz scalar terms cubic and higher in the fields to the Lagrangian density. for scalar fields this is easy to do. For example any term

\[
- \frac{g_n}{n!} \phi^n(x), \quad \text{for } n > 2
\]

is a scalar which produces a term \( g_n \phi^{n-1} / (n - 1)! \) in the field equation. Since such terms do not involve time derivatives, they do not alter the definition of the conjugate momentum nor
the commutation relations. But clearly plane waves no longer solve the new field equations so there will be interactions.

Nonlinear quantum field theory is too complicated to solve exactly, so we shall first assume the nonlinear terms are small so they can be handled using perturbation theory. In a late chapter we review the formalism of time dependent perturbation theory which we use to calculate scattering amplitudes at weak coupling. Here we restrict attention to only the lowest order approximation to the scattering amplitude in the presence of a small quartic term $-\lambda \phi^4/4!$.

We first have to study what the interaction does to the vacuum (ground state) and to a single particle state. The evolution of the vacuum to first order in the perturbation is

$$\langle 0 \bigg| \left( I - i \frac{\lambda}{4!} \int dt d^3 x \phi^4 \right) \bigg| 0 \rangle = 1 - i \frac{\lambda}{4!} VT \langle 0 | \phi^4 | 0 \rangle$$

(1.64)

where we are using interaction picture where the fields have the time dependence of free fields. Since $\langle 0 | \phi^4 | 0 \rangle$ is independent of $x$ by translation invariance of the vacuum, the integral over space and time just provides a factor of $VT$, the volume of spacetime. We can interpret this effect as a shift in the energy of the vacuum $\Delta E_0 = V \lambda \langle 0 | \phi^4 | 0 \rangle / 4!$. We already subtracted the zero point energy of the oscillators in our definition of the free vacuum energy. So here we learn that in the interacting theory we have to subtract a bit more to keep the vacuum energy zero. Later on we shall learn that we can do this order by order in perturbation theory.

Next we study what the interaction does to the evolution of a single particle state. The matrix element of the interaction term involves

$$-i \frac{\lambda}{4!} \langle 0 | a(p') \phi^4 a^\dagger(p) | 0 \rangle$$

$$= -i \frac{\lambda}{3!} \langle 0 | \phi^2 a^\dagger(p_1) | 0 \rangle \frac{e^{-ip' \cdot x - i\omega'_1 t}}{(2\pi)^{3/2} \sqrt{2\omega'_1}} - i \frac{\lambda}{4!} \langle 0 | \phi^4 | 0 \rangle \delta(p'_1 - p_1)$$

$$= -i \frac{\lambda}{2} \langle 0 | \phi^2 | 0 \rangle \frac{e^{-i(p'_1 - p) \cdot x - i(\omega'_1 - \omega_1)t}}{(2\pi)^{3/2} \sqrt{2\omega'_1 \omega_1}} - i \frac{\lambda}{4!} \langle 0 | \phi^4 | 0 \rangle \delta(p'_1 - p_1)$$

(1.65)

Integration over space gives

$$-i \frac{\lambda}{4!} \langle 0 | a(p') \rangle \int d^3 x \phi^4 a^\dagger(p) | 0 \rangle$$

$$= -i \frac{\lambda}{2} \langle 0 | \phi^2 | 0 \rangle \frac{1}{2\omega_1} \delta(p'_1 - p_1) - i \frac{\lambda}{4!} V \langle 0 | \phi^4 | 0 \rangle \delta(p'_1 - p_1)$$

(1.66)

The second term just describes the correction to the vacuum energy which is cancelled after we redefine the energy so that the vacuum energy stays zero. The first term represents a shift in the single particle energy by the amount

$$\Delta E(p) = \frac{\lambda}{4\omega(p)} \langle 0 | \phi^2 | 0 \rangle$$

(1.67)
The change of a single particle energy under a shift in its mass $\Delta m^2$ is

$$\delta \omega = \frac{\Delta m^2}{2\omega(p)}$$

(1.68)

so we see that the shift in single particle energy corresponds to a shift of the mass squared of the particle by an amount

$$\Delta m^2 = \frac{\lambda}{2} \langle 0 | \phi^2 | 0 \rangle.$$  

(1.69)

This shows that the energy correction we have calculated is consistent with Lorentz covariance.

We have to distinguish the actual physical mass $m$ of a particle from the coefficient of the $\phi^2$ term $\mu \to \mu_0$ in the lagrangian. When we do perturbation theory we would like to use the actual physical masses of the particles of the theory at zeroth order. This is given in terms of the “bare mass” $\mu_0$ by a series like

$$m^2 = \mu_0^2 + \lambda \mu_1^2 + \lambda^2 \mu_2^2 + \cdots \equiv \mu_0^2 + \delta m^2.$$  

(1.70)

To do perturbation theory about a zeroth order that describes the actual masses, we can write $\mu_0^2 = m^2 - \delta m^2$, plug it into the lagrangian, and then use the $m^2$ term in the zeroth order lagrangian, and include the $\delta m^2$ term in the interaction part of the lagrangian. Then $\delta m^2$ is determined by the order by order condition that the mass shift is zero. That is it is chosen to cancel the mass shift due to the interactions. Putting the $\delta m^2$ term into the first order correction adds a term $-\delta m^2/(2\omega_1)$ to the shift in energy, so the total change in energy reads

$$\Delta E = \frac{\lambda}{4\omega_1} \langle 0 | \phi^2 | 0 \rangle - \frac{\delta m^2}{2\omega_1}$$

(1.71)

so the requirement that $\Delta E = 0$ to lowest order gives

$$\delta m^2 = \frac{\lambda}{2} \langle 0 | \phi^2 | 0 \rangle + O(\lambda^2)$$

(1.72)

The higher order terms in $\delta m^2$ are chosen to ensure the cancellation of all higher order corrections to $\Delta E$.

Finally we consider the scattering of two particles, which we can infer from the time evolution of the two particle state in lowest order. We need the matrix element

$$-i\frac{\lambda}{4!} \langle 0 | a(p'_1) a(p'_2) \phi^4 a^\dagger(p_1) a^\dagger(p_2) | 0 \rangle$$

$$= -i\frac{\lambda}{3!} \langle 0 | a(p'_1) \phi^3 a^\dagger(p_1) a^\dagger(p_2) | 0 \rangle \frac{e^{-ip'_2 \cdot x - i\omega'_2 t}}{(2\pi)^{3/2} \sqrt{2\omega'_2}}$$

$$+ \delta(p'_2 - p'_1) \left[ i\frac{\lambda}{4!} \langle 0 | a(p'_1) \phi^4 a^\dagger(p_2) | 0 \rangle \right] + (1 \leftrightarrow 2)$$

(1.73)
The terms on the last line contribute only to the mass shift of particle 1 and particle 2 and to the vacuum energy shift. According to our setup these terms will be cancelled: so in practice we can simply drop them. Then we continue the evaluation of the term on the second line

\[ -\frac{i\lambda}{3!}(0|a(p'_1))\phi^3a^\dagger(p_1)a^\dagger(p_2))(0)\frac{e^{-ip'_2\cdot x-i\omega'_2 t}}{(2\pi)^{3/2}\sqrt{2\omega'_2}} \]

\[ = -\frac{i\lambda}{2}(0|\phi^2a^\dagger(p_1)a^\dagger(p_2))(0)\frac{e^{-i(p'_2+p'_1\cdot x-i(\omega'_2+\omega'_1) t)}}{(2\pi)^{3/2}\sqrt{\omega'_1\omega'_2}} + \text{Terms to be dropped} \]

\[ = -i\lambda\frac{e^{-i(p'_2+p'_1-p_1-p_2))}x-i(\omega'_2+\omega'_1-\omega_1-\omega_2) t}}{(2\pi)^6\sqrt{\omega'_1\omega'_2\omega_1\omega_2}} + \text{Terms to be dropped} \]

Integrating over space and time produces the final result for the scattering matrix

\[ [\delta(p'_1-p_1)\delta(p'_2-p_2) + (1 \leftrightarrow 2)] - i\lambda \frac{(2\pi)^4\delta^4(p'_1+p'_2-p_1-p_2)}{(2\pi)^6\sqrt{(2E_1)(2E_2)(2E'_1)(2E'_2)}} \]

Later on we shall see that the structure of this expression is universal for a general two particle scattering process: simply replace the factor $-i\lambda$ by a general Lorentz covariant expression $M$ depending on the momenta of the particles. Here we see that, in lowest order in the $\phi^4$ term of the lagrangian, $M$ is the constant $-i\lambda$. The differential cross section is proportional to $|M|^2$

### 1.5 The Free Electromagnetic Field

#### 1.5.1 Quantum Electromagnetic Field

Treated classically the EM field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfies Maxwell’s equations

\[ \partial_\nu F^{\nu\mu} = J^\mu, \]

which imply current conservation $\partial_\mu J^\mu = 0$ for consistency.

The source free ($j^\mu = 0$) Maxwell’s equations follow from stationarity of the action

\[ S = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \int d^4x (E^2 - B^2), \]

under $A_\mu \to A_\mu + \delta A_\mu$. Then the momentum conjugate to $A_\mu$ is

\[ \Pi^\mu = F^\mu_0 = \partial_0 A^\mu - \partial^\mu A_0. \]
The spatial components of $\Pi$ are just those of minus the electric field strength $\Pi = -E$, but $\Pi^0 = 0$. This last fact poses a difficulty for quantization since it is inconsistent with nonvanishing canonical commutation relations. We know how to assign operator properties to $A$ and $\Pi$, but not to $A_0$. Before facing this difficulty, we construct the canonical Hamiltonian

$$H_{\text{CAN}} = \int d^3x \left[ \Pi \cdot \dot{A} - L \right]$$

$$= \int d^3x \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \times A)^2 + \nabla A_0 \cdot \Pi \right].$$

Notice that the troublesome variable $A_0$ appears only linearly and in the last term. After an integration by parts the coefficient is just $-\nabla \cdot \Pi$, which would vanish if we could use Gauss’ Law $\nabla \cdot E = 0$. Dirac pioneered a way to quantize systems in the presence of such constraints on the phase space, which we shall return to later. However, here we use a more heuristic method, which is to use the constraints to eliminate physically redundant field components. This is called “fixing the gauge”, and is motivated by gauge invariance: the field strengths are unchanged under the transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ where $\Lambda$ is an arbitrary function of space and time. We interpret this ambiguity by postulating that two vector potentials differing by a gauge transformation are physically equivalent: all physical observables are required to be gauge invariant.

To fix the gauge ambiguity one specifies a gauge condition, here we take Coulomb gauge $\nabla \cdot A = 0$. Then Gauss’ law simplifies to

$$\nabla \cdot E = -\nabla \cdot \Pi = -\nabla^2 A_0 = J^0$$

which is solved for $A_0$ by setting

$$A_0(x, t) = -\frac{1}{\nabla^2} J_0(x, t) = \int d^3y \frac{J_0(y, t)}{4\pi |x - y|} \to 0, \quad \text{for} \quad J^\mu = 0. \tag{1.82}$$

The longitudinal component of $\Pi$ is also eliminated because $\nabla \cdot \Pi = \nabla^2 A_0$. One passes to quantum mechanics by promoting only $A = A_T$ and $\Pi_T$ to operators. (If the currents are operators, $A_0$ is an operator by virtue of the constraint, but it is not independent.) Then the transverse projector must appear on the right side of the canonical commutation relations

$$[A_T^k(x), \Pi_T^l(y)] = i \left( \delta_{kl} - \frac{\nabla^k \nabla^l}{\nabla^2} \right) \delta(x - y). \tag{1.83}$$

The subscript $T$ is to remind us that $\Pi$ and $A$ have zero divergence.

### 1.5.2 Energy momentum tensor

From classical electrodynamics a gauge invariant and symmetric energy momentum tensor for the free electromagnetic field can be taken to be

$$T^{\mu\nu} = F_\rho^{\mu} F^{\nu\rho} - \eta^{\mu\nu} \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma}. \tag{1.84}$$
It is conserved by virtue of Maxwell’s equations

\[ \partial_{\nu} T^{\mu\nu} = (\partial_{\nu} F_{\rho}^{\mu}) F^{\nu \rho} + F_{\rho}^{\mu} \partial_{\nu} F^{\nu \rho} - \frac{1}{2} F_{\rho \sigma} \partial^{\mu} F^{\rho \sigma} \]

\[ = (\partial^{\mu} \partial_{\rho} A_{\rho} - \partial_{\rho} \partial_{\rho} A^{\mu}) F^{\nu \rho} - \frac{1}{2} F_{\rho \sigma} \partial^{\mu} F^{\rho \sigma} \]

\[ = (\partial^{\mu} \partial_{\rho} A_{\rho}) F^{\nu \rho} - \frac{1}{2} F_{\rho \sigma} \partial^{\mu} F^{\rho \sigma} = 0 \tag{1.85} \]

the second line uses Maxwell’s equations, and the third line exploits the antisymmetry of \( F^{\rho \sigma} \).

The energy and momentum densities are

\[ T^{00} = E^{2} + \frac{1}{2} (B^{2} - E^{2}) = \frac{1}{2} (B^{2} + E^{2}) \tag{1.86} \]

\[ T^{i0} = F_{k}^{i} F^{0k} = \epsilon_{ikj} B^{j} E^{k} = (E \times B)^{i} \tag{1.87} \]

Since \( T^{\mu\nu} \) is symmetric we can immediately write down the Lorentz generators

\[ M^{0k} = \int d^{3}x (x^{0} T^{k0} - x^{k} T^{00}), \quad M^{kl} = \int d^{3}x (x^{k} T^{l0} - x^{l} T^{k0}) \tag{1.88} \]

We shall need this information in the next section.

### 1.5.3 Polarization and Helicity of Photons.

An explicit realization of the commutation relations can be given in terms of creation and annihilation operators as follows (Recall that for the free EM field \( \omega(k) = |k| \)):

\[ A_{T_{k}}(x,0) = \int \frac{d^{3}k}{\sqrt{(2\pi)^{3}2|k|}} [a_{k}(k)e^{ik \cdot x} + a_{k}^{\dagger}(k)e^{-ik \cdot x}] \tag{1.89} \]

\[ \Pi_{T_{k}}(x,0) = -i \int \frac{d^{3}k \sqrt{|k|}}{(2\pi)^{3}2} [a_{k}(k)e^{ik \cdot x} - a_{k}^{\dagger}(k)e^{-ik \cdot x}] \tag{1.90} \]

The Coulomb gauge condition \( \nabla \cdot A = 0 \) and Gauss’ Law \( \nabla \cdot \Pi = 0 \) imply that \( k \cdot a = 0 \), so the commutation relations for the \( a \)’s read:

\[ [a_{k}(k), a_{m}^{\dagger}(q)] = \left( \delta_{km} - \frac{k_{k}k_{m}}{k^{2}} \right) \delta(k - q). \tag{1.91} \]

Inserting these into \( H_{\text{eff}} \) gives

\[ H_{\text{eff}} \equiv H - E_{0} = \int d^{3}k |k| a_{i}^{\dagger}(k) \cdot a(k), \tag{1.92} \]

where \( E_{0} \) is the usual (infinite) zero point energy of the oscillators which will be dropped from now on. This formula shows us immediately that for \( J^{\mu} = 0 \), the quantum e.m. field
is interpretable as a system of massless bosons (photons). The vacuum $|0\rangle$ is defined by
\[ a_k^\dagger(q_1)a_{m_2}^\dagger(q_2)\cdots a_{m_n}^\dagger(q_n)|0\rangle. \] (1.93)

Because of transversality there are two photon states for each momentum. These two polarization states will next be shown to correspond to the two helicities $\pm 1$ of the photon. The helicity acting on a single particle momentum eigenstate is defined by
\[ h|k\rangle = \frac{k \cdot J}{|k|} |k\rangle. \] (1.94)

The orbital angular momentum part of $J$ does not contribute to the helicity because $p \cdot (x \times p) = 0$. Thus the helicity of a spinless particle is zero.

First, for fixed $k$, let us introduce two (in general complex) basis vectors $\epsilon_a$, $a = 1, 2$ for the plane perpendicular to $k$, satisfying $k \cdot \epsilon_a = 0$ and the orthonormality and completeness relations
\[
\epsilon_a \cdot \epsilon_b^* = \delta_{ab} \quad \sum_a \epsilon_a^m \cdot \epsilon_a^{n*} = \delta_{mn} - \frac{k^m k^n}{k^2}.
\] (1.95) (1.96)

We can then introduce two independent sets of creation and annihilation operators via
\[ a(k) = \sum_a \epsilon_a a_a(k). \] (1.97)

We shall relate the multiplicity associated with the index $a$ to the spin of the photon. First recall the classical expression for the angular momentum carried by the e.m. field,
\[
J = \int d^3x \mathbf{x} \times (\mathbf{E} \times \mathbf{B}) = \int d^3x \sum_k E_k \mathbf{x} \cdot \nabla A_k - \int d^3x \mathbf{x} \times (\mathbf{E} \cdot \nabla) \mathbf{A}.
\] (1.98) (1.99)

We can recognize the first term in the last line as the “orbital” angular momentum, which will not contribute to the helicity of a one photon state. This is because acting on a one photon state the $\nabla$ is replaced by $k$ and because of the cross product the term will be perpendicular to $k$. The second term, after an integration by parts becomes
\[
S = \int d^3x \mathbf{E} \times \mathbf{A} = -\int d^3x \dot{\mathbf{A}} \times \mathbf{A} \quad = -i \int d^3k \epsilon_b \mathbf{a}^\dagger(k) \times \mathbf{a}(k) = -i \int d^3k a_b^\dagger \epsilon^*_b \epsilon_a a_a
\] (1.100) (1.101)
Applying $S$ to a one photon state $a_\alpha^\dagger(k)|0\rangle$, yields

$$Sa_\alpha^\dagger(k)|0\rangle = i \sum_b (\epsilon_\alpha \times \epsilon_b^*)a_b^\dagger(k)|0\rangle. \quad (1.102)$$

Thus we see that the $2 \times 2$ matrix $S_{ab} = i\epsilon_\alpha \times \epsilon_b^*$ acts as a spin matrix on the index of the creation operator. Note that the transverse components of $S$ are zero! To get the helicity interpretation, consider the case of $k = k\hat{z}$. Then the helicity matrix is

$$S_{3ab}^3 = i(\epsilon_\alpha^1 \epsilon_b^{2*} - \epsilon_\alpha^2 \epsilon_b^{1*}). \quad (1.103)$$

This matrix is $\text{diag}\{1, -1\}$ with the choices

$$\epsilon_1 = (1,i,0)/\sqrt{2} \quad \text{and} \quad \epsilon_2 = (1,-i,0)/\sqrt{2}, \quad (1.104)$$

so with this choice of polarization vectors, $a_\alpha^\dagger$ creates a photon with helicity $+1$ and $a_\alpha^\dagger$ creates a photon with helicity $-1$. This establishes that the photon is a spin one particle. There is no zero helicity state for the photon: this is consistent with Poincaré invariance because the photon is massless, and the helicity of a massless particle is Poincaré invariant.

Notice that the other components of photon spin vanish $S^1 = S^2 = 0$ for this momentum. This means that the three matrices $S^1, S^2, S^3$ do not satisfy the Lie algebra of $O(3)$. Instead $[S^k, S^l] = 0$. The latter algebra holds for general $k$. Indeed we can write $S = \hat{k}h$. These peculiar properties of photon spin are due to the photon’s masslessness, the photon cannot be brought to rest. The spin of a massive particle, which can be brought to rest, must generate $O(3)$. In particular a massive spin 1 particle must have all three helicities $\pm 1, 0$.

The polarization vector enters scattering amplitudes multilinearly, with a factor of $\epsilon$ for each incoming photon and a factor $\epsilon^*$ for each outgoing photon. Its four-vector index forms a Minkowski scalar product with that of the vertex coupling the gauge potential to the charged fields. According to gauge invariance, this vertex satisfies current conservation: its scalar product with the momentum entering it gives zero. Thus changing each polarization vector by an amount proportional to its four-momentum leaves the scattering amplitude unaltered. In Coulomb gauge the polarization vector is of the form $\epsilon = (\epsilon, 0)$ with $k \cdot \epsilon = 0$, so $k_\mu \epsilon^\mu = 0$.

But $k_\mu k^\mu = 0$ since the photon is massless. Thus we can characterize the polarization vector completely by the covariant condition $k_\mu \epsilon^\mu = 0$. Any further specification, e.g. $\epsilon^0 = 0$, is merely a gauge choice which can be made at will and exploited to simplify detailed calculations. This is particularly advantageous in the calculation of Compton scattering for polarized photons.
Chapter 2

The Dirac Equation

In our development of relativistic quantum mechanics we shall study intensively the case of spin 1/2 particles solved by Dirac. After this we shall return to the problem of higher spin. As a practical matter, the Dirac equation contains a tremendous amount of the present understanding of elementary particles. In fact, before 2012 all of the known fundamental constituents of matter are either spin 1/2 (quarks and leptons), or spin 1 gauge particles (photon, gluons, weak vector bosons). To these we now have to add a fundamental scalar (Higgs particle, discovered in 2012). Of course there is also the spin 2 graviton that must be understood eventually. But we can go a tremendous distance to understanding the physics of the standard model through study of the Dirac equation interacting with a gauge field. Indeed, for many applications the gauge field doesn’t even need to be quantized!

Dirac’s original motivation for his equation is still useful and inspiring to recall. The most straightforward attempt to write down a relativistic version of the Schrödinger equation

\[ i\hbar \frac{\partial \psi(x,t)}{\partial t} = \sqrt{m^2c^4 - (\hbar c)^2 \nabla^2} \psi(x,t) \]  

\[ \equiv m \sum_{k=0}^{\infty} \left( \frac{1/2}{n} \right) \left( \frac{-\hbar^2 \nabla^2}{m^2} \right)^n \psi(x,t) \] (2.2)

is horribly nonlocal, involving an arbitrarily high number of spatial derivatives. Dirac proposed to get around this by making \( \psi \) a multicomponent wave function and defining the square root in a local way by using matrices, in analogy with the properties of the Pauli spin matrices

\[(\sigma \cdot \mathbf{v})^2 = v^2.\] (2.3)

To achieve this he required the introduction of four anticommuting matrices \( \gamma^\mu \) with \( \mu = 0, 1, 2, 3 \):

\[ \{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta^{\mu\nu}. \] (2.4)

Using (2.4) it is simple to show that

\[ (\gamma^\mu \partial_\mu)^2 = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu = -\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2, \] (2.5)
where we have chosen units for which $c = 1$. We shall also choose units so that $\hbar = 1$. Then the Dirac equation is
\begin{equation}
\frac{1}{i} \gamma^\mu \partial_\mu \psi + m \psi = 0.
\end{equation}
(2.6)

To cast this equation as a relativistic Schrödinger equation we rewrite it as
\begin{equation}
i \frac{\partial \psi(x, t)}{\partial t} = \left( \frac{1}{i} \alpha \cdot \nabla + \beta m \right) \psi(x, t)
\end{equation}
where we have multiplied through by $\gamma^0 \equiv \beta$ using $\beta^2 = I$ and have defined $\alpha \equiv \gamma^0 \gamma$.

**Exercise**: Show that if $x' = \Lambda x = e^{-\lambda x}$ is a Lorentz transformation, then
\[ \psi'(x') = e^{-i\lambda \mu \nu \sigma^\mu \sigma^\nu / 4} \psi(\Lambda^{-1} x') \]
satisfies the Dirac equation in the new frame, with the same $\gamma^\mu$, and $\sigma^{\mu \nu} = (i/2)[\gamma^\mu, \gamma^\nu]$.

## 2.1 Single particle interpretation

### 2.1.1 The Matrices $\gamma^\mu$

So far we have not specified the gamma matrices. It is simple to show that they must be at least $4 \times 4$ (In $D$ space-time dimensions the minimum size is $2^{D/2} \times 2^{D/2}$ for even $D$.)

Let $\sigma^k$, $k = 1, 2, 3$ be the $2 \times 2$ Pauli matrices
\begin{align*}
\sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
(2.8)

Then we have two popular representations:

**Standard Representation of $\gamma^\mu$**

**Chiral (Natural) Representation of $\gamma^\mu$**

\begin{align*}
\gamma^0 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} & \gamma^0 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\
\gamma &= \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} & \gamma &= \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \\
\alpha &= \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} & \alpha &= \begin{pmatrix} -\sigma & 0 \\ 0 & \sigma \end{pmatrix} \\
\gamma_5 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} & \gamma_5 &= \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \\
\sigma_{km} &= \epsilon_{kmn} \begin{pmatrix} \sigma^n & 0 \\ 0 & \sigma^n \end{pmatrix} & \sigma_{km} &= \epsilon_{kmn} \begin{pmatrix} \sigma^n & 0 \\ 0 & \sigma^n \end{pmatrix}
\end{align*}
(2.9, 2.10, 2.11, 2.12, 2.13)

where $\epsilon_{kmn}$ is the completely antisymmetric three tensor with $\epsilon_{123} = +1$. The standard representation is more convenient for slowly moving particles, whereas the chiral one is more convenient for massless fermions that move at the speed of light.
Clearly $\beta = \gamma^0$ is hermitian, $\gamma$ is anti-hermitian, and $\alpha$ is hermitian. Thus the Hamiltonian

$$H = \frac{1}{i} \alpha \cdot \nabla + \beta m$$

(2.14)

is a hermitian operator as it should be.

### 2.1.2 Spin

To justify the interpretation of the Dirac particle as spin $1/2$, we need to construct an angular momentum operator for which $\alpha$ transforms as a vector. This is clearly accomplished by choosing the spin operator to be $\Sigma/2$ with

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$  

(2.15)

Then the Dirac Hamiltonian obviously commutes with the total angular momentum

$$J = r \times \frac{1}{i} \nabla + \Sigma.$$  

(2.16)

The Dirac wave function transforms under the $\frac{1}{2} \oplus \frac{1}{2}$ representation of the rotation group. In particular it describes a spin $1/2$ particle.

To get the energy spectrum of the Dirac particle we must find all the eigenstates of the Hamiltonian (2.14). Clearly, it is best to work with momentum eigenstates

$$\psi_p = u(p)e^{ipx},$$

(2.17)

since the momentum operator commutes with $H$. Then the coefficient spinor must satisfy

$$(\alpha \cdot p + \beta m)u(p) = Eu(p).$$

(2.18)

Since the square of the matrix on the l.h.s. is just $m^2 + p^2$, we immediately learn that

$$E(p) = \pm \sqrt{m^2 + p^2}.$$  

(2.19)

### 2.1.3 Explicit Solutions

It is also fairly simple to find the eigenfunctions in the Standard Representation. It is sufficient to find the positive energy eigenfunctions, because the negative energy ones can be obtained from them by a simple operation: Complex conjugating (2.18) and multiplying both sides by $i\gamma^2$, we learn that $i\gamma^2 u^*(-p)$ is an eigenstate of $H$ with eigenvalue $-E$ if $u(p)$ is an eigenstate with eigenvalue $+E$. An explicit solution for $u$, with $E = +\omega(p) > 0$, is easily shown to be

$$u(p) = \left( \begin{array}{c} \sqrt{m+\omega} \phi \\ \frac{\sigma p}{\sqrt{m+\omega}} \phi \end{array} \right), \quad E = \omega > 0.$$  

(2.20)
where $\omega = \sqrt{m^2 + \mathbf{p}^2}$ and $\phi$ is any two component spinor. The two independent components just represent the two spin states of a spin 1/2 particle. As we have explained the negative energy eigenfunctions are then

$$u_-(\mathbf{p}) = i\gamma^2 u^*(-\mathbf{p}), \quad E = -\omega < 0. \quad (2.21)$$

In the Natural or Chiral representation, the positive energy eigenfunctions can be written

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{\omega(\mathbf{p}) - \mathbf{p} \cdot \mathbf{\sigma}} \phi \\ \sqrt{\omega(\mathbf{p}) + \mathbf{p} \cdot \mathbf{\sigma}} \phi \end{pmatrix} \quad (2.22)$$

and again the negative energy spinor is obtained from the positive energy one by (2.21). In both standard and natural representations, we have $u^\dagger u = 2\omega(\mathbf{p})\phi^\dagger \phi$. The matrices under the square roots in (2.22) have positive eigenvalues, and we always take the positive square root.

### 2.1.4 The Dirac sea

The positive energy states of the Dirac particle give the desired description of a relativistic spin 1/2 particle. The negative energy states are a disaster for the single particle interpretation of the Dirac wave function. As soon as the Dirac particle in a positive energy state is subjected to external forces transitions to negative energy states will be induced and the system is unstable. Dirac himself proposed the remedy for this problem, which exploits the Pauli Exclusion Principle for fermions: If one postulates that the Dirac particle is a fermion, then one can consider the “vacuum” to be the state in which all of the negative energy levels are filled, then any further particles added to this state must by the exclusion principle occupy only positive energy states! This vacuum state is sometimes called the “negative energy sea.” Although we call this state the vacuum, it is clear from its construction that it is far from empty, since it contains an infinite number of particles. The presence of the sea will make itself felt as soon as one considers interactions of the Dirac particles with themselves or with independent force fields. Strictly speaking, there is an enormous (infinitely) negative energy associated with the sea. However in quantum mechanics, only energy differences are measurable (in the absence of gravity) and we might as well measure the energy of all states relative to that of the sea, i.e. we take $E_{\text{sea}} = 0$. The total momentum contained in the sea is automatically zero because momentum states in all directions are occupied.

In addition if the Dirac particle is charged the sea possesses an infinite uniform positive charge density. It is convenient in this case to postulate a compensating negative background charge density, so our vacuum will be neutral.

To understand the dynamical significance of the presence of the sea, it is instructive to consider the excitation spectrum of the whole system. We can first of all add $N$ particles with momenta $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$ to the sea. The resulting state will have energy

$$E_{N \text{ particles}} = \sum_k \sqrt{m^2 + \mathbf{p}_k^2} \quad (2.23)$$
and $N$ units of charge.

Next we can remove $N$ particles with momenta $q_1, q_2, \ldots, q_N$ from the sea. This state, which we describe as a state of $N$ holes, will also have a positive energy

$$E_{N \text{ holes}} = \sum_k \sqrt{m^2 + q_k^2}$$

(2.24)

since we have subtracted negative energy. However the momentum of this state is

$$P_{N \text{ holes}} = -\sum_k q_k$$

(2.25)

for the same reason, and the charge is $-N$ units. Thus this state appears to be a state of $N$ particles of opposite charge to the ordinary (positive energy) Dirac particle and with momenta $-q_1, -q_2, \ldots, -q_N$. This is how antiparticles appear in the theory.

Finally, one can simply excite $N$ particles from the sea to positive energy states. This excitation does not change the number of particles so it is neutral. The energy of this state is the sum of the positive energy eigenvalues occupied minus the sum of the negative energy levels vacated. It is a state with $N$ particles and $N$ antiparticles (holes), with corresponding energy. In principle, this state can be prepared from the sea by delivering sufficient energy, at least $2Nm$, to the system.

If the original Dirac particle is the electron with charge $-e$, then the theory predicts the existence of its antiparticle, the positron, with charge $+e$. The positron was unknown when Dirac discovered his equation, and the idea of identifying the positron with the proton failed because the latter does not have the electron’s mass nor could it annihilate with an electron into photons. Thus when the positron was eventually discovered, it was a spectacular confirmation of Dirac’s theory.

### 2.2 Second Quantization

We have seen that the consistent interpretation of the Dirac theory requires the presence of the sea of an infinite number of negative energy particles. In practice, however, at least if interactions are weak, all but a finite number of particles in the sea are spectators in any calculation. Thus we need a formalism that allows us to concentrate only on the part of the sea that is active in a given process. The formalism which achieves this has been developed, and for historical reasons goes under the name of second quantization. It should be stressed that it is completely equivalent to a description in terms of the many body Schrödinger wave function for the sea, but it is much less cumbersome, and almost indispensable to efficient calculation. The formalism is applicable to any system of identical particles, bosons or fermions. We shall present only the fermion case in lecture since we will only be applying it to the Dirac theory. The changes needed for the boson case will be indicated in a series of exercises.
2.2.1 General Formalism

We begin with the concept of the occupation number basis. Consider a system of fermions, and let an orthonormal complete set of states for a single fermion be labeled by an index $\alpha$:

$$\psi_\alpha(x), \quad \int d^3x \psi_\alpha^*(x) \psi_{\beta}(x) = \delta_{\alpha\beta}. \quad (2.26)$$

Then a multi-fermion antisymmetrized tensor product state is completely specified by the set of numbers $\{n_\alpha\}$ which give the number of fermions (0 or 1) occupying each state $\alpha$.

Now the crucial idea of second quantization is to define a creation operator $b_\alpha^\dagger$ by

$$b_\alpha^\dagger |n_1 \ldots n_\alpha \ldots \rangle = (-)^{\sum_{\gamma<\alpha} n_\gamma} |n_1 \ldots n_\alpha + 1 \ldots \rangle. \quad (2.27)$$

The state dependence of the prefactor is necessary for consistency with the antisymmetry of the state under fermion interchange. Since $b_\alpha^\dagger$ adds a fermion in state $\alpha$ its square must vanish. It is a direct consequence of this fact and its definition that

$$\{b_\alpha^\dagger, b_\beta^\dagger\} = 0 \quad (2.28)$$

By considering

$$\langle\{n\}|b_\beta^\dagger|\{n'\}\rangle^* = \langle\{n'\}|b_\alpha^\dagger|\{n\}\rangle \quad (2.29)$$

it is clear that $b_\alpha$ removes a fermion from the state $\alpha$ and so is an annihilation operator. It is then simple to check that

$$\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}. \quad (2.30)$$

For our application, the state label will include a continuous momentum as well as a discrete label. In that case the Kronecker delta will of course include a factor of a Dirac delta function $\delta(p - p')$.

Having introduced the creation and annihilation operators, we now have a very efficient notation for the occupation number basis. Call the state with no states occupied $|0'\rangle$. Then the state with

$$n_{\alpha_1} = n_{\alpha_2} = \cdots = n_{\alpha_N} = 1 \quad (2.31)$$

and all other occupation numbers zero is just

$$b_{\alpha_N}^\dagger \cdots b_{\alpha_2}^\dagger b_{\alpha_1}^\dagger |0'\rangle. \quad (2.32)$$

We are now in a position to relate all this formalism to the standard many-body Schrödinger wave function description. The wave function describing the state in which the single particle states $\alpha_1 \cdots \alpha_N$ are occupied is just

$$\Psi_{\alpha_1 \cdots \alpha_N}(x_1, \ldots, x_2) = \frac{1}{\sqrt{N!}} \sum_p \delta_p \psi_{\alpha_1}(x_{P(1)}) \psi_{\alpha_2}(x_{P(2)}) \cdots \psi_{\alpha_N}(x_{P(N)}) \quad (2.33)$$
where $\delta_P = -1$ if $P$ is an odd permutation and +1 if $P$ is an even permutation. If we define the field operator

$$\psi(x) \equiv \sum_{\alpha} b_{\alpha} \psi_{\alpha}(x),$$  \hspace{1cm} (2.34)

with anticommutation relations

$$\{\psi(x), \psi(y)\} = \{\psi^\dagger(x), \psi^\dagger(y)\} = 0$$  \hspace{1cm} (2.35)

$$\{\psi(x), \psi^\dagger(y)\} = \sum_{\alpha} \psi_{\alpha}(x) \psi_{\alpha}^*(y) = \delta(x - y),$$  \hspace{1cm} (2.36)

Then we have

$$\Psi_{\alpha_1 \cdots \alpha_N}(x_1, \cdots, x_N) = \frac{1}{\sqrt{N!}} \langle 0' | \psi(x_1) \psi(x_2) \cdots \psi(x_N) b_{\alpha_N}^\dagger \cdots b_{\alpha_2}^\dagger b_{\alpha_1}^\dagger | 0' \rangle.$$  \hspace{1cm} (2.37)

A completely general $N$ body wave function is of course a general superposition of such states

$$\Phi = \sum_{\alpha_1 \cdots \alpha_N} c_{\alpha_1 \cdots \alpha_N} \Psi_{\alpha_1 \cdots \alpha_N} = \frac{1}{\sqrt{N!}} \langle 0' | \psi(x_1) \psi(x_2) \cdots \psi(x_N) | \Phi \rangle$$  \hspace{1cm} (2.38)

where

$$| \Phi \rangle = \sum_{\alpha_1 \cdots \alpha_N} c_{\alpha_1 \cdots \alpha_N} b_{\alpha_N}^\dagger \cdots b_{\alpha_2}^\dagger b_{\alpha_1}^\dagger | 0' \rangle$$  \hspace{1cm} (2.39)

The state $| \Phi \rangle$ thus can describe the most general many body Schrödinger wave function. Indeed, the second quantization formalism allows one to even superpose states of different numbers of particles, so it is the more general description!

We have shown how to describe general quantum states, it remains to show how linear operators are related between the two formalisms. All possible observables in the Schrödinger wave function description are completely symmetric in the operators acting on each particle. Such operators can be classified according to the number of particles involved in each term of the operator. For example, a one body operator acting on $N$ identical particles has the form

$$\Omega^{(1)} = \sum_k \omega_k$$  \hspace{1cm} (2.40)

where $\omega_k$ acts only on the variables of the $k^{th}$ particle. A two body operator would be a sum of terms, each acting on a pair of particles:

$$\Omega^{(2)} = \sum_{k<m} \omega_{km}.$$  \hspace{1cm} (2.41)

A $K$ body operator is obviously a sum of terms each acting on $K$ particles. In a typical nonrelativistic system with $N$ identical particles, e.g. the electrons of a $Z = N$ atom, the
kinetic energy of the electrons is a 1 body operator, while the potential energy is a two body operator (since the electrons interact in pairs). The interaction of the electrons with an external field would be described by another one body operator.

The second quantized description of these operators is straightforward. Starting with the one body case we first represent the single particle operator $\omega$ by its matrix elements in the single particle basis we have introduced

$$\omega_{\alpha\beta} = \langle \alpha | \omega | \beta \rangle. \quad (2.42)$$

Then the one body operator $\Omega^{(1)}$ is just

$$\Omega^{(1)} = \sum_{\alpha\beta} b^\dagger_{\alpha} \omega_{\alpha\beta} b_{\beta}. \quad (2.43)$$

To see that this definition has the correct action on multi-particle states, one simply applies it to one of the basis states

$$\Omega^{(1)} b^\dagger_{a_{N}} \cdots b^\dagger_{a_{2}} b^\dagger_{a_{1}} |0'\rangle = \sum_k \sum_{\gamma_k} (b^\dagger_{\gamma_k} \langle \gamma_k | \omega | \alpha_k \rangle \cdots b^\dagger_{a_1} |0'\rangle \quad (2.44)$$

and we see that the r.h.s. is a sum of terms for the $k^{th}$ of which $\omega$ has the correct action on the $k^{th}$ single particle state label.

Multi-body operators have the obvious analogous second quantized description. We limit the discussion here to the two-body case. Let $|\beta_1\rangle |\beta_2\rangle$ denote the standard unsymmetrized two particle tensor product state. It is convenient to reverse the order of factors in the corresponding bra $\langle \alpha_2 | \langle \alpha_1 |$ so that

$$\langle \alpha_2 | \langle \alpha_1 | \beta_1 \rangle \beta_2 \rangle = \delta_{\alpha_1 \beta_1} \delta_{\alpha_2 \beta_2}. \quad (2.45)$$

Then we define

$$\omega_{\alpha_2 \alpha_1, \beta_1 \beta_2} = \langle \alpha_2 | \langle \alpha_1 | \omega | \beta_1 \rangle | \beta_2 \rangle \quad (2.46)$$

and the two-body operator $\Omega^{(2)}$ is then just

$$\Omega^{(2)} = \frac{1}{2} \sum_{\alpha_1 \alpha_2, \beta_1 \beta_2} b^\dagger_{\alpha_2} b^\dagger_{\alpha_1} \omega_{\alpha_2 \alpha_1, \beta_1 \beta_2} b_{\beta_1} b_{\beta_2}. \quad (2.47)$$

A typical example which requires both one and two body operators is the hamiltonian for a system of $N$ nonrelativistic fermions with an interaction potential energy $V(r_k, r_m) = V(r_m, r_k)$ between each pair $(k, m)$. The Schrödinger picture Hamiltonian for this system is just

$$H = \sum_k \frac{-\hbar^2}{2m} \nabla^2_k + \sum_{k < m} V(r_k, r_m). \quad (2.48)$$
According to the procedure just outlined, the second quantized version of the kinetic term is
\[
\sum_{\alpha, \beta} b_\alpha^\dagger \langle \alpha | \left( -\frac{\hbar^2}{2m} \nabla^2 \right) |\beta \rangle b_\beta = \int d^3r \psi^\dagger(r) \left( -\frac{\hbar^2}{2m} \nabla^2 \right) \psi(r)
\]
\[
= \int d^3r \frac{\hbar^2}{2m} \nabla \psi^\dagger(r) \cdot \nabla \psi(r),
\]
(2.49)
where in the second form we have gone to the coordinate basis, using the definition of the second quantized field operator (2.34). The second quantized version of the potential term is the two-body operator
\[
\frac{1}{2} \sum_{\alpha_1 \alpha_2, \beta_1 \beta_2} b_{\alpha_2}^\dagger b_{\alpha_1}^\dagger \langle \alpha_2 | \langle \alpha_1 | V | \beta_1 \rangle |\beta_2 \rangle b_{\beta_1} b_{\beta_2}
\]
\[
= \frac{1}{2} \int d^3x d^3y V(x, y) \psi^\dagger(y) \psi^\dagger(x) \psi(x) \psi(y).
\]
(2.50)
Two details to note about this expression are the overall factor of 1/2 and the order of operators. These are necessary to arrange that the potential energy has the correct sign and normalization. The complete hamiltonian for this system can now be compactly written as
\[
H = \int d^3r \frac{\hbar^2}{2m} \nabla \psi^\dagger(r) \cdot \nabla \psi(r)
\]
\[
+ \frac{1}{2} \int d^3x d^3y V(x, y) \psi^\dagger(y) \psi^\dagger(x) \psi(x) \psi(y).
\]
(2.51)
This is the hamiltonian for a quantum field theory. The fundamental quantum operators are the local fields \( \psi(x) \) or the corresponding creation and annihilation operators. The operators \( x, p \) etc. of the Schrödinger description have been demoted to \( c \) number labels and derivatives with respect to them.

One final feature of the formalism to explain is the role of the field equations. In the Schrödinger picture the quantum dynamics is given by the Schroedinger equation
\[
i\hbar \frac{\partial}{\partial t} |\Phi, t \rangle = H_S(t) |\Phi, t \rangle
\]
(2.52)
where we stress that we allow time varying external forces to be present (hence the time dependence of \( H \)). The field equations arise in the Heisenberg picture wherein the time dependence resides in the operators rather than in the system states which are constant in time. To pass to the Heisenberg picture we write
\[
|\Phi, t \rangle = U(t) |\Phi, 0 \rangle
\]
(2.53)
where
\[
i\hbar \frac{\partial}{\partial t} U = H_S(t) U \quad U(0) = I,
\]
(2.54)
and give the time independent Schroedinger picture operators $\Omega$ time dependence according to

$$\Omega(t) \equiv U^\dagger \Omega U. \tag{2.55}$$

The Heisenberg picture Hamiltonian is similarly related to the Schrödinger picture one by

$$H(t) \equiv U^\dagger H_S(t) U. \tag{2.56}$$

Then the Heisenberg picture operators corresponding to constant Schrödinger picture operators satisfy the Heisenberg equations

$$i\hbar \dot{\Omega}(t) = [\Omega(t), H(t)]. \tag{2.57}$$

Returning to our system of nonrelativistic fermions we find that the Heisenberg equation for the field operator $\psi$ implies

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(x,t) + \int d^3z V(x,z) \psi^\dagger(z) \psi(z) \psi(x). \tag{2.58}$$

This is the quantum field equation. It is a nonlinear differential equation for a quantum operator, and because of its operator nature it has much more information packed in it than is immediately apparent. The origin of the name “second quantization” for this formalism is that (2.58) looks like a nonlinear version of the Schroedinger equation. Indeed if the particles did not mutually interact, it would be exactly the Schrödinger equation. If “first quantization” produced the Schroedinger equation, we have now reinterpreted the latter as a classical field equation, which is then “second quantized”. Of course we know that all we have really done is given a clever reformulation of ordinary many body quantum mechanics, and the name is really a misnomer, which has stuck.

We should mention that (2.58) is an elegant point of departure for the Hartree-Fock approximation, in which one approximates the nonlinear term by a one body term

$$\int d^3y K(x,y) \psi(y) \tag{2.59}$$

in which $K$ is chosen to make this term as close as possible to the original two body term.

How much of the quantum dynamics is captured in the quantum field equation? The time dependent many body wave function in Schrödinger picture, which summarizes the complete quantum dynamics of the system can be recast

$$\frac{1}{\sqrt{N!}} \langle 0' | \psi(x_1) \cdots \psi(x_N) | \Phi, t \rangle = \frac{1}{\sqrt{N!}} \langle 0' | U(t) \psi(x_1, t) \cdots \psi(x_N, t) | \Phi, 0 \rangle. \tag{2.60}$$

So in addition to the time dependence of $\psi(x,t)$ which we could get from the field equation, we would also need to find the time dependence of the state

$$\langle 0' | U(t) \rangle. \tag{2.61}$$
In our example, the state $\langle 0' \rangle$ is really dynamically inert \textit{i.e.}

$$\langle 0'|U(t) = \langle 0' \rangle,$$  

(2.62)

because each term of $H_S(t)$ has a $b^\dagger$ on the left so that $\langle 0'|H_S(t) = 0$. Thus knowledge of $\psi(x, t)$ at all times allows us to reconstruct the time dependence of the many body Schrödinger wave function completely. If we were transforming a different matrix element, in which $\langle 0' \rangle$ were replaced by a nonempty state, to the Heisenberg picture, the time dependence of $U$ acting on this state would have to be found in addition to the time dependence of the Heisenberg operators. The time dependence of $\langle A|U(t)$ has a simple interpretation if $\langle A \rangle$ is characterized as an eigenstate, say, of some definite Schrödinger picture observable $\Omega$. Then $\langle A|U(t)$ is the corresponding eigenstate of the corresponding Heisenberg picture operator $\Omega(t) = U^\dagger \Omega U$. In particular if the time dependent terms in the Schrödinger picture Hamiltonian $H_S(t) = H_0 + H'(t)$ vanish initially and finally,

$$H'(T) = H'(0) = 0,$$  

(2.63)

and $\langle A \rangle$ is an eigenstate of $H_0 = H_S(0) = H(0)$, then $\langle A|U(T)$ is an eigenstate of the Heisenberg picture operator $H_0(T) = H(T)$.

Let us consider what is gained and lost in the alternative formulations of many body quantum physics. The second quantization formalism contains all the information contained in the (anti)symmetrized wave functions. The (unphysical) non-symmetrized wave functions are of course lost, but that is desirable. The second quantization machinery allows a broader range of dynamical options. For example, particle number conservation is built into the wave function description. This conservation law is reflected in the existence of the number operator

$$N = \sum_\alpha b_\alpha^\dagger b_\alpha = \int d^3x \psi^\dagger(x)\psi(x)$$  

(2.64)

which counts the number of particles in a given state. The conservation law is the statement that $N$ commutes with the Hamiltonian: $[N, H] = 0$. We can also identify $\psi^\dagger(x)\psi(x)$ as the number density operator. There is, however, no principle which excludes considering a Hamiltonian with terms that don’t commute with $N$. For example, a term

$$\int d^3x d^3x' A(x, x')(\psi(x)\psi(x') - \psi^\dagger(x)\psi^\dagger(x'))$$  

(2.65)

would not commute with $N$. In that case energy eigenstates would not have a definite number of particles!
2.2.2 Second Quantized Dirac Equation

Single particle basis

The formalism developed in the last chapter can now be fruitfully applied to the Dirac equation, cast as a Schrödinger equation,

\[ i \frac{\partial \psi(x, t)}{\partial t} = \left( \frac{1}{i} \alpha \cdot \nabla + \beta m \right) \psi(x, t) \equiv H_1(x, t) \psi. \] (2.66)

Let us regard the component label \( a \) as an additional coordinate: \( \psi_a(x) = \psi(x, a) \). For definiteness let us choose our single particle basis to be momentum eigenstates, with an additional label \( \lambda \) for spin and \( \pm \) to distinguish positive and negative energy states. Thus the role of \( \psi_\alpha(x) \) of the previous chapter will be played by

\[ \psi^{(\pm)}_\lambda(x, a) = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega(p)}} u^a_{\lambda \pm} e^{ip \cdot x}. \] (2.67)

The prefactors are conventional and with them in place the condition of orthonormality implies that

\[ u_{\lambda, \pm}^\dagger u_{\lambda, \pm} = \sum_a u^a_{\lambda, \pm} u^a_{\lambda, \pm} = 2\omega(p) \delta_{\lambda, \lambda} \delta_{\pm, \pm}. \] (2.68)

We therefore write the explicit solution (2.20) for \( u = u^+ \) in the rescaled form

\[ u_\lambda(p) = \sqrt{\omega(p) + m} \left( \begin{array}{c} \phi_\lambda \\ \sigma_p/m + \omega(p) \phi_\lambda \end{array} \right) \] (2.69)

so that the normalization condition on the two spinor \( \phi \) is

\[ \phi_\lambda^\dagger \phi_\lambda = \delta_{\lambda', \lambda}. \] (2.70)

There are two widely used choices for \( \phi_\lambda \). One is to simply choose the two orthogonal spinors

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (2.71)

In the rest frame \( p = 0 \) these are just eigenstates of \( \Sigma_3 \) with eigenvalues +1, −1 respectively. The other choice is to pick them to be eigenstates of helicity \( h = p \cdot \sigma/2|p| \) denoted \( \chi_\lambda(p) \):

\[ h \chi_\lambda(p) = \lambda \chi_\lambda(p) \quad \lambda = \pm \frac{1}{2}. \] (2.72)

Explicit forms for \( \chi_\lambda \) are developed in the exercises. Notice that for the helicity basis the expression for \( u \) simplifies to

\[ u_\lambda(p) = \sqrt{\omega(p) + m} \left( \begin{array}{c} \chi_\lambda \\ 2\lambda|p| / (m + \omega(p)) \chi_\lambda \end{array} \right). \] (2.73)
and furthermore $u_\lambda$ is itself an eigenstate of helicity $p \cdot \Sigma/2|p|$ with eigenvalue $\lambda$.

In all cases we maintain our choice (2.21)

$$u_-(p) = i\gamma^2u^*(-p)$$

for the negative energy basis functions. For the helicity basis choice for $u_\lambda$, this construction gives a negative energy spinor with the same helicity, as can easily be shown by applying $h$ to both sides. For the rest frame $\Sigma_3$ basis this construction reverses the sign of $\Sigma_3$ in the rest frame.

The properties of $\psi_{\lambda, p}^{(\pm)}(x, a)$ needed for the second quantization formalism are **Orthonormality**:

$$\int d^3x \sum_\lambda \psi^{(\pm)*}_{\lambda, p}(x, a)\psi^{(\pm)'}_{\lambda, p'}(x, a) = \delta_{\lambda'\lambda}\delta(\pm')\delta(p' - p)$$

(2.75)

and **Completeness**:

$$\int d^3p \sum_{\lambda \pm} \psi^{(\pm)}_{\lambda, p}(x, a)\psi^{(\pm)*}_{\lambda, p'}(x', a') = \delta_{aa'}\delta(x' - x).$$

(2.76)

### Creation and annihilation operators

To pass to the second quantized formalism, we simply introduce creation and annihilation operators $b^\dagger_{\lambda \pm}(p), b_{\lambda \pm}(p)$ with anticommutation relations

$$\{b_{\lambda \pm}(p), b^\dagger_{\lambda' \pm'}(p')\} = \delta_{\lambda\lambda'}\delta(\pm)\delta(p' - p),$$

(2.77)

and define the Dirac quantum field operator

$$\psi^a(x) = \int d^3p \sum_{\lambda \pm} b_{\lambda \pm}(p)\psi^{(\pm)}_{\lambda, p}(x, a).$$

(2.78)

By virtue of (2.76) the field operators satisfy

$$\{\psi^a(x), \psi^{a\dagger}(x')\} = \delta_{aa'}\delta(x - x').$$

(2.79)

A principal virtue of second quantization is the efficiency with which we can construct the state describing the negative energy sea. We simply apply to the empty state all of the creation operators for negative energy states:

$$|\text{sea}\rangle \equiv |0\rangle = \mathcal{N} \prod_{p, \lambda} b^\dagger_{\lambda -}(p)|0\rangle.$$  

(2.80)

This looks terribly complicated, but we can uniquely characterize this state very simply: It is annihilated by all of the positive energy annihilation operators and by all the negative energy creation operators

$$b_{\lambda +}(p)|0\rangle = b^\dagger_{\lambda -}(p)|0\rangle = 0.$$  

(2.81)
These conditions tell us everything we need to know about the sea. The annihilation operator for a negative energy electron creates a hole in the sea. Thus the construction of the sea is completely equivalent to interchanging the role of the creation and annihilation operators for the negative energy Dirac particles.

Hamiltonian

To see the consequences of this interchange of roles, let us consider a few of the observables of the theory. The Hamiltonian is just

$$H = \int d^3x \sum_a \psi_a^\dagger(x) \left( \frac{1}{i} \alpha \cdot \nabla + \beta m \right) \psi^a(x)$$  \hspace{1cm} (2.82)

$$= \int d^3p \omega(p) \sum_\lambda (b_{\lambda+}^\dagger(p)b_{\lambda+}(p) - b_{\lambda-}^\dagger(p)b_{\lambda-}(p))$$  \hspace{1cm} (2.83)

$$= \int d^3p \omega(p) \sum_\lambda (b_{\lambda+}^\dagger(p)b_{\lambda+}(p) + b_{\lambda-}(p)b_{\lambda-}^\dagger(p))$$

where in the last form we have reordered the creation and annihilation operators of the negative energy contributions, the nonzero anticommutator producing the negative infinite constant term. Notice that thanks to the Fermi statistics both contributions to the energy are positive. This constant is just the energy of the sea. the factor of $\delta(0)$ can be identified with $\text{Volume}/(2\pi)^3$ so the sea has an infinite negative energy density. As we have already stressed we can and will choose to measure all energies relative to that of the sea which amounts to dropping this constant$^1$, so henceforth we shall take the free Dirac second quantized Hamiltonian to be

$$H_{\text{Dirac}} = \int d^3p \omega(p) \sum_\lambda (b_{\lambda+}^\dagger(p)b_{\lambda+}(p) + b_{\lambda-}(p)b_{\lambda-}^\dagger(p)).$$  \hspace{1cm} (2.85)

When we pass to the Heisenberg picture we find the field equation for $\psi$ to be nothing other than the Dirac equation

$$(\frac{1}{i} \gamma \cdot \partial + m)\psi = 0.$$  \hspace{1cm} (2.86)

Since we have selected our single particle basis to be eigenstates of $\frac{1}{i} \alpha \cdot \nabla + \beta m$ the time dependence of $\psi$ in Heisenberg picture is simply

$$\psi^a(x, t) = \int d^3p \sum_{\lambda \pm} b_{\lambda \pm}(p) \psi_{\lambda, p}^{(\pm)}(x, a)e^{\mp i \omega(p)t}$$  \hspace{1cm} (2.87)

$^1$Alternatively we could introduce a bare cosmological constant to cancel it.
The annihilation operators for the positive and negative energy Dirac particles are thus identified with the positive and negative frequency components of the Dirac field in Heisenberg picture. This is a useful observation because when we introduce time dependent external fields which are switched off at early and late times, it will allow us to easily relate the operators that characterize the sea at late times to the ones that characterize the sea at early times.

**Momentum operator**

Returning to our survey of observables, the momentum operator is just

\[ P = \int d^3x \sum_a \psi^\dagger_a(x) \frac{1}{i} \nabla \psi^a(x) \]  

\[ = \int d^3p \sum_\lambda (b^{\dagger}_{\lambda+}(p)b_{\lambda+}(p) + b^{\dagger}_{\lambda-}(p)b_{\lambda-}(p)) \]  

\[ = \int d^3p \sum_\lambda (b^{\dagger}_{\lambda+}(p)b_{\lambda+}(p) + b_{\lambda-}(-p)b^{\dagger}_{\lambda-}(-p)) \]

The term \( 2 \int d^3p \delta(0) \) arising from reordering the negative energy operators automatically vanishes and need not be dropped. We see from the explicit form of the momentum operator that \( b_{\lambda-}(-p) \) creates from the sea a particle of momentum \(+p\).

**Charge operator**

The charge operator is just \( Q = qN \) where \( q \) is the unit of charge carried by the Dirac particle and \( N \) is the number operator

\[ Q = q \int d^3x \sum_a \psi^\dagger_a(x) \psi^a(x) \]  

\[ = q \int d^3p \sum_\lambda (b^{\dagger}_{\lambda+}(p)b_{\lambda+}(p) - b_{\lambda-}(-p)b^{\dagger}_{\lambda-}(-p)) + 2q \int d^3p \delta(0) \]

from which we see that \( b_{\lambda-}(-p) \) creates a state of charge \(-q\). We shall also in future drop the constant term in \( Q \) so the charge of the sea is then zero\(^2\) There is a convenient way to make this subtraction. Instead of taking the charge density to be \( \rho(x) = q\psi^\dagger \psi \), take it to be the symmetrized form

\[ \rho(x) = \frac{q}{2} \sum_a [\psi^\dagger_a(x)\psi_a(x) - \psi_a(x)\psi^\dagger_a(x)] \]

\(^2\)If there is more than one species of fermion in the universe, the coefficient of this term is \( \sum_f q_f \). One way of getting rid of the sea charge is to insist that this sum of charges vanishes. As it happens the standard model of strong weak and electromagnetic interactions has this property, which is required for cancellation of the axial anomaly. Thanks to Mr. Yan-Bo Xie for drawing my attention to this circumstance. In a similar vein supersymmetry is often proposed so that the zero point energy of bosons exactly cancels the sea energy of fermions.
Then when the operators in \( Q = \int d^3 x \rho \) are suitably reordered the piece coming from the positive energy term exactly cancels that from the negative energy term. At this point we can also identify the current operator from local current conservation
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0. 
\] (2.94)

Inserting (2.93) into (2.94) and using the Dirac equation, we identify
\[
j = \frac{q}{2} \sum_a [\psi_a^\dagger(x)(\alpha \psi)_a(x) - (\alpha \psi)_a(x)\psi_a^\dagger(x)]. 
\] (2.95)

We can assemble \((\rho, j)\) in a four vector \(j^\mu\):
\[
 j^\mu(x, t) = \frac{q}{2} \sum_a \left[ \psi_a^\dagger(x, t)(\beta \gamma^\mu \psi)_a(x, t) - (\beta \gamma^\mu \psi)_a(x, t)\psi_a^\dagger(x, t) \right] 
\] (2.96)
\[
 = \frac{q}{2} \sum_a \left[ \bar{\psi}_a(x, t), (\gamma^\mu \psi)_a(x, t) \right] 
\] (2.97)

We have made use of the Dirac adjoint
\[
\bar{\psi}_a \equiv \sum_b \psi_b^\dagger \beta_{ba} 
\] (2.98)
in the last form, which we will also sometimes shorten even more by suppressing the spinor indices, \(\frac{q}{2}[\bar{\psi}, \gamma^\mu \psi]\). Current conservation \(\partial_\mu j^\mu = 0\) is an immediate consequence of the Dirac equation and its Dirac adjoint
\[
\bar{\psi}(-i\gamma \cdot \frac{\partial}{\partial t} + m) = 0. 
\] (2.99)

**Angular momentum and helicity**

The final observable we mention is the angular momentum
\[
 J = \int d^3 x \sum_{a,b} \psi^a_\dagger(x) \left( \frac{1}{i} (x \times \nabla) + \frac{1}{2} \Sigma \right)_{ab} \psi^b(x). 
\] (2.100)

The sea is of course rotationally invariant
\[
J|0\rangle = 0, 
\] (2.101)
as can be confirmed by direct calculation. Of particular interest is the action of the helicity on the single particle states. On the particle states we find
\[
p \cdot J_{b_a^\dagger}(p)|0\rangle = \frac{1}{2\omega} \sum_{a,b} u_{a^*}(p) \Sigma_{ab} b^\dagger_{b_a}(p)|0\rangle = \lambda |p| b^\dagger_{b_a}(p)|0\rangle, 
\] (2.102)
confirming that this state carries helicity $\lambda$. The helicity of the one hole state $b_{\lambda^-}(-p)|0\rangle$ which possesses momentum $+p$ is also $\lambda$ but the reason is slightly subtle. First of all

$$p \cdot J_{b_{\lambda^-}(-p)}|0\rangle = -\frac{1}{2\omega} \sum_{a,b,\lambda'} u^{a*}_{\lambda^-}(-p) \frac{1}{2} p \cdot \Sigma_{ab} u^b_{\lambda'}(-p)b_{\lambda'}(-p)|0\rangle$$

(2.103)

where the minus sign arises because $b^{\dagger}_{\lambda^-}$ occurs in $\psi^\dagger$ and must anticommute with $\psi$ before it can contract against $b_{\lambda^-}$. But then

$$\frac{1}{2} (-p) \cdot \Sigma u_{\lambda^-}(-p) = \lambda p |u_{\lambda^-}(-p)\rangle.$$  

(2.104)

Thus

$$\frac{p \cdot J}{|p|} b_{\lambda^-}(-p)|0\rangle = \lambda b_{\lambda^-}(-p)|0\rangle$$

(2.105)

as we claimed.

**Summary**

This survey of single particle observables has established:

1. The state $b^{\dagger}_{\lambda^+}(p)|0\rangle$ is a one particle state of momentum $p$, energy $\omega = \sqrt{p^2 + m^2}$, charge $q$, and helicity $\lambda$.

2. The state $b_{\lambda^-}(-p)|0\rangle$ is a one particle state of momentum $p$, energy $\omega = \sqrt{p^2 + m^2}$, charge $-q$, and helicity $\lambda$.

In particular if the first state is an electron of charge $-e$, then the second is a positron of charge $+e$. To emphasize these facts it is traditional to rename the creation and annihilation operators for negative energy particles. So define

$$b_{\lambda^-}(-p) \equiv d^{\dagger}_{\lambda}(p) \quad b^{\dagger}_{\lambda^+}(p) \equiv b^{\dagger}_{\lambda}(p).$$

(2.106)

So $b^{\dagger}_{\lambda}(p)$ creates a particle and $d^{\dagger}_{\lambda}(p)$ creates an antiparticle. Similarly it is useful to define the Dirac spinor

$$v_{\lambda}(p) \equiv u_{\lambda^-}(-p) = i\gamma^2 u^{\dagger}_{\lambda}(p).$$

(2.107)

Note that $u$ and $v$ satisfy

$$\begin{align*}
(\gamma \cdot p + m) u_{\lambda}(p) &= 0 \\
(\gamma \cdot p - m) v_{\lambda}(p) &= 0
\end{align*}$$

(2.108)

$$\begin{align*}
(h - \lambda) u_{\lambda}(p) &= 0 \\
(h + \lambda) v_{\lambda}(p) &= 0
\end{align*}$$

(2.109)

with opposite signs in front of the mass and opposite helicities.
With these definitions the free Dirac field operator in Heisenberg picture has the representation

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum \lambda \left( b^\dagger_\lambda(p)u_\lambda(p) e^{ix \cdot p} + d_\lambda(p)v_\lambda(p) e^{-ip \cdot x} \right),$$

(2.110)

where $p \cdot x = p \cdot x - \omega(p)t$ is the Minkowski scalar product. A point to bear in mind with this new interpretation is that one body operators will generally contain terms like $b^\dagger d^\dagger$ which create a particle antiparticle pair and terms like $bd$ which destroy such a pair. Thus when we couple currents to the electromagnetic field we will have charge conservation, but not particle number conservation.

The Dirac field is now expressed in a way parallel to the scalar and electromagnetic fields:

$$\phi(x, t) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \left( a(p) e^{ix \cdot p} + a^\dagger(p) e^{-ip \cdot x} \right)$$

$$A^k(x, t) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum \lambda \left( \epsilon^k_\lambda(p)a_\lambda(p) e^{ix \cdot p} + \epsilon^{k\dagger}_\lambda(p)a^\dagger_\lambda(p) e^{-ip \cdot x} \right).$$

(2.111)
Chapter 3

The Discrete Symmetries of the Dirac Equation

3.1 Parity

The parity transformation $x \rightarrow -x$ can be extended to a symmetry of the Hamiltonian. Consider the following transformation on the field:

$$
\psi(x, t) \rightarrow -1 \beta \psi(-x, t)
$$

where we have allowed a multiplicative phase. Then the Hamiltonian transforms to

$$
P^{-1}HP = \int d^3x \beta \psi(-x) \psi(-x)
$$

after changing integration variables, so it is parity invariant. From the parity transformation (3.1) we can infer how parity acts on the particle states:

$$
e^{i\phi} \beta \psi(-x, t) = e^{i\phi} \int \frac{d^3p}{(2\pi)^{3/2} \sqrt{2\omega}} \sum_\lambda \left( b_\lambda(-p) \beta u_\lambda(-p) e^{ix \cdot p} + d_\lambda^\dagger(-p) \beta v_\lambda(-p) e^{-ip \cdot x} \right),
$$

where we have reversed the sign of $p$ by a change of variables. Next we note that depending on the spin basis we choose,

Helicity Basis:

$$
\beta u_\lambda(p) = -ie^{i\lambda(\pi + 2\phi_p)} u_{-\lambda}(-p)
$$

$$
\beta v_\lambda(p) = -ie^{-i\lambda(\pi + 2\phi_p)} v_{-\lambda}(-p)
$$

where we have used the formula

$$
\chi_\lambda(p) = -ie^{i\lambda(\pi + 2\phi_p)}\chi_{-\lambda}(-p)
$$
obtained in the exercises, or

\begin{align*}
\text{Rest Frame } \Sigma_3: \quad \beta u_\mu(p) &= u_\mu(-p) \quad (3.8) \\
\beta v_\mu(p) &= -v_\mu(-p). \quad (3.9)
\end{align*}

Using these spinor properties, we learn that

\begin{align*}
\text{Helicity Basis:} \quad P^{-1}b_\lambda(p)P &= e^{i\phi}ie^{-i\lambda(\pi+2\phi_p)}b_{-\lambda}(-p) \quad (3.10) \\
P^{-1}d_\mu(p)P &= e^{i\phi}ie^{i\lambda(\pi+2\phi_p)}d_{-\lambda}(-p) \quad (3.11) \\
P^{-1}b_\mu(p)P &= e^{-i\phi}(-i)e^{i\lambda(\pi+2\phi_p)}b_{-\lambda}(-p) \quad (3.12)
\end{align*}

where the last equation is just the hermitian conjugate of the first. And

\begin{align*}
\text{Rest Frame } \Sigma_3: \quad P^{-1}d_\mu(p)P &= -e^{i\phi}d_\mu(-p) \quad (3.13) \\
P^{-1}b_\mu(p)P &= e^{-i\phi}b_\mu(-p). \quad (3.14)
\end{align*}

It should be noted that the arbitrary phase we allowed in the definition of parity cancels out for neutral states, i.e. those with an equal number of \(b^\dagger\)'s and \(d^\dagger\)'s acting on the sea, which we can take to be parity invariant. This means that whereas the intrinsic parity of a single particle is conventional, that of a particle antiparticle pair is not. For example the above formulae imply that the parity of the ground state \(s\) wave of positronium is odd, i.e. the ground states are \(0^-\) and \(1^-\). These assignments also apply to positronium like states: muonium, charmonium, borromonium, etc.

### 3.2 Charge Conjugation

It is apparent that a Dirac particle and its antiparticle are closely related to each other. They have identical mass and spin, but exactly opposite charges. In fact this relationship is a reflection of a symmetry in the dynamics under interchange of a particle with its antiparticle. To explore this symmetry, first define a unitary transformation \(C, CC^\dagger = I\) by the rules

\[ C^{-1}b^\dagger C = d^\dagger \quad C^{-1}d^\dagger C = b^\dagger \quad C|0\rangle = |0\rangle. \quad (3.15) \]

(The last of these equations implies an extremely complicated transformation of the empty state, but we shall never see those complications.) From the definition of \(C\) we can work out how the field transforms

\[ C^{-1}\psi(x)C = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_\lambda \left( d_\lambda(p)u_\lambda(p)e^{ip\cdot x} + b^\dagger_\lambda(p)v_\lambda(p)e^{-i\phi_p} \right) \quad (3.16) \]

which we can relate to \(\psi^\dagger\):

\[ \psi^\dagger_a(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_\lambda \left( b^\dagger_\lambda(p)\omega_{\lambda a}^*(p)e^{-i\phi_p} + d_\lambda(p)\omega^*_{\lambda a}(p)e^{i\phi_p} \right). \quad (3.17) \]
But, in the standard and chiral representations, $v^* = i\gamma^2 u$ and $u^* = i\gamma^2 v$ so we can infer

$$C^{-1}\psi_a(x)C = i(\gamma^2)_{ab}\psi_b^\dagger(x).$$

(3.18)

When we suppress spinor indices, it is usually convenient to think of $\psi^\dagger$ as a row vector and $\psi$ as a column vector. To write (3.18) with suppressed indices we want the r.h.s. to be a column vector which we could indicate by $(\psi^\dagger)^T$ meaning the transpose on spinor indices. In that case (3.18) can be written

$$C^{-1}\psi(x)C = i\gamma^2(\psi^\dagger)^T(x).$$

(3.19)

It should also be noted that the occurrence of $\gamma^2$ in the charge conjugation transformation law is specific to the standard and chiral representations; with other representations a different matrix would appear. For instance, in the Majorana representation the $i\gamma^2$ is replaced by the identity matrix.

The invariance of $H$ is obvious from its expression in terms of creation and annihilation operators, but it is also instructive to see it using the local definition

$$C^{-1}HC = \int d^3x \psi^T i\gamma^2 \left(\frac{1}{i}\alpha \cdot \nabla + \beta m\right) i\gamma^2 (\psi^\dagger)^T$$

$$= -\int d^3x \psi^T \left(-\frac{1}{i}\alpha^T \cdot \nabla + \beta^T m\right) (\psi^\dagger)^T$$

(3.20)

$$= H$$

(3.21)

where the last step involves an integration by parts, a transposition of Dirac indices, and a reordering of the order of $\psi$ and $\psi^\dagger$ giving a minus sign which cancels the overall minus sign in the second line.

The transformation of the charge and current densities under $C$ should simply change their signs. This is not hard to see:

$$C^{-1}j^\mu C = \frac{q}{2} \left(\psi^T i\gamma^2 \beta^\mu \gamma^2 (\psi^\dagger)^T - \psi^\dagger i\gamma^2 \gamma^T \beta i\gamma^2 \psi\right)$$

(3.22)

$$= -j^\mu$$

(3.23)

where use is made of

$$i\gamma^2 \gamma^\mu i\gamma^2 = -\gamma^\mu$$

$$\beta^\gamma \beta = \gamma^\dagger.$$ 

(3.24)

Note that because we have used the symmetrized definition of the current, there is no reordering of operators necessary in arriving at this result.

### 3.3 Majorana Fermions

In our discussion so far it has seemed inevitable that the Dirac particle carries charge. More precisely, it carries a conserved fermion number $N$ which could be identified with charge. If
there were several species $f$ of Dirac particle the fermion number $N_f$ of each species might be separately conserved. Including terms of the form

$$\bar{\psi}_f \Gamma \psi_{f'}$$

(3.25)

with $f' \neq f$ in the Hamiltonian could violate the individual $N_f$ but $\sum_f N_f$ would still be conserved. Majorana pointed out that even with only one species of fermion it is possible to make it totally neutral, i.e. carry no conserved quantum number at all.

Starting with the Dirac theory we can see that this is possible, because we can consider redefining creation and annihilation operators to be eigenoperators under charge conjugation:

$$b_{\lambda\pm}(p) = \frac{1}{\sqrt{2}}(b_{\lambda}(p) \pm d_{\lambda}(p)) \quad \text{with} \quad C^{-1}b_{\lambda\pm}(p)C = \pm b_{\lambda\pm}(p).$$

(3.26)

Then the Hamiltonian is the sum of two commuting pieces

$$H = H_+ + H_- \quad \text{with} \quad H_\pm = \int d^3 p \, \omega(p) \sum_{\lambda} b_{\lambda\pm}^\dagger(p)b_{\lambda\pm}(p).$$

(3.27)

Clearly, it is perfectly consistent to consider the quantum system defined by $H_+$ (or $H_-$) alone. The number operator of the Dirac theory

$$N = \int d^3 p \sum_{\lambda} (b_{\lambda-}^\dagger(p)b_{\lambda+}(p) + b_{\lambda+}^\dagger(p)b_{\lambda-}(p)),$$

(3.28)

clearly has no meaning in the truncated theory, but that is to be expected.

One might worry that truncating the theory in this way might spoil locality, but this is not the case. We can just as easily redefine the local fields to be eigenoperators of charge conjugation: in standard of chiral representations,

$$\psi_\pm(x) = \frac{1}{\sqrt{2}}(\psi(x) \pm i\gamma^2(\psi^\dagger)^T(x)) \quad \text{with} \quad C^{-1}\psi_\pm C = \pm \psi_\pm,$$

(3.29)

Which satisfy anticommutation relations

$$\{\psi_\pm^a(x), \psi_{\pm}^b(x')\} = \pm(i\gamma^2)_{ab}\delta(x-x').$$

(3.30)

Clearly

$$\psi_\pm(x) = \int \frac{d^3 p}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_{\lambda} \left(b_{\lambda\pm}(p)\psi_{\lambda}(p) e^{ixp} \pm b_{\lambda\pm}^\dagger(p)\psi_{\lambda}^T(p) e^{-ipx}\right),$$

(3.31)

and in terms of these fields

$$H_\pm = \frac{1}{2} \int d^3 x \psi_\pm \left(\frac{1}{i} \alpha \cdot \nabla + \beta m \right) \psi_\pm$$

(3.32)

$$= \frac{1}{2} \int d^3 x \left(\pm \psi_\pm^T i\gamma^2 \left(\frac{1}{i} \alpha \cdot \nabla + \beta m \right) \psi_\pm.\right.$$

(3.33)
In the second line we have used $\psi_\pm^\dagger = \pm \psi_\mp^T i\gamma^2$, a consequence of (3.29).

The appearance of $i\gamma^2$ in the above discussion is due to our choice of representation for the gamma matrices. The Majorana representation is characterized by the condition that the gamma matrices be pure imaginary $\gamma^\mu^* = -\gamma^\mu$. In that case the charge conjugation transformation does not involve a matrix at all and all of the $i\gamma^2$’s disappear.

### 3.4 Weyl Fermions

In the case of massless fermions $m = 0$, it is possible to describe relativistic spin 1/2 particles with only one helicity. In the Dirac theory the easiest way to see this is to consider the matrix

$$
\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3,
$$

which anticommutes with the $\gamma^\mu$. It therefore commutes with the Lorentz matrices

$$
\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu].
$$

$\gamma_5$ commutes with the $\alpha \cdot \nabla$ term of the Dirac Hamiltonian but not with the $m\beta$ term. So when $m = 0$, and only then, the energy eigenstates can be simultaneously eigenstates of $\gamma_5$. Since $\gamma_5^2 = 1$ the eigenvalues of $\gamma_5$, called chirality, are $\pm 1$ and

$$
\frac{I \pm \gamma_5}{2}
$$

are projectors onto orthogonal two dimensional subspaces with chirality $\pm 1$ respectively. Defining

$$
R = \frac{I + \gamma_5}{2}\psi, \quad L = \frac{I - \gamma_5}{2}\psi, \quad \psi = R + L
$$

$R$ for “right-handed” and $L$ for “left-handed,” the Dirac Hamiltonian for $m = 0$ decomposes into two commuting terms

$$
H = \int d^3x \left( R^\dagger \frac{1}{i}\alpha \cdot \nabla R + L^\dagger \frac{1}{i}\alpha \cdot \nabla L \right)
$$

either of which could define a consistent dynamics of the corresponding subsystem. These subsystems are called Weyl fermions. The corresponding momentum space spinors of definite chirality are

$$
\frac{I \pm \gamma_5}{2} u_\lambda(p) = \frac{1}{2} \sqrt{|p|}(1 \pm 2\lambda) \left( \frac{\chi_\lambda}{\pm \chi_\lambda} \right) = \delta_{\lambda, \pm 1/2} \frac{I \pm \gamma_5}{2} u_\lambda(p),
$$

from which it is clear that helicity is identical to Chirality/2. In other words right-handed Weyl fermions have helicity $+1/2$ and left-handed ones have helicity $-1/2$. Since $\gamma_5$ is real and $\gamma^2$ anticommutes with $\gamma_5$, the antiparticle spinors

$$
\frac{I \pm \gamma_5}{2} v_\lambda(p) = i\gamma^2 \left( \frac{I \pm \gamma_5}{2} u_\lambda(p) \right)^* = \delta_{\lambda, \mp 1/2} \frac{I \pm \gamma_5}{2} v_\lambda(p)
$$

(3.40)
have the opposite correlation between chirality and helicity. So if the Weyl particle is right
d-handed the particle has helicity \(+1/2\) and the antiparticle has helicity \(-1/2\). Since charge
conjugation interchanges the role of particle and antiparticle one can even choose by con-
vention all Weyl particles to be left(right)-handed. One can do this, for example, by writing
\(R^\dagger = L'^T i\gamma^2\). Of course, such an \(L'\) has charge opposite to \(R\).

The Weyl particle with helicity \(\pm 1/2\) comes along with its antiparticle with helicity \(\mp 1/2\).
Thus the Weyl system has the same helicity content as the massless Majorana system. In
fact one can describe the Majorana system using Weyl fields. First, separate the Majorana
field \(\psi^\dagger = \psi^T i\gamma^2\) into two fields of definite chirality

\[
\psi_{R,L} = \frac{I \pm \gamma_5}{2} \psi, \tag{3.41}
\]

Then notice that

\[
\psi_R^\dagger = \psi^\dagger I + \gamma_5 \tag{3.42}
\]

\[
= \psi^T i\gamma^2 I + \gamma_5 \tag{3.43}
\]

\[
= \psi_L^T i\gamma^2 \tag{3.44}
\]

so that the right-handed component of the Majorana field can be eliminated in favor of
the hermitian conjugate of the left-handed component. When this is done the Majorana
Hamiltonian simplifies to

\[
H_{Maj} = \int d^3x \psi_L^\dagger \frac{1}{i} \alpha \cdot \nabla \psi_L + \frac{m}{2} \int d^3x \psi_L^T i\gamma^2 \beta \psi_L + \underbrace{\psi_L^T i\gamma^2 \beta \psi_L}_\Delta F = -2 + \underbrace{(\psi_L^T i\gamma^2 \beta \psi_L)^\dagger}_\Delta F = +2 \tag{3.45}
\]

which reduces to the Weyl Hamiltonian for \(m = 0\). Notice that the massless limit conserves
fermion number but the mass term violates fermion number conservation by \(\pm 2\) units. This
is the so-called Majorana mass term for a Weyl fermion. If the system contains several
left handed Weyl fermions, \(L_k\) then one can consider a general Majorana mass term which
includes off diagonal couplings

\[
\frac{m_{kl}}{2} L_k^T i\gamma^2 \beta L_l + \frac{m^*_l}{2} (L_k^T i\gamma^2 \beta L_l)^\dagger. \tag{3.46}
\]

Since the matrix \(i\gamma^2 \beta\) is antisymmetric and the \(L_k\) obey anticommutation relation,
the matrix \(m_{kl}\) can be taken to be symmetric although it is allowed to be complex.

To construct the Dirac mass term which conserves fermion number, one must add a
right-handed Weyl fermion to the theory. Writing the Dirac field \(\psi = L + R\) we find that
\(LL = RR = 0\) because \((I \mp \gamma_5)\beta(I \pm \gamma_5) = 0\) and

\[
m\bar{\psi}\psi = m\bar{R}L + m\bar{L}R. \tag{3.47}
\]
But we can relate $R$ to a left handed field by $R^\dagger = L^T i \gamma^2$ after which the Dirac mass term takes the form

$$m \bar{\psi} \psi = mL^T i \gamma^2 \beta L - mL^\dagger i \gamma^2 \beta L^\dagger T = mL^T i \gamma^2 \beta L + (mL^T i \gamma^2 \beta L)^\dagger$$  \quad (3.48)

Written this way in terms of a pair of left handed Weyl fields, the Dirac mass is seen as a special case of the Majorana mass (3.46) with $m_{11} = m_{22} = 0$ and $m_{12} = m_{21} = m$.

The Weyl theory violates parity invariance, essentially because $\beta$ fails to commute with $\gamma_5$:

$$P^{-1} L(x) P = \frac{I - \gamma_5}{2} \beta \psi(-x) = \beta R(-x).$$  \quad (3.49)

How then have we managed to show that it is equivalent to the parity conserving Majorana theory? The answer is that although parity is violated in the Weyl theory, $CP$ the product of parity times charge conjugation remains a symmetry. The Majorana field is inert under charge conjugation, and has only parity as a nontrivial symmetry. It is the $CP$ symmetry of the Weyl theory that corresponds to the parity of the Majorana theory.

Notice that it is impossible to have a fermion that is simultaneously Majorana and Weyl: even at zero mass one always must have both helicities. This is true in four space-time dimensions but in other dimensions it need not be so. For example in 10 dimensions one can define Majorana-Weyl fermions. A Dirac fermion in $D = 2k$ dimensions has $2^k$ degrees of freedom, $2^{k-1}$ states for the particle and the same number for the antiparticle. In some dimensions (including 4 and 10) one can have Majorana fermions with only $2^{k-1}$ degrees of freedom. In the massless case one can define Weyl fermions in all even dimensions giving $2^{k-1}$ degrees of freedom. In $2 + 8n$ dimensions one can have Majorana-Weyl fermions with only $2^{k-2}$ degrees of freedom. For example in 10 dimensions a Dirac fermion has 32 states, a Majorana or Weyl fermion has 16 states, and a Majorana-Weyl fermion has only 8 states. This possibility is crucial for the consistency of superstring theory.

### 3.5 Time Reversal

The last discrete symmetry we discuss is time reversal $T$. It is well-known that $T$ must be an antiunitary transformation, meaning that it is antilinear, and furthermore

$$\langle T \Phi | T \Psi \rangle = \langle \Psi | \Phi \rangle.$$  \quad (3.50)

With this in mind, we search for a transformation of the form

$$T^{-1} \psi(x, t) T = \mathcal{T} \psi(x, -t),$$  \quad (3.51)

with $\mathcal{T}$ an appropriate matrix. From antiunitarity we have

$$\langle \Phi | T^{-1} \psi T \Psi \rangle = \langle \psi T \Psi | T \Phi \rangle = \langle T \Psi | T^{-1} \psi T \Phi \rangle = \langle T^{-1} \psi T \Phi | \Psi \rangle,$$  \quad (3.52)
from which it follows that

\[ T^{-1}\psi^\dagger(t)T = (T^{-1}\psi^\dagger T) = \psi^\dagger(-t)T^\dagger. \]  (3.53)

Thus

\[ T^{-1}HT = \int d^3x\psi^\dagger(x,-t)T^\dagger \left(-\frac{1}{i}\alpha^* \cdot \nabla + \beta m\right) T\psi(x,-t). \]  (3.54)

If we choose \( T \) to be unitary, then invariance of \( H \) will be achieved (given conservation of energy \( dH/dt = 0 \), which follows from the field equation) if and only if \( T \) commutes with \( \beta \) and \( \alpha^2 \) and anticommutes with \( \alpha^1 \) and \( \alpha^3 \). Clearly the most general solution of these conditions is

\[ T = e^{i\tau} \gamma^1 \gamma^3 = ie^{i\tau} \Sigma_2, \]  (3.55)

so that the transformation law becomes

\[ T^{-1}\psi(x,t)T = ie^{i\tau} \Sigma_2 \psi(x,-t). \]  (3.56)

This transformation law on \( \psi \) implies that for \( b,d \). It is easiest to do this for the rest frame \( \Sigma_3 \) basis, because then the two-spinors \( \phi_\mu \) are real. Thus

\[ T^{-1}\psi(x,t)T = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega}} \left[T^{-1}b_\lambda(p)Tu^*_\lambda(p)e^{-ix\cdot p} + T^{-1}d^*_\lambda(p)Tv^*_\lambda(p)e^{ix\cdot p}\right] \]  (3.57)

After changing integration variables \( p \rightarrow -p \) we need

\[ u^*_\mu(-p) = \sqrt{\omega(p)} + m \left(\frac{\phi_\mu}{m+\omega(p)}\phi_\mu\right) \]  (3.58)
\[ = \Sigma_2 \sqrt{\omega(p)} + m \left(\frac{\sigma_2 \phi_\mu}{m+\omega(p)}\sigma_2 \phi_\mu\right) \]  (3.59)
\[ = i^{2\mu} \Sigma_2 u_\mu(p) \]  (3.60)

Using \( v = i\gamma^2 u^* \), it is just a few steps to show that

\[ v^*_\mu(-p) = i^{2\mu} \Sigma_2 v_\mu(p). \]  (3.61)

Because \( T \) is antilinear, the l.h.s. of (3.56) involves \( u^*e^{-ix\cdot p}T^{-1}b^\dagger T \) and \( v^*e^{ix\cdot p}T^{-1}d^\dagger T \), so (3.60) and (3.61) allow us to infer from (3.56) that

\[ T^{-1}b_\mu(p)T = ie^{i\tau} i^{-2\mu} b_{-\mu}(-p), \quad T^{-1}b^\dagger_\mu(p)T = -ie^{-i\tau} i^{2\mu} b^\dagger_{-\mu}(-p) \]  (3.62)
\[ T^{-1}d^*_\mu(p)T = ie^{i\tau} i^{-2\mu} d^\dagger_{-\mu}(-p), \quad T^{-1}d_\mu(p)T = -ie^{-i\tau} i^{2\mu} d_{-\mu}(-p) \]  (3.63)
The reversal of signs of momentum and spin label is intuitively correct since time reversing a motion reverses both momentum and angular momentum. The \( \mu \) dependence of the phase is perhaps less intuitive, but follows straightforwardly by using angular momentum raising and lowering operators together with the action of time reversal on angular momentum. If we had used the helicity basis, the helicity label would not be reversed by time reversal (remember it is \( \mathbf{J} \cdot \mathbf{p}/|\mathbf{p}| \)); unfortunately the phases that are induced are angle dependent and not very illuminating.

### 3.6 Violation of the Discrete Symmetries and the CPT Theorem

Having shown that \( C \), \( P \), and \( T \) are symmetries of the Dirac equation it is instructive to contemplate how things must be changed to violate these symmetries. For example, we have seen that both parity and charge conjugation are violated with Weyl fermions, but in such a way that \( C P \) remains a symmetry. More generally, one can consider adding noninvariant terms to the Hamiltonian. In exercises, it is shown how the bilinears \( \bar{\psi}_A \Gamma \psi_B \) transform under these symmetries for \( \Gamma = (I, i\gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}) \). Under parity they transform with a factor of \( (+, -, +, +, -)(-)^S \) times the bilinear evaluated with \( \mathbf{x} \to -\mathbf{x} \), and where \( S \) is the number of spatial indices in the tensor component. So, examples of parity odd Lorentz invariants would be

\[
\bar{\psi}i\gamma_5\psi \quad \bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_5\gamma_\mu\psi. \tag{3.64}
\]

Adding such terms to the energy density would appear to violate parity. One must be careful that the violation is not an illusion. For example the term \( \psi^\dagger \mathbf{\alpha} \cdot \nabla \psi \) is invariant under the chiral symmetry

\[
\psi \to e^{i\alpha\gamma_5}\psi \tag{3.65}
\]

under which

\[
\bar{\psi}\psi \to \cos 2\alpha \bar{\psi}\psi + \bar{\psi}i\gamma_5\psi \sin 2\alpha \tag{3.66}
\]

so the added term \( \bar{\psi}i\gamma_5\psi \) can be rotated into \( \bar{\psi}\psi \) and parity violation disappears.

With regard to charge conjugation, \( \bar{\psi}_A \Gamma \psi_B \) transforms to \( (+, +, -, +, +) \) times \( \bar{\psi}_B \Gamma \psi_A \). For example the first of (3.64) is invariant under charge conjugation (if \( A = B \)) but the second is odd. Under \( CP \) the result is \( (+, -, -, +, -)(-)^S \). Thus it is the first of (3.64) that would violate \( CP \). Since such a term can be rotated away by a chiral transformation we see that \( CP \) is a bit tricky to violate. For example in the Standard Model one needs at least three generations of quarks and leptons to frustrate the ability to transform away apparently \( CP \) violating couplings! Fortunately there is solid evidence for this number of generations.\(^1\)

---

\(^1\)The simplest version of \( QCD \), the strong interaction sector of the standard model, violates the chiral
Finally we come to time reversal, under which the bilinears can be shown to transform into 
\((+,−,+,+)(−)^{S}\) times the bilinear with \(t \rightarrow −t\). Note that \(\bar{\psi}\gamma^{\mu}\psi\) transforms as expected for a current and \(\bar{\psi}\sigma^{kl}\psi\) as expected for an angular momentum. It is only the first of (3.64) that violates time reversal: it is as tricky to violate as \(CP\). In fact there is a deep connection between \(T\) and \(CP\) known as the \(CPT\) theorem.

Composing the three discrete symmetries we find

\[
(CPT)^{-1}\psi(x)CPT = e^{-i\tau(i\gamma^{2}\gamma^{0}\gamma^{1}\gamma^{3})(\psi^{\dagger})T}(-x) = e^{-i\gamma_{5}(\psi^{\dagger})T}(-x).
\]

(3.67)

Now applying this transformation to the bilinears, remembering the antilinear property of \(CPT\), we find

\[
(CPT)^{-1}\bar{\psi}_{A}(x)\Gamma\psi_{B}(x)CPT = \psi_{A}^{T}(-x)\gamma_{5}\beta^{\dagger}\gamma_{5}(\psi_{B})^{T}(-x)
\]

(3.68)

\[
= −\bar{\psi}_{B}(-x)\beta\gamma_{5}\beta^{\dagger}\gamma_{5}\psi_{A}(-x)
\]

(3.69)

\[
= (−)^{n_{R}}\bar{\psi}_{B}(-x)\beta\gamma_{5}\beta^{\dagger}\psi_{A}(-x)
\]

(3.70)

\[
= (−)^{n_{R}}(\bar{\psi}_{A}(-x)\Gamma\psi_{B}(-x))^{\dagger}
\]

(3.71)

where \(n_{R}\) is the number of Lorentz indices carried by \(\Gamma\). The \(CPT\) theorem states the impossibility of violating this symmetry in quantum field theory. We shall not go through the rigorous proof here, but from the transformation law of the bilinears it is clear what is behind the theorem. Since each Lorentz index must be contracted with another in forming a Lorentz scalar polynomial of the bilinears, all of the \((−)^{n_{R}}\)’s will cancel in the \(CPT\) transform of the polynomial. If we denote the Hamiltonian by some function \(H(\bar{\psi}_{A}(x)\Gamma\psi_{B}(x))\) of the bilinears, we have

\[
(CPT)^{-1}H(\bar{\psi}_{A}(x)\Gamma\psi_{B}(x))CPT = H^{\ast}((CPT)^{-1}\bar{\psi}_{A}(x)\Gamma\psi_{B}(x)CPT)
\]

(3.72)

\[
= H^{\ast}(\bar{\psi}_{A}(-x)\Gamma\psi_{B}(-x))^{\dagger}
\]

(3.73)

where by \(H^{\ast}\) we mean that all of the complex numbers appearing in the formation of \(H\) as a function of the bilinears are complex conjugated. Apart from ordering of operators, the last line is just what we mean by the hermitian conjugate of \(H\), if we set \(t = 0\) (conservation of energy means \(H\) is constant) and integrate over \(x\). So up to operator ordering questions (which can be sorted out for local interactions), a hermitian Hamiltonian must be \(CPT\) invariant.

\[\text{symmetry used to rotate away } \bar{\psi}i\gamma^{5}\psi. \text{ Then one could get } CP \text{ violation with a smaller number of generations. To be compatible with the experimental size of } CP \text{ violation the coefficient of such a term would have to be so tiny that a modified form of } QCD \text{ which restores this symmetry (and predicts axions) is usually postulated. Then one is back to the three generation requirement.}\]
Chapter 4

Representations of the Poincaré Group for General Spin

We have so far encountered via simple quantum field theories the relativistic quantum description of free particles of spin 0 or spin 1/2. It is useful at this point to realize that much of what we have found is not really tied to field theory but rather to simple requirements of Poincaré invariance. In quantum mechanics it is a general fact, established by Eugene Wigner, that a symmetry group must be realized by a unitary or antiunitary representation. The latter possibility only occurs for some discrete symmetries (time reversal being the physical example). Our goal in this chapter is to obtain the unitary realization of the Poincaré group for multiparticle states via a construction also done by Wigner.

The Poincaré group consists of Lorentz transformations together with translations
\[ x^\mu \rightarrow \Lambda^\mu_\nu x^\nu + a^\mu. \]  
where \( \Lambda \) preserves Minkowski scalar products
\[ \eta_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu = \eta_{\mu\nu}. \]  
The \( \Lambda \)'s can be divided into 4 disjoint sets according to the signs of \( \det \Lambda \) and \( \Lambda^0_0 \). This is because it is easy to show from the above property that \( (\det \Lambda)^2 = 1 \) and \( (\Lambda^0_0)^2 \geq 1 \). Thus a continuous variation of \( \Lambda \) always stays within one of these sets. In the following we restrict ourselves to the proper Lorentz Group, \( i.e. \) with \( \det \Lambda = +1 \) and \( \Lambda^0_0 \geq +1 \). The complete Lorentz group is then obtained by adjoining parity and time reversal.

Lorentz transformations with \( \Lambda^0_k = \Lambda^k_0 = 0 \) are simply rotations and form a subgroup. We know from basic quantum mechanics all the unitary irreducible representations of the Rotation group, namely those labeled by angular momentum \( j = 0, 1/2, 1, 3/2, \ldots \). The unitary representations of the Lorentz group must be extensions of these. Let us ask then how to construct this extension for a free particle of mass \( m > 0 \) and spin \( s \). Such a particle must be described by a set of at least \( 2s + 1 \) momentum space wave functions \( f_a(p) \). This much follows just from the Rotation group. We shall find a representation of the Poincaré group with this minimal number of components.
The basic idea, due to Wigner, is to exploit the fact that one can always bring a massive particle to rest by a Lorentz transformation. Define a “standard boost” \( B_p \) which boosts a particle at rest to one with momentum \( p \). Let us introduce momentum eigenstates via

\[
|f\rangle = \int d^3 p \sum_a |p, a\rangle f_a(p).
\] (4.3)

Then a momentum eigenstate of the particle can be related to the state at rest by

\[
|p, a\rangle = \sqrt{\frac{m}{\omega(p)}} U(B_p)|0, a\rangle.
\] (4.4)

The multiplicative constant is necessary because we want \( U \) to be unitary. To understand this point, notice that the relation of the three momentum \( p' \) of a boosted particle to its initial momentum is nonlinear:

\[
p'^k = \Lambda^k_l p^l + \Lambda^k_0 \omega(p).
\] (4.5)

The Jacobian of this nonlinear transformation of variables is \( \partial(p'')/\partial(p) \). The easiest way to see this is to observe that \( \int d^4 p \delta(p^2 - m^2) \) is a Lorentz invariant; integrating over \( p^0 \) then shows that \( \int d^3 p/\omega(p) \) is an invariant, which implies the above value for the Jacobian. A general Lorentz transformation on a particle state of momentum \( p \), which boosts it to \( p' \), can be expressed

\[
\Lambda = B_{p'}(B_p^{-1} \Lambda B_p)B_p^{-1}.
\] (4.6)

The transformation in parentheses leaves a particle at rest at rest, and is therefore simply a rotation. Applying \( U(\Lambda) \) to (4.4) we have

\[
U(\Lambda)|p, a\rangle = \sqrt{\frac{m}{\omega(p)}} U(B_{p'})U(B_p^{-1} \Lambda B_p)|0, a\rangle
\] (4.7)

\[
= \sqrt{\frac{m}{\omega(p)}} U(B_{p'})|0, b\rangle D^s_{ba}(B_{p'}^{-1} \Lambda B_p)
\] (4.8)

\[
= \sqrt{\frac{\omega(p')}{\omega(p)}} |p', b\rangle D^s_{ba}(B_{p'}^{-1} \Lambda B_p).
\] (4.9)

Here \( D^s_{ba}(R) \) is just the standard representation matrix of the rotation group with spin \( s \). It is now easily checked that this defines a unitary representation of the Lorentz group on single particle states of spin \( s \). Of course, on momentum eigenstates of a free particle, space-time translations are trivially realized by multiplication of the state by the phase \( e^{-ia \cdot p} \) for the space-time translation by amount \( a^\mu \).

Let us return to the “standard boost” \( B_p \). It is clearly not uniquely determined since it can be preceded by an arbitrary rotation and followed by a rotation about the axis parallel to \( p \). There are two widely used choices for this boost. (1) The simplest choice is the pure
boost parallel to \( p \) which we will call \( B^0_p \). (2) The second choice is dictated by choosing helicity states. It is described as follows. First agree that the spin states of the particle at rest be labeled by \( \lambda \) the eigenvalue of \( J_3 \). Then first boost the particle along the \( z \) axis to momentum \(|p|\hat{z}\). Then apply a rotation that carries the \( z \) axis to the direction of \( p \). This latter rotation can be taken to be

\[
R_0(p) \equiv e^{-i\phi J_3}e^{-i\theta J_2}e^{+i\phi J_3}
\]

where \((\theta, \phi)\) are the polar angles of \( p \). Then the helicity preserving standard boost is given by

\[
B^h_p = R_0(p)B^0(|p|\hat{z}).
\]

Then, clearly, \(|p, \lambda \rangle \equiv \sqrt{m/\omega}B^h_p|0, \lambda \rangle\) is an eigenstate of momentum \( p \) with helicity \( \lambda \), since rotations do not change helicity. This is easy to see from the general transformation law. For a massive particle, helicity is changed by a general Lorentz transformation, i.e. \( B^{-1}_p \Lambda B_p \) can be any rotation. However, if one considers the massless limit of this rotation for any fixed \( \Lambda \), he discovers that it always approaches a rotation about the \( z \) axis. Thus for massless particles helicity is actually a Lorentz invariant. Thus it can be consistent for massless particles with spin to exist in only one helicity state.

Having understood how a single particle state transforms under Lorentz transformations it is straightforward to find the transformation law for states with any number of free particles of varying mass and spin, which can be viewed as tensor products of single particle states. We can incorporate bose or fermi statistics by introducing a vacuum state \(|0\rangle\) and creation and annihilation operators for each species of particle, e.g.:

\[
|p, \lambda \rangle \equiv b^{\dagger \lambda}(p)|0\rangle,
\]

where \( i \) labels the species. When we are dealing with only a limited number of species, we typically choose different letters for different types of particles, e.g. \( a, a^\dagger \) for neutral scalar particles, \( b, b^\dagger \) \((d, d^\dagger)\) for particles (antiparticles), etc. Then the transformation law for a general multiparticle state is completely defined by

\[
U(\Lambda)|0\rangle = |0\rangle
\]

\[
U(\Lambda)b^{\dagger \lambda}(p)U^{-1}(\Lambda) = b^{\dagger \lambda}(\Lambda p)D^\ast_{\lambda \lambda}(B^{-1}_p \Lambda B_p)\sqrt{\frac{\omega(p')}{\omega(p)}}
\]

\[
U(\Lambda)b^\lambda(p)U^{-1}(\Lambda) = b^\lambda(\Lambda p)D_{\lambda \lambda}(B^{-1}_p \Lambda B_p)\sqrt{\frac{\omega(p')}{\omega(p)}}
\]

Of course the last transformation law is just the Hermitian conjugate of the second one since \( U \) is supposed to be unitary. We complete the description by writing down the energy and
momentum operators

\[
H = \int d^3p \sum_i \sqrt{p^2 + m^2} \sum_\lambda b_\lambda^i(p) b_\lambda^i(p) \tag{4.16}
\]

\[
P = \int d^3p \sum_i \sum_\lambda b_\lambda^i(p) b_\lambda^i(p). \tag{4.17}
\]

The above discussion might mislead one into thinking that the problem of free quantum field theory for any spin is completely solved. Indeed, we have solved the problem of constructing a relativistic quantum description of any number of free particles with any spins. However, the equivalence of this description to local field theory is not yet transparent. We have explicitly seen how this works for spin 0 and 1/2, which are described by scalar and Dirac fields respectively. The scalar field is supposed to have the Lorentz transformation properties \( \phi'(x') = \phi(x) \). We relate this to the general discussion by first identifying the momentum eigenstates with a creation operator applied to the vacuum \( |p\rangle = a^\dagger(p)|0\rangle \). Assuming the vacuum is Lorentz invariant, we conclude from the general discussion that

\[
U(\Lambda)a^\dagger(p)U^{-1}(\Lambda) = \frac{\sqrt{\omega(p')}}{\omega(p)}a^\dagger(p'). \tag{4.18}
\]

With this result we can then evaluate how the scalar field transforms

\[
U^\dagger(\Lambda)\phi(x)U(\Lambda) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{\sqrt{\omega(p')}}{\omega(p)}(a(p')e^{ixp} + a^\dagger(p')e^{-ixp}) \]

\[
= \int \frac{d^3p'}{(2\pi)^{3/2}} \frac{\sqrt{\omega(p')}}{\omega(p')} (a(p')e^{ix\Lambda p'} + a^\dagger(p')e^{-ix\Lambda p'}) 
\]

\[
= \phi(\Lambda^{-1}x) \tag{4.19}
\]

as desired.

Notice that Lorentz covariance alone is achieved by the positive frequency part of the field

\[
\phi^+(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(p)}} a(p)e^{ixp}. \tag{4.20}
\]

But such a field would not commute with its adjoint at space-like separations as a local field must.

\[
[\phi^+(x), \phi^-(y)] = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{2\omega} e^{i(x-y)p}. \tag{4.21}
\]

Compare this to the result for the total field

\[
[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{2\omega} (e^{i(x-y)p} - e^{-i(x-y)p}). \tag{4.22}
\]
If $(x - y)^2 > 0$ there is a Lorentz frame for which $x^0 = y^0$. In that frame the r.h.s. is manifestly zero. It must be zero in all frames by Lorentz covariance. Notice also that the all important minus sign on the r.h.s came because we chose commutation relations for $a, a^\dagger$. Had we tried to make the scalar particles fermions by imposing anticommutation relations, the two terms would add and locality would be lost. This is the famous spin-statistics connection for the scalar field. For the Dirac field fermi statistics was necessary for stability rather than locality, but the spin-statistics connection is nonetheless fixed.

The Dirac field shows us how we must generalize these considerations to develop a field theory for particles with spin. As shown in an exercise the Dirac field transforms under Lorentz transformations as

$$\psi'(x') = e^{-i\lambda_\mu\sigma^{\mu\nu}/4}\psi(x),$$

(4.23)

where $(e^{-\lambda})^\mu_\nu = \Lambda^\mu_\nu$. The $4 \times 4$ matrices $\sigma^{\mu\nu}$ provide a finite dimensional representation of the Lorentz group which is necessarily not unitary. This nonunitarity is associated with the noncompactness of the Lorentz group. The nonunitarity of these matrices does not conflict with the unitarity of the action of the Lorentz group on the state space which is just that on multi-particle states we have just discussed. By expressing the field in terms of creation and annihilation operators we can see how the unitary representation on particle states induces the desired field transformation.

From the transformation properties of a spin 1/2 particle it follows that

$$U^\dagger(\Lambda)d_\lambda^\dagger(p)U(\Lambda) = \sqrt{\frac{\omega(p')}{\omega(p)}}d_\lambda^\dagger(p')D_{\lambda\lambda}^{1/2}(B_p^{-1}\Lambda^{-1}B_p^*)$$

(4.24)

$$U^\dagger(\Lambda)b_\lambda(p)U(\Lambda) = \sqrt{\frac{\omega(p')}{\omega(p)}}b_\lambda(p')D_{\lambda\lambda}^{1/2*}(B_p^{-1}\Lambda^{-1}B_p).$$

(4.25)

where $p' = \Lambda^{-1}p$. Focus on the way $b$ enters the Dirac field:

$$U^\dagger(\Lambda)\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(p)}}\sum_\lambda b_\lambda(p)u_\lambda(p)e^{ix\cdot p}U(\Lambda)$$

(4.26)

$$= \int\frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega(p')}}b_\lambda(p')u_\lambda(\Lambda p')e^{ix\cdot (p' - p)}D_{\lambda\lambda}^{1/2*}(B_p^{-1}\Lambda^{-1}B_p)$$

(4.27)

$$= \int\frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega(p')}}b_\lambda(p')u_\lambda(p')e^{ix\cdot (p' - p)}D_{\lambda\lambda}^{1/2}(B_p^{-1}\Lambda B_p)$$

(4.28)

$$= e^{-i\lambda_\mu\sigma^{\mu\nu}/4}\int\frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega(p')}}\sum_\lambda b_\lambda(p')u_\lambda(p')e^{ix\cdot (p' - p)}.$$  

(4.29)

which is exactly the desired field transformation. The term involving $d^\dagger$ works in an exactly similar way. In obtaining the last line we used the identity

$$e^{-i\lambda_\mu\sigma^{\mu\nu}/2}u_\lambda(p) = \sum_{\lambda'} u_{\lambda'}(\Lambda p)D_{\lambda\lambda'}^{1/2}(B_p^{-1}\Lambda B_p),$$

(4.30)
which is a simple consequence of the way Lorentz covariance works in the first quantized interpretation of the Dirac wave function. But this transformation law actually determines the $p$ dependence of $u_\lambda$, and in particular implies that $u_\lambda$ satisfies the Dirac equation in momentum space: $(p \cdot \gamma + m)u_\lambda = 0$.

Clearly the first step in generalizing to higher spin fields is to classify all of the finite dimensional representations of the Lorentz group. We would like to find the possible representation matrices so that a multi-component field $\psi_\alpha(x)$ will have the transformation law

$$\psi'_\alpha(x) = D_{\alpha\beta}(\Lambda)\psi_\beta(\Lambda^{-1}x). \quad (4.31)$$

Let us first cast the algebra of Lorentz generators

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho}) \quad (4.32)$$

in terms of the generators for rotations $J_k = \epsilon_{klm}M_{lm}/2$ and for boosts $K_k = M_{0k}$:

$$[J_k, J_l] = i\epsilon_{klm}J_m$$

$$[J_k, K_l] = i\epsilon_{klm}K_m$$

$$[K_k, K_l] = -i\epsilon_{klm}J_m. \quad (4.33)$$

Now notice that the linear combinations $J_\pm \equiv (J \pm iK)/2$ satisfy the algebra of two mutually commuting angular momentum algebras

$$[J_\pm^k, J_\pm^l] = i\epsilon_{klm}J_\pm^m \quad (4.34)$$

$$[J_\pm^k, J_\mp^l] = 0. \quad (4.35)$$

We know from elementary quantum mechanics what all of the finite dimensional representations of the rotation group are: they are labeled by the eigenvalues $j(j + 1)$ of the Casimir operator $\sum_k J_k^2$ with $j = 0, 1/2, 1, 2, \ldots$ all nonnegative integers and half integers. The representation $j$ has dimension $2j + 1$. It follows that all the finite dimensional representations of the Lorentz group are labeled by the pair of eigenvalues $j_+(j_+ + 1), j_-(j_- + 1)$ of the pair of Casimir operators $\sum_k J_k^2$ where $2j_+, 2j_-$ are any pair of nonnegative integers. All of these representations are equivalent to a unitary representation, i.e. $J_\pm$ are both represented by hermitian matrices. This means that the rotation generators $J = J_+ + J_-$ are represented by hermitian matrices, but the boost generators $K = -i(J_+ - J_-)$ are represented by antihermitian matrices. Thus the finite dimensional representations of the Lorentz group are not equivalent to unitary ones. This is associated with the fact that the Lorentz group is noncompact. We already encountered this nonunitarity in the representation of the Lorentz group by gamma matrices.

We denote the representation matrices by $D(j_+, j_-)$. The simplest nontrivial representations are $D(1/2, 0)$ and $D(0, 1/2)$. In the first the generators $J_-$ are represented by 0 and the generators $J_+$ by $\sigma/2$. This means that the angular momentum is represented by $J = \sigma/2$ and the boost generators by $K = -i\sigma/2$. The other two dimensional representation has
the same representative for $J$ but the boost is represented by $K = +i\sigma/2$. It is clear that these representations are not equivalent to each other, since any similarity transformation which could reverse the sign of $K$ would do the same to $J$. However these two inequivalent representations can be related by complex conjugation. In fact it is easy to see from the properties of the Pauli matrices that $D(1/2, 0)^* = \sigma_2 D(0, 1/2)\sigma_2$. In general, the representation $D(k, m)^*$ is equivalent to $D(m, k)^1$, so the only real irreducible representations have $j_+ = j_-$. Of course $D(k, m) \oplus D(m, k)$ is real but it is also reducible.

Notice that parity reverses the sign of the boost generators but not the sign of the angular momentum. Thus the representations $D(k, m)$ and $D(m, k)$ are also related by parity. We encountered this fact with the Dirac field which admits the parity symmetry. It exploits the reducible representation $D(1/2, 0) \oplus D(0, 1/2)$ to achieve this. To make this quite explicit we note that the representatives of the Lorentz generators are the $4 \times 4$ matrices

$$
J = \frac{1}{2} \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \tag{4.36}
$$

$$
K = \frac{-i}{2} \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \tag{4.37}
$$

which are just the components of $\sigma^{\mu\nu}/2$ constructed out of gamma matrices in the so-called natural representation

$$
\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{4.38}
$$

$$
\gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}. \tag{4.39}
$$

Defining $\sigma^{\mu} = (I, \sigma)$, and $\bar{\sigma}^{\mu} = (I, -\sigma)$ we can unify these two in the single equation

$$
\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}. \tag{4.40}
$$

The potential spin content of a field in a given representation is typically richer than one might desire. For example, since $J = J_+ + J_-$ the representations of the rotation group contained in $D(k, m)$ include all spins that arise from adding spin $k$ to spin $m$: $|k - m|, |k - m| + 1, \ldots, k + m$. Thus if our desire is to describe a given spin, depending on our choice of representation, we might bring in several other spins as well. The choice of field content is not unambiguous. Even for spin 1/2 we have noted various possibilities, e.g. Dirac, Majorana, and Weyl. Weyl fermions make use of $D(1/2, 0)$, but since this is not a real representation, the hermitian conjugate field, which transforms under $D(0, 1/2)$, must also be introduced and represents the anti-particle.

When we come to spin 1, two possibilities come to mind. $D(1, 0) \oplus D(0, 1)$ or $D(1/2, 1/2)$. The latter contains potentially both spin 1 and spin 0 and is in fact the representation of a

---

1In $D(k, m)^*$ the generators are $-J^*, -K^*$ so e.g. $J_+$ is represented by $-J^* - iK^* = -(J^-)^*$ and vice versa.
four-vector field. The former seems to contain spin 1 twice. It is easy to see that it is the antisymmetric tensor product of $D(1/2, 1/2)$ with itself and thus represents an antisymmetric second rank tensor. The field strengths $F_{\mu\nu}$ of electromagnetism spring to mind, so we might decide that the first choice is best. However, we know that in quantum mechanics it is necessary to use the potentials $A_\mu$ which transform under the second choice. Then gauge invariance is essential to eliminate unwanted spin states.

This general discussion of higher spin serves to indicate some of the subtleties and complexities that must be confronted. In fact, consistent fully interacting quantum field theories have never been constructed for spins higher than 2 (the graviton). Furthermore, ultraviolet divergences have so far caused incurable difficulties for theories with spin higher than 1 including quantum gravity. Since gravity is very much present in the real world, it is clear that there is much to do before we can claim that quantum field theory can describe all of physics.
### 4.1 Particles and “Particles” of the Standard Model

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</table>

The preceding table lists all of the particles in the standard model for which fundamental fields are introduced. We have listed quarks and gluons as “particles” because none of them can exist in isolation: they are trapped (confined) inside of hadrons. Quarks always come in groups of three or quark antiquark pairs. The gluons come in groups of two or more. Because of confinement the notion of mass is not precise and hence we put the mass values in quotes. They are theoretical parameters that roughly correspond to our heuristic notions of mass.
Chapter 5

Time Dependent Perturbation Theory

Although there are special quantum field theories, for example some in 2 space-time dimensions, that can be solved exactly, the exact solution of realistic interacting quantum field theories in 4 space-time dimensions is beyond reach. There are important cases for which linear QFT’s in the presence of certain external fields can be solved. For example the Dirac equation in a Coulomb potential admits an exact solution for which one can find all the energy eigenvalues and eigenstates. The Coulomb potential is of special importance since the exact solution in that case is the starting point for the relativistic theory of atomic energy levels. Still the complete dynamics is never exactly given by these special cases and perturbation theory is the important tool for evaluating corrections to the exactly soluble (idealized) case, which can be free field theory with no external fields or one of the above cases. When we quantize the electromagnetic field, perturbation theory is essentially our only tool for computing radiative corrections due to the quantum nature of the electromagnetic field.

5.1 Heisenberg and Schrödinger Pictures

In the Schrödinger picture the quantum dynamics is given by the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |\Phi(t)\rangle = H_S(t) |\Phi(t)\rangle \] (5.1)

where we stress that we allow time varying external forces to be present (hence the time dependence of \( H \)). The field equations arise in the Heisenberg picture wherein the time dependence resides in the operators rather than in the system states which are constant in time. To pass to the Heisenberg picture we write

\[ |\Phi(t)\rangle = U(t) |\Phi,0\rangle \] (5.2)

where

\[ i\hbar \frac{\partial}{\partial t} U = H_S(t) U \quad U(0) = I, \] (5.3)

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and give the time independent Schrödinger picture operators $\Omega_S$ time dependence according to
\[
\Omega(t) \equiv U^\dagger \Omega_S U. \quad (5.4)
\]
The Heisenberg picture Hamiltonian is similarly related to the Schrödinger picture one by
\[
H(t) \equiv U^\dagger H_S(t) U. \quad (5.5)
\]
Then the Heisenberg picture operators corresponding to constant Schrödinger picture operators satisfy the Heisenberg equations
\[
i\hbar \dot{\Omega}(t) = [\Omega(t), H(t)]. \quad (5.6)
\]

It is most natural to formulate the time dependence in quantum field theory using Heisenberg picture, since the field operators will then satisfy equations of motion that are the direct quantum analogue of the classical field equations. We shall therefore always understand $H(t)$ without subscripts to be the Hamiltonian in Heisenberg picture. When we work with any other picture we will attach a subscript to $H$, e.g. $H_S(t)$ is the Hamiltonian in Schrödinger picture.

### 5.2 Asymptotic States and Matrix Elements

In discussing time dependent processes, it is convenient to introduce asymptotic states which are eigenstates of $H(\pm \infty)$. We denote by $|\text{in}\rangle$ the ground state of $H(-\infty)$ and by $|\text{out}\rangle$ the ground state of $H(+\infty)$. We shall almost always assume that all external fields vanish at sufficiently early and late times. Then $|\text{in}\rangle$ and $|\text{out}\rangle$ will typically be ground states of $H_{\text{ext}=0}(-\infty)$ and $H_{\text{ext}=0}(+\infty)$ respectively. Although these operators are not the same (because their time evolution is governed by $H$ not $H_{\text{ext}=0}$), the spectra of the two Hamiltonians are identical: $H_{\text{ext}=0}(t) = U^{-1}(t, -\infty)H_{\text{S,ext}=0}U(t, -\infty)$. By convention we are identifying the Schrödinger and Heisenberg pictures at $t = -\infty$. Thus, if $|\text{in}\rangle$ is the ground state of $H_{\text{ext}=0}(-\infty) = H_{\text{S,ext}=0}$, the state $\langle \text{in}|U(\infty, -\infty)$ is an eigenstate of $H_{\text{ext}=0}(+\infty)$ with the same eigenvalue and hence the ground state. Thus we can and shall fix phases by defining
\[
\langle \text{out}| \equiv \langle \text{in}|U(\infty, -\infty). \quad (5.7)
\]

We stress that this is the true “out” state only when $H_S(\infty) = H_S(-\infty) \equiv H_{\text{ext}=0}$. In QFT we can treat the coupling constants as external fields which can be turned on and off at early and late times, in which case $H_{\text{ext}=0}$ would be the free field part of the Hamiltonian.

If the time dependence of $H_S$ is adiabatic, i.e. very slow on the time scale set by the level spacings, the Adiabatic Theorem assures us that an eigenstate of $H_S(-\infty)$ evolves to an eigenstate of $H_S(t)$ for all $t$ for which adiabatic conditions apply, even after a long enough time to change $H_S$ by a finite amount. For example, the state $|\text{in}\rangle$ will be an eigenstate of $H(t)$ for all $t$ for which adiabatic time variation applies. In particular, the ground state
eigenvalue $E_G(t)$ must not get close to the next higher eigenvalue as $t$ varies. If this situation holds for all time, it follows that the state $|in\rangle$ is a phase times the state $|out\rangle$, or $\langle out| is this same phase times $\langle in|$. This phase is easily evaluated in terms of the time dependent ground state energy $E_G(t)$ of $H_S(t)$ by applying the Schrödinger equation to $\langle in|U(t, -\infty)|in\rangle$ and using the adiabatic theorem $H_S(t)U(t, -\infty)|in\rangle = E_G(t)U(t, -\infty)|in\rangle$:

$$\langle out|in\rangle = \exp\left\{-i \int_{-\infty}^{\infty} dt E_G(t) \right\} \quad \text{Adiabatic Conditions.} \quad (5.8)$$

Note carefully that adiabatic conditions would not apply if the ground state energy got close to an excited level as time evolved. In particular, it would not apply in processes with pair production when $|\langle out|in\rangle| < 1$.

### 5.3 General Formalism

We shall keep the initial discussion completely general and consider the situation in which the Heisenberg picture Hamiltonian is the sum of two pieces,

$$H(t) = H_0(t) + H'(t) \quad (5.9)$$

where $H_0$ can be exactly dealt with, e.g. in QFT could be the free field part of the Hamiltonian; and $H'$ is “small” in an appropriate sense. Note that even when $H$ is independent of time, $H_0$ and $H'$ still depend on time through the time dependence of the Heisenberg operators that enter it. The Heisenberg equations for the dynamical variables have the form

$$i \frac{d\Omega}{dt} = [\Omega, H(t)]. \quad (5.10)$$

The goal of time dependent perturbation theory is to expand the evolution operator $U(t, t_0)$ which carries the time dependence of the Heisenberg picture operators $\Omega = U^{-1}\Omega_S U$, or alternatively the time dependence of the Schrödinger picture system states $|\psi(t)\rangle = U|\psi(t_0)\rangle$, in a power series in $H'$. A complication is that there is time dependence in $U$ even when the perturbation vanishes. To systematically deal with this complication a new Interaction Picture (sometimes called the Dirac Picture) has been devised in which the operators carry the (known) time dependence due to $H_0$ and the perturbation only enters the modified evolution operator $U_I$, which is constructed to be the identity in the absence of $H'$.

Thus for each Heisenberg picture operator with no explicit time dependence, we define an interaction picture operator by

$$\Omega_I(t) \equiv U_I(t)\Omega(t)U_I^{-1}(t) \quad (5.11)$$

and require that $\Omega_I$ satisfies the Heisenberg equation with Hamiltonian $H_{0I} = U_I H_0 U_I^{-1}$

$$i \frac{\partial \Omega_I}{\partial t} = [\Omega_I, H_{0I}(t)]. \quad (5.12)$$

---

1Note that if $H_0 = H_0(\Omega_k(t), t)$, then $H_{0I} = H_0(\Omega_{Ik}(t), t)$. 

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differentiating (5.11) we find the requirement

\[
[\Omega_I, H_{0I}(t)] = i\dot{U}_I \Omega U_I^{-1} + iU_I \Omega \dot{U}_I^{-1} + U_I[\Omega, H(t)] U_I^{-1} = iU_I \Omega U_I^{-1} \Omega + i\dot{U}_I U_I^{-1} + [\Omega_I, H_{0I}(t)] + [\Omega_I, H_I'(t)]
\]

(5.13)

Thus the equation for \(U_I\) is just

\[i\dot{U}_I(t) = H_I'(t)U_I(t) = U_I(t)H'(t).\]  

(5.16)

We choose the initial condition \(U_I(t_0) = I\), in which case it is a good idea to display two time arguments \(U_I(t, t_0)\) as we did for \(U\). Notice that since \(U\) relates Heisenberg and Schrödinger pictures, the equation (5.3) for \(U\) can be also written

\[i\dot{U} = UH(t)\]  

(5.17)

from which it is clear that we can express \(U = U_0 U_I\) where

\[i\dot{U}_0 = U_0 H_{0I}(t)\]  

(5.18)

To expand \(U_I\) in powers of \(H'\) it is convenient first to incorporate initial condition information by writing the integral equation

\[U_I(t, t_0) = I - i \int_{t_0}^{t} dt' H'_I(t')U_I(t', t_0),\]  

(5.19)

and then to generate the perturbation series by iteration

\[U_I(t, t_0) = I - i \int_{t_0}^{t} dt' H'_I(t')(I - i \int_{t_0}^{t} dt'' H'_I(t'')U_I(t'', t_0))\]  

(5.20)

and then

\[= I - i \int_{t_0}^{t} dt' H'_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' H'_I(t') H'_I(t'') + \cdots.\]  

(5.21)

There is a useful way to summarize the entire perturbation series, which employs the concept of the time ordered product of operators. Consider a set of operators each associated with a different time, \(A_1(t_1), A_2(t_2), \ldots, A_N(t_N)\). The time ordered product of these operators is defined as the ordinary product with the operators ordered according to the time argument: the operator \(A_k(t_k)\) to the left of \(A_l(t_l)\) if \(t_k > t_l\). If there are any anticommuting operators in the set, there is also an overall \(-1\) if one achieves the time ordering by an odd permutation of fermionic operators. Thus, for example,

\[
T[A(t_1)B(t_2)] = \begin{cases} 
A(t_1)B(t_2) & t_1 > t_2 \\
\pm B(t_2)A(t_1) & t_2 > t_1 
\end{cases}
\]

(5.22)

with the \(-\) for both operators fermionic. Now the factors of \(H'_I\) in the series for \(U_I\) are all time ordered due to the limits of integration. If we make use of the time ordering symbol, we
can extend all integrations to the full range \( t_0 < t' < t \) provided we divide the \( n^{th} \) term by \( n! \) to account for the overcounting due to the \( n! \) orderings of the \( t' \)s. Thus the entire series becomes

\[
U_I(t, t_0) = \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n \int_{t_0}^{t} dt_1 dt_2 \cdots dt_n T[H'_I(t_1) H'_I(t_2) \cdots H'_I(t_n)].
\] (5.23)

If it weren’t for the time ordering symbol this would be just the exponential series. It is therefore a useful mnemonic to write

\[
U_I(t, t_0) = T e^{-i \int_{t_0}^{t} dt' H'_I(t')}
\] (5.24)

where it is understood that \( t > t_0 \). This equation is known as the Dyson Formula. Since the formula just reflects the equation \( U_I \) satisfies, we can write a similar formula for the full \( U \):

\[
U(t, t_0) = T e^{-i \int_{t_0}^{t} dt' H_0(t')} = T e^{-i \int_{t_0}^{t} dt' H_0(t')} T e^{-i \int_{t_0}^{t} dt' H'_I(t')}
\] (5.25)

where \( \bar{T} \) denotes anti-time ordering (later times to the right).

It will be useful to extend the definition of \( U_I(t, t_0) \) to times earlier than \( t_0 \). We shall do this in a way to preserve the closure property

\[
U_I(t, t_1) U_I(t_1, t_0) = U_I(t, t_0),
\] (5.26)

which follows from the differential equation and initial condition for \( t > t_1 > t_0 \). If we set \( t = t_0 \) in (5.26), the r.h.s. is just \( I \) so we have to define

\[
U_I(t, t_0) \equiv U_I^{-1}(t_0, t) = U_I^\dagger(t_0, t) \quad \text{for } t < t_0.
\] (5.27)

(Note that \( U_I^\dagger U_I = I \) is a simple consequence of the differential equation and the hermiticity of \( H'_I \).) It is then simple to check that (5.26) holds for all time orderings.

Next let us show how to express various physical quantities in the Interaction picture. One interesting quantity is the so called vacuum persistence amplitude \( \langle \text{out} | \text{in} \rangle \), given by

\[
\langle \text{out} | \text{in} \rangle = \langle \text{in} | U(\infty, -\infty) | \text{in} \rangle = \langle \text{in} | U_0(\infty, -\infty) U_I(\infty, -\infty) | \text{in} \rangle.
\] (5.28)

We shall identify all pictures at \( t = -\infty \). If external fields vanish at early times and the couplings are treated as external fields vanishing at earl and late times, the state \( | \text{in} \rangle \) will be the ground state of \( H_0(-\infty) = H_{0f} \). Furthermore, with no external fields in \( H_{0f}(t) \) the latter will be time independent for all time (since its time evolution is governed by \( H_{0f} \) itself). In this situation

\[
U_0(t, t_0) = e^{-i H_{0f}(t-t_0)}
\] (5.29)

and \( \langle \text{in} | \) is an eigenstate of \( U_0(t, t_0) \) with eigenvalue \( e^{-i E_0(t-t_0)} \), where \( E_0 \) is the ground state energy of \( H_{0f} \). By convention we can choose our zero of energy so that \( E_0 = 0 \), in which case we have

\[
\langle \text{out} | \text{in} \rangle = \langle 0, I | U_I(\infty, -\infty) | 0, I \rangle \quad E_0 = 0.
\] (5.30)
The persistence amplitude carries a lot of information, because it can be defined for any choice of external fields. Its dependence on these external fields can then be exploited to obtain numerous matrix elements relevant to the zero field situation. We shall see many applications of this remark in the course of our studies. One can also get the energy spectrum for static external fields from this amplitude by switching them on at some early time keeping them constant for a long time $2T$ and then switching them off. The $T$ dependence of $\langle \text{out}|\text{in}\rangle$ will then display the dependence $e^{-iE\cdot 2T}$ from which the energy eigenvalues can be read off. The states that are probed by this device will depend on the manner of the switching on procedure. For adiabatic switching on, only the ground state in the presence of the static field will contribute.

5.4 Scattering in an External Field: Born Approximation

5.4.1 Scattering of scalar particles

One can equally well choose initial and final states that contain particles. For example, the amplitude for a scalar particle with momentum $p$ initially making a transition to $p'$ at very late times is

$$\langle 0, I|a(p')U_I(\infty, -\infty)a^\dagger(p)|0, I\rangle \approx \delta(p' - p) - i \int_{-\infty}^\infty dt \langle 0, I|a(p')H'_I(t)a^\dagger(p)|0, I\rangle,$$  

(5.31)

where we kept only terms to first order. As a concrete example, consider the interaction $\int dt H'_I(t) = -\int d^4 x \phi^2_I(x)B(x)/2$, where $B(x)$ is a fixed external scalar field. Since the $\phi_I$ are free fields, they can be expressed in terms of creation and annihilation operators and the matrix element evaluated:

$$\int_{-\infty}^\infty dt \langle 0, I|a(p')H'_I(t)a^\dagger(p)|0, I\rangle = -\int d^4 x B(x)e^{i(p' - p)'x} \frac{1}{(2\pi)^3 \sqrt{2\omega(p)}\sqrt{2\omega(p')}}$$  

(5.32)

$$= -\frac{1}{(2\pi)^3 \sqrt{2\omega(p)}\sqrt{2\omega(p')}} \tilde{B}(p' - p),$$  

(5.33)

where $\tilde{B}(q) = \int d^4 x B(x)e^{-iq'x}$ is the Fourier transform of the external field. Note that in the case $B$ is static the time integral gives a factor of $2\pi\delta(\omega' - \omega)$.

Recall from basic scattering theory that if the scattering matrix for a particle from a static potential is written

$$\langle q, \text{out}|p, \text{in}\rangle = \delta(q - p) - 2\pi i\delta(\omega(q) - \omega(p))T(q, p)$$  

(5.35)
then the differential scattering cross section is given by

\[
\frac{d\sigma}{d\Omega} = \frac{d^3q}{d\Omega} \delta(\omega(q) - \omega(p)) \frac{(2\pi)^4}{v} |T(q, p)|^2,
\]

\[
= q^2 dq \delta(\omega(q) - \omega(p)) \frac{(2\pi)^4}{v} |T(q, p)|^2,
\]

\[
= p \omega(p) \frac{(2\pi)^4}{v} |T(q, p)|^2,
\]

\[
= \omega(p)^2 (2\pi)^4 |T(q, p)|^2,
\]

(5.36)

where \( v = p/\omega \) is the speed of the incident particle. Defining the spatial Fourier transform \( \tilde{B}(k) \equiv \int d^3xe^{-ik\cdot x}B(x) \), we then obtain

\[
T^{\text{Born}}(q, p) = -\frac{1}{(2\pi)^3 2\omega(p)} \tilde{B}(q - p),
\]

(5.37)

giving the cross section

\[
\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} |\tilde{B}(q - p)|^2.
\]

(5.38)

### 5.4.2 Scattering of Dirac particle

Another scattering example, the amplitude for a Dirac particle with momentum and helicity \( p, \lambda \) initially making a transition to \( p', \lambda' \) at very late times is

\[
\langle 0, I | b_\lambda(p') U_I(\infty, -\infty) b_\lambda^\dagger(p) | 0, I \rangle
\]

\[
= \langle \text{out} | \text{in} \rangle \delta_{\lambda, \lambda'} (p' - p) + \langle 0, I | [b_\lambda(p') U_I(\infty, -\infty)] b_\lambda^\dagger(p) | 0, I \rangle
\]

\[
\approx \langle \text{out} | \text{in} \rangle \left[ \delta_{\lambda, \lambda'} (p' - p) - i \int_{-\infty}^{\infty} dt \langle 0, I | [b_\lambda(p'), H^I(t)] b_\lambda^\dagger(p) | 0, I \rangle \right],
\]

(5.39)

where we kept only terms to first order in \( H^I \). The overall \( \langle \text{out} | \text{in} \rangle \) factor just describes the independent evolution of the vacuum and is irrelevant to particle scattering. In the case of a weak external electromagnetic field \( \int dt H^I(t) = -\int d^4x j^\mu_\gamma(x) A_\mu(x) \), where \( j^\mu_\gamma = \frac{\gamma^\mu}{2} [\bar{\psi}_I, \gamma^\mu \psi_I] \). Since the \( \psi_I \) are free fields, they can be expressed in terms of creation and annihilation operators and the matrix element evaluated:

\[
\int_{-\infty}^{\infty} dt \langle 0, I | b_\lambda(p') H^I(t) b_\lambda^\dagger(p) | 0, I \rangle = \delta(p' - p)^\dagger \langle 0, I | H^I(t) | 0, I \rangle
\]

\[
- q \int d^4 x A_\mu(x) e^{i(p' - p) \cdot x} \frac{1}{(2\pi)^2 \sqrt{2\omega(p') \sqrt{2\omega(p)}}} \bar{u}_\lambda(p') \gamma^\mu u_\lambda(p)
\]

(5.40)

\[\text{The second term in square brackets should be divided by } \langle \text{out} | \text{in} \rangle, \text{ but since } \langle \text{out} | \text{in} \rangle - 1 \text{ is of first order, so this would only contribute to second and higher order terms.}\]
The first term on the right contributes the first order correction to a factor \( \langle \text{out}|\text{in} \rangle \) multiplying \( \delta(p' - p) \) the term describing no scattering. Since the electric current is defined so that \( \langle 0, i|j^\mu|0, I \rangle = 0 \) this first order correction vanishes, but higher order corrections can be nonzero as we shall see. Dropping this term the lowest order scattering is controlled by

\[
\langle 0, I|b_{\lambda'}(p')H'_I(t)b_{\lambda}(p)|0, I \rangle = -q \frac{1}{(2\pi)^3 \sqrt{2\omega(p)} \sqrt{2\omega(p')}} \bar{u}_{\lambda'}(p') \gamma^\mu \tilde{A}_\mu(p' - p) u_{\lambda}(p),
\]

where \( \tilde{A}(q) = \int d^4x A(x) e^{-iq\cdot x} \) is the Fourier transform of the potential. Note that in the case \( A \) is static the time integral gives a factor of \( 2\pi \delta(\omega' - \omega) \).

Recall from basic scattering theory that if the scattering matrix for a particle from a static potential is written

\[
\frac{\langle q, \text{out}|p, \text{in} \rangle}{\langle \text{out}|\text{in} \rangle} = \delta_{\lambda',\lambda} \delta(q - p) - 2\pi i \delta'(\omega - \omega) \pi T_{\lambda'\lambda}(q, p), \tag{5.41}
\]

We divide by \( \langle \text{out}|\text{in} \rangle \) to make the coefficient of the delta function term unity.] then the differential scattering cross section is given by

\[
\frac{d\sigma}{d\Omega} = \frac{\theta^2}{4} \frac{\delta(\omega - \omega)}{v} |T_{\lambda'\lambda}(q, p)|^2, \tag{5.42}
\]

\[
= q^2 \frac{\theta^2}{4} \frac{\delta(\omega - \omega)}{v} |T_{\lambda'\lambda}(q, p)|^2, \tag{5.43}
\]

\[
= (\omega(p))^2 \frac{\theta^2}{4} |T_{\lambda'\lambda}(q, p)|^2, \tag{5.44}
\]

\[
= \omega(p)^2 (2\pi)^4 |T_{\lambda'\lambda}(q, p)|^2, \tag{5.45}
\]

where \( v = p/\omega \) is the speed of the incident particle.

Defining \( \tilde{A}(k) \equiv \int d^3x e^{-ik\cdot x} A(x) \), we then obtain

\[
T_{\lambda'\lambda}^{\text{Born}}(p', p) = -q \frac{1}{(2\pi)^3 2\omega(p)} \bar{u}_{\lambda'}(p') \gamma \cdot \tilde{A}(p' - p) u_{\lambda}(p), \tag{5.46}
\]

giving the cross section

\[
\frac{d\sigma}{d\Omega}^{\text{Born}} = \frac{q^2}{16\pi^2} |\bar{u}_{\lambda'}(p') \gamma \cdot \tilde{A}(p' - p) u_{\lambda}(p)|^2. \tag{5.47}
\]

Consider the example of the scattering of an electron with \( q = -e \) from the Coulomb potential of a nucleus of atomic number \( Z \) (Mott Scattering), \( A^0 = Ze/4\pi r, A = 0 \). Then \( \tilde{A}^0(k) = Ze/k^2 \) and

\[
\frac{d\sigma}{d\Omega}^{\text{Born}} = \frac{e^4 Z^2}{16\pi^2 (p' - p)^4} |\bar{u}_{\lambda'}(p') \gamma^0 u_{\lambda}(p)|^2 = \frac{\alpha^2 Z^2}{(p' - p)^4} |\bar{u}_{\lambda'}(p') \gamma^0 u_{\lambda}(p)|^2. \tag{5.48}
\]
Here we have introduced the fine structure constant $\alpha = e^2/4\pi \approx 1/137$. Evaluating the spinor matrix element in terms of two component helicity spinors leads to (using $|p'| = |p|$)

$$\bar{u}_\lambda(p')\gamma^0u_\lambda(p) = \left[\omega(p) + m + \frac{4\lambda'\lambda p^2}{\omega + m}\right]\chi_\lambda'(p')\chi_\lambda(p). \quad (5.49)$$

The absolute square of $\chi_\lambda'\chi$ can be evaluated by noting that the $2 \times 2$ matrix $\chi\chi^\dagger$ is a projector onto the spin state of definite helicity:

$$\bar{u}_\lambda(p')\chi_\lambda(p) = \frac{1 + 2\lambda\hat{p} \cdot \sigma}{2}. \quad (5.50)$$

Thus we have

$$|\chi_\lambda'(p')\chi_\lambda(p)|^2 = \frac{1}{4}\text{tr}[(1 + 2\lambda\hat{p} \cdot \sigma)(1 + 2\lambda'\hat{q} \cdot \sigma)] = \frac{1}{2}(1 + 4\lambda\lambda'\hat{p} \cdot \hat{q}) \quad (5.51)$$

Inserting all this into the formula for the differential cross section, we obtain after simplifying

$$\frac{d\sigma^\text{Born}}{d\Omega} = \frac{\alpha^2 Z^2}{(p'-p)^4}\left[\omega^2 + m^2 + p' \cdot p + 4\lambda\lambda'(p^2 + (\omega^2 + m^2)\hat{p} \cdot \hat{p}')\right] \quad (5.53)$$

To compare all of the details of this formula with experiment we would have to prepare a polarized beam of electrons with definite helicity and also measure the spin of the final electron. A noteworthy feature of such a complete experiment is that at high energies there is an overall factor of $(1 + 4\lambda\lambda') = 2\delta_{\lambda\lambda'}$, which means that helicity is conserved at high energy. If we don’t measure the final spin we should sum over $\lambda' = \pm 1/2$ to obtain

$$\frac{d\sigma^\text{Born}}{d\Omega_\text{Unobserved spin}} = 2\frac{\alpha^2 Z^2}{(p'-p)^4}\left[\omega^2 + m^2 + p' \cdot p\right]. \quad (5.54)$$

Similarly, if we have a completely unpolarized beam, we need to average over $\lambda$ to obtain

$$\frac{d\sigma^\text{Born}}{d\Omega_\text{unpol}} = \frac{\alpha^2 Z^2}{(p'-p)^4}\left[\omega^2 + m^2 + p' \cdot p\right], \quad (5.55)$$

independent of the final spin.

Two simplifying limits can be considered. The nonrelativistic or low energy limit $p^2 << m^2$ is

$$\frac{d\sigma^\text{NR}}{d\Omega} \sim 2m^2\frac{\alpha^2 Z^2}{(p'-p)^4}\left[1 + 4\lambda\lambda'\hat{p} \cdot \hat{p}'\right] \quad (5.56)$$

Apart from the helicity dependence due to the spin of the electrons this is just the Rutherford formula. The opposite limit, the ultrarelativistic or high energy limit $p^2 >> m^2$ is (assume $\hat{p} \cdot \hat{p}' \neq -1$)

$$\frac{d\sigma^\text{UR}}{d\Omega} \sim \frac{\alpha^2 Z^2}{(p'-p)^4}\left[p^2 + p' \cdot p\right]\left[1 + 4\lambda\lambda'\right] = 2\delta_{\lambda\lambda'}\frac{\alpha^2 Z^2 p^2}{(p'-p)^4}\left[1 + \cos \theta\right]. \quad (5.57)$$

where the high energy helicity conservation is transparent.
5.5 Pair production in a time varying external field

Let us return to the scalar field external field perturbation $H'_I(t) = -\int d^3x \phi_I^2 B(x)/2$ in the case where the initial state is the ground state of $H_{0I}$, $|0\rangle_I$, and the final state contains two particles: $\langle 0, I|a(\vec{p}_1)a(\vec{p}_2)$. The transition amplitude to first order is

\[ T_{fi} = \frac{i}{2} \int d^4x \langle 0, I|a(\vec{k}_1)a(\vec{k}_2)\phi_I^2(x)B(x)|0, I \rangle \]
\[ = \frac{i}{(2\pi)^3 2\sqrt{\omega_1\omega_2}} \int d^4xe^{-i(k_1+k_2)x} B(x) = \frac{i\tilde{B}(k_1 + k_2)}{(2\pi)^3 2\sqrt{\omega_1\omega_2}} \]  

5.6 Perturbation theory for Time Ordered Products

In quantum field theory it is much more natural to calculate expectations of local fields rather than matrix elements between particle states. As we shall see, particle matrix elements can be extracted from out in matrix elements of time ordered products of fields in Heisenberg picture:

\[ \langle \text{out}|T[A_1(t_1)A_2(t_2)\cdots A_N(t_N)]|\text{in}\rangle. \]  

The simplest way to transcribe this matrix element to interaction picture is to first assume the ordering $t_1 > t_2 > \cdots > t_N$ so that the $T$ symbol can be removed. Then

\[ \langle \text{out}|A_1(t_1)A_2(t_2)\cdots A_N(t_N)|\text{in}\rangle \]
\[ = \langle 0, I|U_I(t_1, -\infty)U_I^{-1}(t_1, -\infty)A_{I1}(t_1)U_I(t_1, -\infty)U_I^{-1}(t_2, -\infty)A_{I2}(t_2) \]
\[ U_I(t_2, -\infty)\cdots U_I^{-1}(t_N, -\infty)A_{IN}(t_N)U(t_N, -\infty)|0, I \rangle \]
\[ = \langle 0, I|U_I(t_1, t_1)A_{I1}(t_1)U_I(t_1, t_2)A_{I2}(t_2)U_I(t_2, t_3)\cdots U_I(t_{N-1}, t_N)A_{IN}(t_N) \]
\[ U_I(t_N, -\infty)|0, I \rangle, \]
Dyson formula for each $U_I$ are time ordered. Thus if we insert the time ordering symbol in front of all the operators we can combine all of the $U_I$’s into a single $U_I(\infty, -\infty)$ arriving at

\[
\langle \text{out} | T[A_1(t_1)A_2(t_2)\cdots A_N(t_N)] | \text{in} \rangle = \langle 0, I | T[U_I(\infty, -\infty)A_{I1}(t_1)A_{I2}(t_2)\cdots A_{IN}(t_N)] | 0, I \rangle.
\]  

(5.65)

Finally, we simply note that had the time ordering been any other, the same steps would have led to the same final result.

In the usual situation where $H_{0I}$ is the Hamiltonian for free fields, all of the interaction picture operators are free fields, and to evaluate each finite order in perturbation theory one only needs to master the computation of the vacuum expectation values of the time ordered product of a finite number of free fields. Free fields can always be expressed as a linear functional of creation and annihilation operators. Thus if $\phi_k(x)$ is a free field, it can be written

\[
\phi_k(x) = \phi^+_k(x) + \phi^-_k(x)
\]  

(5.66)

where $\phi^+_k(x)$ annihilates $|0\rangle$ and $\phi^-_k(x)$ annihilates $\langle 0|$. Thus we have

\[
\langle 0 | T[\phi_k(x)\phi_l(x')]|0\rangle = \theta(t - t')\langle 0 | \phi^+_k(x)\phi^-_l(x')|0\rangle \pm \theta(t' - t)\langle 0 | \phi^+_l(x')\phi^-_k(x)|0\rangle
\]  

(5.67)

\[
\equiv \theta(t - t')C_{kl}(x - x') \pm \theta(t' - t)C_{lk}(x' - x)
\]  

(5.68)

where $C_{kl}(x - x') = [\phi^+_k(x), \phi^-_l(x')]_\pm$ is a $c$ number since the fields are free.

Now consider a general time ordered product of $N$ free fields

\[
\langle 0 | T[\phi_1(x_1)\cdots \phi_N(x_N)]|0\rangle
\]  

(5.69)

and first assume $t_1 > t_2 > \cdots > t_N$. Then the leftmost field is $\phi_1$ and it can be replaced by its annihilation part $\phi^+_1$, which is then moved via the commutation relations all the way to the right where it kills the vacuum. The (anti)commutators

\[
[\phi^+_1(x_1), \phi_k(x_k)]_\pm = [\phi^+_1(x_1), \phi^-_k(x_k)]_\pm = \langle 0 | T[\phi_1(x_1)\phi_k(x_k)]|0\rangle
\]  

(5.70)

since the (anti)commutators are $c$ numbers and $t_1 > t_k$ by assumption. Thus

\[
\langle 0 | T[\phi_1(x_1)\cdots \phi_N(x_N)]|0\rangle = \langle 0 | T[\phi_1(x_1)\phi_2(x_2)]|0\rangle \langle 0 | T[\phi_3(x_3)\cdots \phi_N(x_N)]|0\rangle
\]

\[
\pm \langle 0 | T[\phi_1(x_1)\phi_3(x_3)]|0\rangle \langle 0 | T[\phi_2(x_2)\phi_4(x_4)\cdots \phi_N(x_N)]|0\rangle \pm \cdots
\]  

(5.71)

where the sign in front of each term is dictated by the number of times the order of fermionic operators is switched. The time ordering symbol is not needed with our assumed ordering of times. But now we notice that if the time ordering had been any other the same steps would have led to the same result provided we keep the $T$ symbol in place. Thus we have
related the vacuum expectation value of the time ordered product of $N$ free fields to those of 2 free fields and $N - 2$ free fields. By induction we can therefore express the vacuum expectation value of the time ordered product of $N$ free fields as sums of products of the vacuum expectation values of the time ordered product of pairs of free fields.

The result, known as Wick’s Theorem, can be expressed as follows. First note that the answer is 0 unless $N$ is even. Then the vacuum expectation value of the time ordered product of $N$ free fields is the sum of terms, one for each distinct pairing off of all the $N$ fields. The term for each such pairing off is simply $\pm$ the product of the vacuum expectation values of the time ordered product of each pair of fields in the given pairing off. The sign is determined by comparing the ordering of the $N$ operators in the original time ordered product with the order they appear in the given term after being paired off. If the latter ordering is achieved by an odd permutation of fermionic operators the sign is $-;$ otherwise it is $+$. It doesn’t matter what order we display the factors within a given term, since switching their order would always be an even permutation: a pairing of a boson field with a fermion field would always contribute zero!

5.7 A Technical Comment on Time Derivatives in Time Dependent Perturbation Theory (Skip on first reading)

It is important to appreciate some subtle differences between time derivatives of operators in different pictures. For example, interaction picture depends on a specific breakup of the Heisenberg picture Hamiltonian $H = H_0 + H'$, so in Heisenberg picture

$$\dot{\Omega} = \frac{1}{i}[\Omega, H_0] + \frac{1}{i}[\Omega, H'].$$ (5.72)

The transformation to interaction picture, being a purely algebraic similarity transformation shows that

$$(\dot{\Omega})_I = \frac{1}{i}[\Omega_I, H_0] + \frac{1}{i}[\Omega_I, H'_I]$$

$$= \dot{\Omega}_I + \frac{1}{i}[\Omega_I, H'_I],$$ (5.74)

so there is in general a discrepancy between the interaction picture operator corresponding to $\dot{\Omega}$ and the time derivative of the operator $\Omega_I$. When we use the Dyson formula for time dependent perturbation theory to calculate a matrix element involving $\dot{\Omega}$, care must be taken about this difference. However there is a very simple prescription to keep things straight. This is to always think of time derivatives of operators in the Dyson formula as acting outside the time ordering symbol. Note the following identity

$$\frac{\partial}{\partial t} T[e^{-i \int^\infty_{-\infty} dt' H'_I(t')} \Omega_I(t)] = T[e^{-i \int^\infty_{-\infty} dt' H'_I(t')} (\dot{\Omega}_I(t) + \frac{1}{i}[\Omega_I(t), H'_I(t)])] T[e^{-i \int^\infty_{-\infty} dt' H'_I(t')}].$$ (5.75)
This comment becomes particularly useful in cases such as scalar electrodynamics where the relation between \( \pi^I(x) \) and \( i\partial_t \phi(x) = \pi^I(x) + iQA_0 \phi(x) \) involves the interaction. Since the interaction picture fields are free, the relationship in that picture is \( i\partial_t \phi_I(x) = \pi^I_\phi(x) \). Thus the exponent in the Dyson formula shows a disquieting asymmetry between space and time:

\[
\int dt H'_I(t) = \int d^4x \left( iQA \cdot (\phi_I^\dagger \nabla \phi_I - (\nabla \phi_I^\dagger)\phi_I) + Q^2 A^2 \phi_I^\dagger \phi_I - iQA_0 (\phi_I^\dagger \phi_I - \phi_I^\dagger \phi_I) \right). \tag{5.76}
\]

However, it is possible to prove a “reshuffling theorem” that if all time derivatives in the Dyson formula are understood to be taken outside the time ordering symbol, covariance is restored. In other words there are two sources of apparent non-covariance: the form of \( H'_I \) and the time ordering operation itself. To present the results of the reshuffling theorem, we introduce the symbol \( T^* \) to signify time ordering in which all time derivatives are taken outside the time ordering symbol. Then the reshuffling theorem for scalar electrodynamics can be stated

\[
T[e^{-i \int_{-\infty}^{\infty} dt H'_I(t')} = T^* \left[ e^{-i \int d^4x (iQA^\mu (\phi_I^\dagger \partial_\mu \phi_I - (\partial_\mu \phi_I^\dagger)\phi_I) + Q^2 A^\mu A_\mu \phi_I^\dagger \phi_I) + iQA_0 (\phi_I^\dagger \phi_I - \phi_I^\dagger \phi_I) + Q^2 A^\mu A_\mu \phi_I^\dagger \phi_I) \right] 0, I]. \tag{5.77}
\]

When employing the Wick expansion to the r.h.s. one simply needs to remember that one never uses a quantity such as \( \langle 0|T[\partial_\mu \phi(x)\phi^\dagger(y)]|0 \rangle \), namely all derivatives occur outside not inside the time ordering symbols.

As an illuminating example of these ideas we quote the improved Dyson formula for the outin matrix element of the current operator:

\[
\langle \text{out} | \mu(x) | \text{in} \rangle = \langle 0, I | T^* \left[ e^{-i \int d^4x (iQA^\mu (\phi_I^\dagger \partial_\mu \phi_I - (\partial_\mu \phi_I^\dagger)\phi_I) + Q^2 A^\mu A_\mu \phi_I^\dagger \phi_I) + iQA_0 (\phi_I^\dagger \phi_I - \phi_I^\dagger \phi_I) - Q^2 A^\mu A_\mu \phi_I^\dagger \phi_I) \right] 0, I \rangle. \tag{5.78}
\]

Take particular note of the manifest covariance of the r.h.s. of this formula.

### 5.8 Propagators for Scalar and Dirac Fields

Wick’s Theorem assures us that to obtain a general time ordered product of free fields, we only need to know the two field case, \( \langle 0|T[\phi_1(x)\phi_2(x)]|0 \rangle \), which is also called the two point function and sometimes the propagator.

Let us first work out the propagator for a free scalar field which has the representation

\[
\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{2\omega(p)} e^{ip \cdot x} \left( a(p) e^{ip \cdot x} + b^\dagger(p) e^{-ip \cdot x} \right), \tag{5.80}
\]

where \( a \) annihilates a particle and \( b^\dagger \) creates an antiparticle. These operators satisfy the commutation relations

\[
[a(p), a^\dagger(p')] = [b(p), b^\dagger(p')] = \delta(p' - p), \tag{5.81}
\]

\( \odot 1992, 2017 \) by Charles Thorn
with all other commutators vanishing\(^3\). The Hamiltonian for the free scalar field is easy to write down

\[ H = \int d^3p \omega(p)(a^\dagger(p)a(p) + b^\dagger(p)b(p)) \]  
\[ = \int d^3x : (\dot{\phi}^\dagger \dot{\phi} + \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi) : \]  

where the double colons \( \langle \cdots \rangle \) denotes normal ordering, \( i.e. \) all creation operators to the left of all annihilation operators.

Clearly the vacuum expectation of the time ordered product of two \( \phi \)'s or two \( \phi^\dagger \)'s vanishes, and

\[ \langle 0 | T[\phi(x_1)\phi^\dagger(x_2)] | 0 \rangle = \theta(t_1 - t_2) \int \frac{d^3p}{(2\pi)^32\omega(p)} e^{ip(x_1-x_2)} \]
\[ + \theta(t_2 - t_1) \int \frac{d^3p}{(2\pi)^32\omega(p)} e^{ip(x_2-x_1)}, \]

where we recall that \( p \cdot x = p \cdot x - \omega t \). To make this expression less unwieldy, it is helpful to use the following integral representation for the step function

\[ \theta(t) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} e^{-ip^0t} \frac{1}{-p^0 - i\epsilon} \quad \epsilon \to 0^+. \]  

Including the factor \( e^{-i\omega t} \) gives

\[ \theta(t)e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} e^{-ip^0t} \frac{1}{\omega - p^0 - i\epsilon}, \]  

after a shift of \( p^0 \). Inserting this representation into (5.86), gives

\[ \langle 0 | T[\phi(x_1)\phi^\dagger(x_2)] | 0 \rangle = \int \frac{d^3p}{(2\pi)^32\omega(p)} \left( e^{ip(x_1-x_2)} \frac{-i}{\omega - p^0 - i\epsilon} + e^{ip(x_2-x_1)} \frac{-i}{\omega - p^0 - i\epsilon} \right), \]

where now \( p \cdot x = p \cdot x - p^0t \). Thus we can change \( p \to -p \) in the second term and then combine it with the first to obtain finally

\[ \langle 0 | T[\phi(x_1)\phi^\dagger(x_2)] | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x_1-x_2)} \frac{-i}{p^2 + m^2 - i\epsilon} \equiv \Delta_F(x_1 - x_2). \]  

\(^3\)Notice that the commutator

\[ [\phi(x_1), \phi^\dagger(x_2)] = \int \frac{d^3p}{(2\pi)^32\omega(p)} (e^{ip(x_1-x_2)} - e^{ip(x_2-x_1)}) \]

vanishes for space-like separations \((x_2 - x_1)^2 > 0\). To see this go to a Lorentz frame where \( t_2 = t_1 \) (always possible for space-like separations). Then the second term cancels the first after the variable change \( p \to -p \). If \( a, b \) satisfied anticommutation relations, the anticommutator would not have this locality property since the two terms would then add. This is the spin-statistics connection for scalar fields. Also notice that if the \( \omega(p) \) were absent anticommutation relations \( would \) be local.
From its definition the propagator should have the property that only positive frequency components should be present as \( t_k \to +\infty \) and negative frequency components as \( t_k \to -\infty \). This property is assured in (5.89) by the \(-i\epsilon\) in the denominator. The propagator is a Green function for the Klein-Gordon differential \(-\partial^2 + m^2:\)

\[
(-\partial^2 + m^2)\Delta_F(x, y) = -i\delta(x - y).
\] (5.90)

The \( i\epsilon \) prescription tells us which boundary conditions to impose.

This prescription also follows if we define \( \Delta_F \) by analytically continuing the Euclidean space Green function to Minkowski space. To continue from Minkowski space to Euclidean space, one rotates the \( p^0 \) integration contour to the imaginary axis in the counterclockwise direction (to avoid the poles at \( \pm (\omega - i\epsilon) \)). In order to preserve convergence at infinity, \( x^0 \) must be simultaneously rotated in the opposite (clockwise) direction. Changing variables \( p^0 = -ip^4 \) and calling \( x^0 = -ix^4 \) (positive \( p^0 \) rotates to negative \( p^4 \) but positive \( x^0 \) rotates to positive \( x^4 \)) gives the Euclidean Green function

\[
\Delta_F \to \Delta_E \equiv \int \frac{d^4p}{(2\pi)^4} e^{i(x-p+x^4p^4)} \frac{1}{p^2 + (p^4)^2 + m^2}.
\] (5.91)

Clearly \( \Delta_E \) satisfies

\[
(-\partial_4^2 - \nabla^2 + m^2)\Delta_E(x - y) = \delta^4(x - y).
\] (5.92)

Next we turn to the evaluation of the propagator for the Dirac field. Remembering that \( \psi \) is fermionic we have

\[
S_F(x_1 - x_2)_{ab} \equiv \langle 0 | T[\psi_a(x_1)\bar{\psi}_b(x_2)] | 0 \rangle
= \theta(t_1 - t_2)\langle 0 | \psi_a(x_1)\bar{\psi}_b(x_2) | 0 \rangle - \theta(t_2 - t_1)\langle 0 | \bar{\psi}_b(x_2)\psi_a(x_1) | 0 \rangle
= \int \frac{d^4p}{(2\pi)^32\omega(p)} \left[ \theta(t_1 - t_2) e^{ip(x_1-x_2)} \sum_\lambda u^a_\lambda(p)u^b_\lambda(p) - \theta(t_2 - t_1) e^{ip(x_2-x_1)} \sum_\lambda v^a_\lambda(p)v^b_\lambda(p) \right],
\] (5.93)
where in the integrand we have \( p^0 = \omega(p) \).

To simplify the expression for \( S_F \) we need to evaluate \( \sum_\lambda u^a_\lambda(p)\bar{u}^b_\lambda(p) \) and \( \sum_\lambda v^a_\lambda(p)\bar{v}^b_\lambda(p) \).

If we regard them as matrices with indices \( a, b \), we know we can write each as a linear combination of the 16 matrices \( \Gamma \) we used in constructing the bilinears. By virtue of the sum over helicity, conjugation by the Lorentz transformation matrices simply does the corresponding Lorentz transformation on \( p^\mu = (p, \omega) \). Thus they must be scalars formed from \( p \) and the matrices \( \Gamma \). The only possibilities are \( I \) and \( p \cdot \gamma \), so

\[
\sum_\lambda u^a_\lambda(p)\bar{u}^b_\lambda(p) = A\delta_{ab} + Bp \cdot \gamma_{ab}, \tag{5.94}\]

and we only need to determine \( A, B \). First notice that multiplying by the matrix \( m + p \cdot \gamma \) must give 0, which determines \( A = -mB \). Then, multiplying both sides by \( \gamma^0 \) and taking the trace determines \( 2\omega \sum_\lambda B = -4\omega \) or \( B = -1 \). Thus

\[
\sum_\lambda u^a_\lambda(p)\bar{u}^b_\lambda(p) = (m - p \cdot \gamma)_{ab}. \tag{5.95}\]

The definition \( v = i\gamma^2 u^* \) then determines

\[
\sum_\lambda u^a_\lambda(p)\bar{v}^b_\lambda(p) = [i\gamma^2(m - p \cdot \gamma^\gamma)(-i\gamma^2)]_{ab} = -(m + p \cdot \gamma)_{ab}. \tag{5.96}\]

Inserting these relations into (5.93) yields

\[
S_F(x_1 - x_2)_{ab} = \int \frac{d^4p}{(2\pi)^42\omega(p)}\left[\theta(t_1 - t_2)e^{ip(x_1 - x_2)}(m - p \cdot \gamma)_{ab} + \theta(t_2 - t_1)e^{ip(x_2 - x_1)}(m + p \cdot \gamma)_{ab}\right]. \tag{5.97, 5.98}\]

The final step is to employ the integral representation for the step functions as we did for the scalar propagator. This process results in a four dimensional momentum integral with \( p^0 \) substituted for \( \omega \) in the exponents but not in front of \( \gamma^0 \). Then the two terms involving \( m \) and those involving the spatial \( \gamma^k \) combine, after the change of variable \( p \to -p \) in the second term, exactly as in the scalar case. The two terms involving \( \gamma^0 \) have a factor of \( \omega \) which cancels that in the denominator, but then they combine with the opposite relative sign to produce a \( 2p^0 \) in the numerator. Thus the net result is simply

\[
S_F(x_1 - x_2)_{ab} = -i\int \frac{d^4p}{(2\pi)^4} e^{ip(x_1 - x_2)} \left( \frac{m - p \cdot \gamma}{m^2 + p^2 - i\epsilon} \right)_{ab}. \tag{5.99}\]

Just as with the scalar propagator \( S_F \) may be recognized as a Green function for the differential Dirac operator \( \frac{1}{i}\gamma \cdot \partial + m \):

\[
\left( \frac{1}{i} \gamma \cdot \partial + m \right) S_F(x - y) = -i\delta^4(x - y), \tag{5.100}\]

with boundary condition dictated by the \( i\epsilon \) prescription. As before this boundary condition can be enforced by defining \( S_F \) as the continuation of the Euclidean Green function \( S_E \).
continuation from \(S_F\) to \(S_E\) proceeds by rotating the \(p^0\) integration contour to the imaginary axis in the counterclockwise direction (of course rotating \(x^0\) in the opposite direction), changing variables \(p^0 = -ip^4\), and defining \(x^0 = -ix^4\), \(\gamma^0 = -i\gamma_E^4\):

\[
S_F(x - y) \rightarrow S_E(x - y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x_1 - x_2)} \left( \frac{m - p \cdot \gamma_E}{m^2 + p^2} \right)_{ab} .
\] (5.101)

\(S_E\) satisfies the equation

\[
\left( \frac{1}{i} \gamma \cdot \partial + m \right) S_E(x - y) = \delta^4(x - y).
\] (5.102)

5.9 Vacuum expectations from large time limits of general transition amplitudes.

One obstacle to formulating an efficient perturbation theory for systems with interacting quantum fields is that one can’t “turn off” the interactions at early and late times as is possible with externally applied fields. Thus \(\text{out}\) and \(\text{in}\) states are eigenstates of complicated interacting Hamiltonians. There are two approaches to this difficulty.

5.9.1 Adiabatic switching

The simplest is to temporarily make the coupling constants time dependent such that they vanish at early and late times, adiabatically if at all possible, and let them be constant for all times \(-T < t < T\). We take \(T\) large enough so that all of the times in the time ordered product are later than \(-T\) and earlier than \(T\). Then the coupling constants are a kind of external field, for which the passage to interaction picture proceeds as we have discussed, with \(H_0\) just the free field Hamiltonian.

Of course, such a procedure introduces unwanted dependence on the switching procedure. To isolate and remove this dependence, recall that the adiabatic theorem assures us that \(U(-T, -\infty)|\text{in}\rangle\) is the ground state of \(H_S(-T)\). As for \(\langle \text{in} | U(\infty, T) | E_r \rangle\) we can say that the adiabatic evolution any eigenstate of \(H_S(T), U(\infty, T)|E_r\) will be an eigenstate of \(H_S(+\infty)\), and further since level crossings are forbidden in adiabatic evolution, \(\langle \text{in} | U(\infty, T) | E_r \rangle\) will be zero unless \(r = 0\) the ground state.

Next we convert the \(\text{outin}\) matrix element to Schrödinger picture:

\[
\langle \text{out} | T[\Omega(t_1) \cdots \Omega(t_N)] | \text{in} \rangle = \langle \text{in} | \left( U(\infty, -\infty)U(t_1, -\infty)\cdots\Omega S U(t_N, -\infty) \right) | \text{in} \rangle
\]

\[
= \langle \text{in} | U(\infty, t_1)\Omega_S U(t_1, t_2) \cdots \Omega S U(t_N, -\infty) | \text{in} \rangle
\]

\[
= \langle \text{in} | U(\infty, T) | 0, T \rangle \langle 0, T | U(t_1, t_1)\Omega S U(t_1, t_2) \cdots \Omega(t_N)U(t_N, -T) | 0, -T \rangle
\]

\[
\langle 0, -T | U(-T, -\infty) | \text{in} \rangle
\] (5.103)
In the last step we used closure to write $U(\infty, t_1) = U(\infty, T)U(T, t_1)$ and $U(t_N, -\infty) = U(t_N, -T)U(-T, -\infty)$ inserted a complete set of eigenstates of $H_S(T)$ and of $H_S(-T)$ respectively. Then the adiabatic theorem dictates that only the ground state of each sum is kept, $U(T, t_1)$.

The switching dependence of the matrix element is entirely contained in the first and last factors. But the same factors appear in

$$\langle \text{out}|\Omega(t_1)\cdots\Omega(t_N)|\text{in}\rangle = \langle 0, T|U(T, t_1)\Omega_S^1U(t_1, t_2)\Omega_S^2\cdots\Omega_S^NU(t_N, -T)|0, -T\rangle e^{2iEGT}$$

In this line we return to Heisenberg picture but this time coinciding with Schrödinger picture at $t = -T$ when the Hamiltonian has the final set of coupling constants. The left side of the equation can be expressed in terms of the interaction picture with all external fields and coupling constants vanishing at $t = \pm\infty$. In this picture all the fields are free.

$$\langle \text{out}|\Omega(t_1)\cdots\Omega(t_N)|\text{in}\rangle = \langle 0|T[U(\infty, -\infty)\Omega_1(t_1)\cdots\Omega_I(t_N)]|0T\rangle = \langle 0\rangle_{Ext=0}$$

Here the ground eigenvalue of $H_{0,I}$ is taken to be zero.

### 5.9.2 Long time suppression of excited states. (Optional)

A more general approach, which we shall favor, is to relax the requirement that the initial and final states be eigenstates of the Hamiltonian with vanishing external fields. Then one calculates in first instance a quantity that is not of immediate interest, but which can be simply related to such quantities.

A quantity of more or less direct physical interest is the vacuum expectation value of the time ordered product of several quantum fields. More generally the $\text{out}\text{in}$ matrix element of such a time ordered product is relevant if time varying external fields are present. So let us consider how to obtain this quantity in perturbation theory by first calculating with general initial and final states. Using the evolution operator and assuming $t_1 > t_2 > \cdots > t_n$, we therefore consider

$$\langle f|U(\infty, -\infty)T[\Omega_1(t_1)\cdots\Omega_S(t_n)]|i\rangle = \langle f|U(\infty, t_1)\Omega_S^1U(t_1, t_2)\cdots U(t_n-1, t_n)\Omega_S^nu(t_n, -\infty)|i\rangle.$$  

Choose the time $T$ so that all external fields vanish for times earlier than $-T$ and later than $T$. Then

$$U(t_n, -\infty)|i\rangle = U(t_n, -T)e^{-i(\infty-T)H_S}|i\rangle$$

$$= U(t_n, -T)e^{-i(\infty-T)EG} \sum_r e^{-i(\infty-T)(E_r-E_G)}\langle r|\langle r|i\rangle.$$
We would now like to argue that the infinite oscillations wash out all contributions but the (assumed nondegenerate\(^4\)) ground state. In a field theory this is quite plausible since the excited states correspond to particles so the sum over \(r\) is really an integral over a range of continuous energies. But even without this smearing, we can make the washing out rigorous by calculating with imaginary time: \(it = \beta > 0\). Then \(i\infty\) is really \(+\infty\) and all excited states are damped exponentially. Massless particle states could introduce a subtlety here, but the part of phase space that is not exponentially damped is infinitesimal: this has the effect of changing exponential damping to a power law damping. If we buy this argument, then we can assert quite generally that \(U(t_n, -\infty)|i\rangle = U(t_n, -\infty)|0\rangle\langle 0|i\rangle\) and similarly \(\langle f|U(\infty, t_1) = \langle f|0\rangle\langle 0|U(\infty, t_1)\).

Since we take (as usual) the Heisenberg and Schrödinger pictures to coincide at \(t = -\infty\), then \(|in\rangle = |0\rangle\) and \(|out\rangle = \langle 0|U(\infty, -\infty)\). Thus we have obtained the relation
\[
\langle f|U(\infty, -\infty)T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|i\rangle = \\
\langle f|0\rangle\langle 0|i\rangle \langle out|T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|in\rangle.
\]
(5.109)
(5.110)

In other words calculating with any initial and final states that have finite overlap\(^5\) with the true ground state gives us a constant times the desired matrix element. We can easily evaluate the multiplicative constant by considering by the same reasoning
\[
\langle f|U(\infty, -\infty)|i\rangle = \langle f|0\rangle\langle 0|i\rangle \langle out|in\rangle \\
\to e^{-2i\infty E_G} \langle f|0\rangle\langle 0|i\rangle, \quad \text{External Fields = 0.}
\]
(5.111)
(5.112)

Putting this into our relation we obtain
\[
\langle out|T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|in\rangle = \\
e^{-2i\infty E_G} \langle f|U(\infty, -\infty)T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|i\rangle / \langle f|U(\infty, -\infty)_{\text{Ext}=0}|i\rangle,
\]
(5.113)
(5.114)

where the subscript on \(U\) in the denominator denotes vanishing external fields. In field theory applications \(E_G\) is the energy of the vacuum, which is zero if we measure all energies relative to that of the vacuum. In the absence of gravity all physical quantities depend only on energy differences, so we lose nothing by doing this. Gravity couples directly to the energy density and therefore is sensitive to the energy as opposed to energy differences, but then \(E_G\) only appears in the combination \(\Lambda \equiv E_G + \Lambda_0\), with \(\Lambda_0\) the “bare” cosmological constant. Replacing \(\Lambda_0\) by \(\Lambda\) in effect sets \(E_G = 0\).

\(^4\)There are interesting cases of degenerate vacua, when there is “spontaneous symmetry breakdown.” In such cases the choice of initial and final states determines which of the degenerate vacua is picked out.

\(^5\)The infinite number of degrees of freedom in quantum field theory requires care here: the overlap between different states in a theory with \(n\) degrees of freedom is typically \(f^n\) with \(f < 1\). Since \(n = \infty\), we should expect \(\langle f|0\rangle\langle 0|i\rangle \sim e^{-\infty}\). In field theory \(n = \infty\) because the volume of space is infinite and because space is continuous. Thus strict application of the above relation should be done in the presence of both an infrared and ultraviolet cutoff, which can then be removed after extracting the desired amplitude.
The formula (5.114) is a convenient starting point for developing perturbation theory. Any breakup

\[ H_S(t) = H_0(t) + H'(t) \]  

(5.115)
determines an interaction picture defined by

\[ \Omega_I(t) = U_0^{-1}(t, -\infty)\Omega_S U_0(t, -\infty) = U_I(t, -\infty)\Omega(t)U_I^{-1}(t, -\infty), \]  

(5.116)
where

\[ i\dot{U} = H_S(t)U = UH(t) \]  

(5.117)
\[ i\dot{U}_0 = U_0H_0(t) \]  

(5.118)
\[ i\dot{U}_I = H_I(t)U_I \]  

(5.119)
and all \( U \)'s are the identity at \( t = -\infty \). Then the evolution operator satisfies

\[ U(t_1, t_2) = U(t_1, -\infty)U^{-1}(t_2, -\infty) = U_0(t_1, -\infty)U_I(t_1, t_2)U_0^{-1}(t_2, -\infty). \]  

(5.120)
Plugging these relations into (5.114) then gives

\[ \langle \text{out}|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|\text{in}\rangle = e^{-2i\omega E_0}\frac{\langle f|U_0(\infty, -\infty)T[U_I(\infty, -\infty)\Omega_1(t_1)\cdots\Omega_n(t_n)]|i\rangle}{\langle f|U_0(\infty, -\infty)\Omega(\infty, -\infty)\Gamma_{\text{Ext}}=0|\dot{i}\rangle}. \]  

(5.121)
This formula is completely general: we have even allowed \( H_0 \) to contain time varying external fields, which is hardly ever done in practice. Since all operators in this formula are in interaction picture, it is most convenient to choose \(|i\rangle, |f\rangle\) to have simple properties with respect to \( H_{0I}(\infty, -\infty) \). Let us call the ground state of this operator \(|\text{in}, 0\rangle\). Then \(|\text{in}, 0\rangle U_0(\infty, -\infty)\) is the ground state of \( H_{0I}(\infty) \) and therefore deserves the name \(|\text{out}, 0\rangle\). When all external fields vanish, \( H_{0I} \) is time independent and we call its ground state \(|0, I\rangle \equiv |\text{in}, 0\rangle\) and its ground state energy \( E_0 \). Then \(|\text{in}, 0\rangle U_0(\infty, -\infty)\Gamma_{\text{Ext}=0} = e^{-2i\omega E_0}|0, I\rangle\). Thus choosing \(|i\rangle = |f\rangle = |0, I\rangle = |\text{in}, 0\rangle\) we obtain the useful formula

\[ \langle \text{out}|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|\text{in}\rangle = e^{-2i\omega (E_G-E_0)}\frac{\langle \text{out}, 0|T[U_I(\infty, -\infty)\Omega_1(t_1)\cdots\Omega_n(t_n)]|\text{in}, 0\rangle}{\langle 0, I|U_I(\infty, -\infty)\Gamma_{\text{Ext}=0}|0, I\rangle}. \]  

(5.122)
In the usual case where we do not include external fields in \( H_0 \), the formula simplifies further

\[ \langle \text{out}|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|\text{in}\rangle = e^{-2i\omega E_G}\langle G|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|G\rangle \]  

(5.124)

\[ e^{-2i\omega E_G}\frac{\langle 0, I|T[U_I(\infty, -\infty)\Omega_1(t_1)\cdots\Omega_n(t_n)]|0, I\rangle}{\langle 0, I|U_I(\infty, -\infty)\Gamma_{\text{Ext}=0}|0, I\rangle}. \]  

(5.125)
Using the Wick expansion one can describe the perturbation series for the numerators and denominators of these formulas using Feynman diagrams. The diagrams contributing to the denominator are all those completely disconnected from either external fields or from the points assigned to the operators in the numerator. The numerator contains this same sum of diagrams as a multiplicative factor. Thus the division by the denominator is achieved by simply deleting all such disconnected “vacuum” diagrams from the expansion of the numerator.
5.10 Tree diagrams in momentum space

At any vertex the \( \int d^4x \) delivers a 4 momentum conserving delta function. For example

\[
\int d^4x \Delta_F(x_1 - x) \cdots \Delta_F(x_N - x) = \int \frac{d^4q_1}{(2\pi)^4} \cdots \frac{d^4q_N}{(2\pi)^4}
\]

\[
(2\pi)^4 \delta \left( \sum q_k \right) e^{i \sum q_k \cdot x_k} \frac{-i}{q^2_k + m^2} \cdots \frac{-i}{q^2_N + m^2}
\]

(5.126)

In momentum space external propagators have fixed momentum, but internal momenta are potentially integrated. Connected tree diagrams have the property that the number of vertices is 1 more than the number of internal propagators. Thus all internal momenta are determined by the external momenta. Diagrams with \( L \) loops have \( L \) undetermined momenta which are each integrated with measure \( d^4p/(2\pi)^4 \). It is these momentum integrals that produce the notorious divergences of QFT.

5.10.1 Scattering amplitudes from time ordered products

Instead of the \( S \)-matrix for two to two scattering suppose we calculate the Fourier transform of \( \langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle \). Then in \( g\phi^3 \) theory one lowest order tree diagram is

\[
A = (ig)^2 \int d^4xd^4y \Delta_F(x_1 - x) \Delta_F(x_2 - x) \Delta_F(x_3 - y) \Delta_F(x_4 - y)
\]

Then

\[
\int \prod_k \left( d^4x_k e^{iq_k \cdot x_k} \right) A
\]

\[
= (ig)^2 \prod_k \left( - \frac{i}{q^2_k + m^2} \right) \int d^4xd^4y e^{i(q_1 + q_2) \cdot x + i(q_3 + q_4) \cdot y} \Delta_F(x - y)
\]

\[
= (ig)^2 \prod_k \left( - \frac{i}{q^2_k + m^2} \right) \int d^4xe^{i(q_1 + q_2 - q) \cdot x} \Delta_F(x)(2\pi)^4 \delta \left( \sum q_k \right)
\]

(5.128)

We see that we have for each external propagator a factor \(-i/(q_k^2 + m^2)\) which contains poles at \( q_k^0 = \pm \sqrt{q_k^2 + m^2} \). The residue of either of these poles is proportional to the desired scattering amplitude. The F.T. of a suitable time ordered product thus contains information about scattering. The prescription is to “amputate” the external propagators and send \( q_k^0 \rightarrow \pm \omega(q_k) \). The + choice gives an incoming particle with momentum \( q_k \) and the − choice to an outgoing particle of momentum \( -q_k \).

The analogous procedure for Dirac particles leads to an external propagator \( S_F(x - y) \) where the external coordinate could be \( x \) or \( y \). If it is \( x \) the F.T. would be

\[
\int d^4xe^{iq \cdot x} S_F(x - y) = \frac{-i(m + q \cdot \gamma)}{m^2 + q^2} e^{iq \cdot y}
\]

(5.129)
If $q^0 > 0$ the numerator on shell goes to $-i(m + q \cdot \gamma) = +i \sum_\lambda v_\lambda(q) \bar{v}_\lambda(q)$ and corresponds to an incoming antiparticle of momentum $q$. Note the extra $-$ in this case. If $q^0 < 0$ it is outgoing with momentum $p = -q$ so the numerator is $-i(m - p \cdot \gamma) = -i \sum U \bar{u}$, showing it is an outgoing particle.

On the other hand if $y$ is the external coordinate the F.T. would be

$$\int d^4 y e^{i\mathbf{q} \cdot \mathbf{y}} S_F(x - y) = \frac{-i(m - q \cdot \gamma)}{m^2 + q^2} e^{i\mathbf{q} \cdot x}$$

(5.130)

If $q^0 > 0$ the numerator on shell goes to $-i(m - p \cdot \gamma)$ and corresponds to an incoming particle with momentum $q$. If $q^0 < 0$ one sees that it is an outgoing antiparticle with momentum $p = -q$ since the numerator is $i \sum v \bar{v}$. 
Chapter 6

Cross Sections and Rates and Spin Sums

6.1 Cross section for $2 \rightarrow N$

The scattering matrix is written as

$$\langle \text{out, } f | \text{in, } i \rangle = \delta_{fi} - i(2\pi)^4 \delta(P_f - p_1 - p_2) \mathcal{T}$$  \hspace{1cm} (6.1)

And we define the Feynman amplitude $\mathcal{M}$ by

$$\mathcal{T} \equiv \frac{\mathcal{M}}{(2\pi)^3 \sqrt{4\omega_1 \omega_2} \prod_f ((2\pi)^{3/2} \sqrt{2\omega_f})}$$  \hspace{1cm} (6.2)

For a scattering process the initial state is two separated wave packets aimed at each other so they will meet and scatter. This situation can be arranged by a momentum space wave function $f(\vec{p}_1, \vec{p}_2)$, narrowly peaked about $\vec{p}_i = \hat{z} p_i$. For the lab frame $p_2 = 0$, and for the CM frame $p_2 = -p_1$. Then

$$\text{Amplitude} = \int d^3 p_1 d^3 p_2 (2\pi)^4 \delta(P_f - p_1 - p_2) \mathcal{T} f(\vec{p}_1, \vec{p}_2)$$

$$\approx (2\pi)^4 \mathcal{T} \int d^3 p_1 \delta(E_f - \omega_1 - \omega_2) f(\vec{p}_1, \vec{P}_f - \vec{p}_1)$$  \hspace{1cm} (6.3)

To deal with the last delta function write $d^3 p_1 = d^2 p_1^z dp_1^\perp = d^2 p_1^z d(\omega_1 + \omega_2)/v_{12}$ where $v_{12} = |\partial \omega_1 / \partial p_1^z + \partial \omega_2 / \partial p_1^z| = |v_1 - v_2|$ is the relative velocity of the two initial packets. Then

$$\text{Amplitude} \approx \frac{(2\pi)^4}{v_{12}} \mathcal{T} \int d^2 p_1 f(\vec{p}_1, \vec{P}_f - \vec{p}_1)$$  \hspace{1cm} (6.4)

This function is very sharply peaked about $P_f = p_1^0 + p_2^0$ with width given by the initial packet. Its square will also be sharply peaked, and safely approximated by a delta function:

$$|\text{Amp}|^2 \approx \frac{(2\pi)^8}{v_{12}^2} |\mathcal{T}|^2 \delta(P_f - p_1^0 - p_2^0) \int d^4 P \left| \int d^2 p_1 f(\vec{p}_1, \vec{P} - \vec{p}_1) \right|^2 \hspace{1cm} (6.5)$$
Now consider

$$\psi(x, y, p_z, \vec{P}) = \int \frac{d^2 p}{2\pi} e^{ixp^x + iyp^y} f(p_1, \vec{P} - \vec{p}_1)$$

(6.6)

It’s square is the probability per $dp_zd^3P$ that the relative transverse displacement of the incident particles is in $dxdy$ of the point $(x, y)$. This quantity appears in our formula for the case $x = y = 0$ and integrated over $d^3PdE_f \approx \frac{v_{12}d^3P}{2\pi}$. \[\int d^4P \left| \int d^2p f(p_1, \vec{P} - \vec{p}_1) \right|^2 \approx (2\pi)^2v_{12} \times \text{the total probability the incident particles are within } dxdy \text{ of } x = y = 0.\]

The coefficient of this probability is just what we mean by the differential cross section

$$d\sigma = \prod \frac{d^3p_f}{(2\pi)^3/2} 1/(2\pi)^{3/2}/\sqrt{2\phi}, \quad \frac{1}{\omega_1\omega_2v_{12}}(2\pi)^4\delta(P_f - p_1 - p_2)|\mathcal{M}|^2$$

(6.7)

The last form is the one that is most convenient to remember. In it $\mathcal{M}$ has all external wave function factors $1/(2\pi)^{3/2}/\sqrt{2\phi}$ removed. It is the Fourier transform of the appropriate time ordered product, with the propagator on each external line amputated and replaced with the appropriate $\sqrt{Z}$ factor.

### 6.2 Decay rate of a single metastable particle

An apparently simpler process than scattering is the decay of a single particle into any final state. In this case we write the decay probability amplitude as

$$2\pi \delta(E_f - \omega_1) \equiv 2\pi(2\pi)^4\delta(P_f - p_1)\frac{\mathcal{M}}{(2\pi)^{3/2}/\sqrt{2\phi_1} \prod |(2\pi)^{3/2}/\sqrt{2\phi_f}|}$$

(6.8)

However in this case putting the initial particle in a wave packet $f(p)$ only takes care of the spatial delta function, and after this we still can’t meaningfully square the amplitude to get a probability. The problem is that the formula assumes that the initial particle has existed for an infinite time, which is not consistent with the fact that it can decay. It has a finite lifetimes $\tau$, and it can only approximately be described as a particle with definite mass for a time $2T \ll \tau$. The correct way to deal with unstable systems involves including their production as well as decay. To deal with the application of the above formula, we can replace the energy conserving delta function with a finite time version

$$2\pi \delta(E_f - \omega_1) \rightarrow \int_{-T}^T dt e^{-i(E_f - \omega_1)t} \frac{2\sin(E_f - \omega_1)T}{E_f - \omega_1}$$

(6.9)

where we understand that $2T \ll \tau$. Application of the formula implicitly assumes that $T \gg 1/\Delta E$ where $\Delta E$ characterizes the scale over which $\mathcal{T}$ varies. Thus we should only try
to apply this formula for the decay of a particle with lifetime $\tau \gg 1/\Delta E$. In this situation we can approximate the square of the finite time delta function with a sharp delta function:

$$\left| \frac{2 \sin(E_f - \omega_1) T}{E_f - \omega_1} \right|^2 \approx \delta(E_f - \omega_1) 4 \int dE \frac{\sin^2 \theta E T}{E^2} = 2\pi \delta(E_f - \omega_1)(2T)$$  \hspace{1cm} (6.10)

With this interpretation we arrive at the following formula for the differential decay probability

$$dP = \prod_f d^3p_f (2\pi)^7 |T|^2 \delta(E_f - \omega_1)(2T) |M|^2$$

$$\approx \prod_f d^3p_f (2\pi)^7 |T|^2 \delta(P_f - p_1)(2T) \int d^3p |f(p)|^2$$

$$\approx \prod_f d^3p_f (2\pi)^7 |T|^2 \delta(P_f - p_1)(2T)$$

$$\approx 2T \prod_f \frac{d^3p_f}{(2\pi)^3 2\omega_f} \frac{1}{2\omega_1} (2\pi)^4 \delta(P_f - p_1) |M|^2$$ \hspace{1cm} (6.11)

We see that in this approximate description the decay probability is proportional to the duration of the process $2T$. This linear dependence is clearly only meaningful for $2T \ll \tau$. However the probability per unit time, the decay rate, is independent of $T$:

$$d\Gamma = \frac{dP}{2T} = \prod_f \frac{d^3p_f}{(2\pi)^3 2\omega_f} \frac{1}{2\omega_1} (2\pi)^4 \delta(P_f - p_1) |M|^2.$$  \hspace{1cm} (6.12)

Summing over all allowed final states, including integrating over phase space gives the total decay rate $\Gamma$. We can use it to determine the long time behavior of the decay process. Let $P(t)$ be the probability that the particle has not decayed in a time $t$ after its creation $P(0) = 1$. Then

$$\frac{dP}{dt} = -\Gamma P(t), \quad \rightarrow \quad P(t) = e^{-\Gamma t}$$ \hspace{1cm} (6.13)

This provides a precise definition for the lifetime of the particle, namely, $\tau = 1/\Gamma$. Also the probability that the particle has decayed in time $t$ is $1 - P(t) = 1 - e^{-\Gamma t} = 1 - e^{-t/\tau} \approx t/\tau = t\Gamma$, when $t \ll \tau$, showing the consistency of our rate formula.

As a very useful example, we specialize these formulas to the case of two particle final states. Four of the six phase space integrals can be evaluated using the delta functions:

$$d^3p_1^f d^3p_2^f \delta(p_1^f + p_2^f - P_f) = d^3p_1^f \delta(\omega_1^f(p_1^f) + \omega_2^f (P_f - p_1^f) - E_i)$$

$$= d\Omega \frac{p_1^2 \omega_1^f \omega_2^f}{p_i^f E_i - P_i \omega_1^f \cos \theta}$$ \hspace{1cm} (6.14)
then we have

\[
\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2} \frac{p_1^f}{\omega_1 \omega_2 v_{12} p_1^f E_i - P_i \omega_1^f \cos \theta} \quad (6.15)
\]

\[
\frac{d\Gamma}{d\Omega} = \frac{|M|^2}{32\pi^2} \frac{p_1^f}{\omega_1 - p_1 \omega_1^f \cos \theta} \quad (6.16)
\]

These formulas dramatically simplify in the center of mass system when \( \vec{P}_i = 0 \). Then \( \omega_1 \omega_2 v_{12} = p_1 (\omega_1 + \omega_2) \) and

\[
\frac{d\sigma}{d\Omega}_{\text{CM}} = \frac{p_1^f}{p_1} \frac{|M|^2}{64\pi^2 (\omega_1 + \omega_2)^2} \quad (6.17)
\]

\[
\frac{d\Gamma}{d\Omega}_{\text{CM}} = \frac{|M|^2}{32\pi^2 m_1^2} p_1^f \quad (6.18)
\]

We finally note that when calculating total rates and cross sections with identical particles in the final state, the result of integrating over phase space must be divided by \( \prod_c n_c! \) when there are \( n_c \) identical particles of type \( c \).

### 6.3 Spin Sums

Cross section and rate formulas simplify when we either choose not to measure final particle spins or use an unpolarized beam. To accomplish this for spin 1/2 particles the following projectors onto definite spin states are useful

\[
u\bar{u} = \frac{1}{2} (m - \gamma \cdot p)(1 - \gamma_5 \gamma \cdot s) \quad (6.19)
\]

\[
v\bar{v} = \frac{1}{2} (-m - \gamma \cdot p)(1 - \gamma_5 \gamma \cdot s) \quad (6.20)
\]

where the spin state information is carried by the four vector \( s^\mu \), given by either

\[
s^0 = \frac{p \cdot \hat{s}}{m} \quad (6.21)
\]

\[
s = \hat{s} + \frac{p \cdot \hat{s}}{m(m + \omega)} \quad (6.22)
\]

and \( \hat{s} \) is a unit vector in the direction of the polarization in the rest frame, i.e.

\[
\hat{s} \cdot \sigma u_{\bar{s}}(0) = u_{\bar{s}}(0),
\]

or, if spins are labelled by helicity,

\[
s^0 = 2h \frac{|p|}{m} \quad (6.23)
\]

\[
s = 2h \frac{\omega p}{m|p|}. \quad (6.24)
\]
Note: in the case of zero mass the projectors have a smooth limit only for the helicity basis:

\[ u_h \bar{u}_h = -\frac{1}{2} (1 + 2 h \gamma_5) \gamma \cdot p \]  
(6.25)

\[ v_h \bar{v}_h = -\frac{1}{2} (1 - 2 h \gamma_5) \gamma \cdot p. \]  
(6.26)

Then summation over spin states leads to

\[ \sum_\lambda u_\lambda \bar{u}_\lambda = m - \gamma \cdot p, \quad \sum_\lambda v_\lambda \bar{v}_\lambda = -m - \gamma \cdot p \]  
(6.27)

For example,

\[ \sum_\lambda' \lambda, |\bar{u}_\lambda'(p') \Gamma u_\lambda(p)|^2 = Tr \left[(m - \gamma \cdot p) \gamma^0 \Gamma^\dagger \gamma^0 (m - \gamma \cdot p') \Gamma \right] \]  
(6.28)

For each \( \Gamma = (I, i \gamma_5, \gamma^\mu, \gamma_5 \gamma^\mu, \sigma^{\mu\nu}) \), \( \gamma^0 \Gamma^\dagger \gamma^0 = \Gamma \),

### 6.3.1 Gamma matrix identities

In evaluating traces of products of gamma matrices the following identities are extremely useful:

\[ Tr[\text{odd number of } \gamma' \text{'s}] = 0 \]  
(6.29)

\[ Tr[\gamma^\mu \gamma^\nu] = -4 \eta^{\mu\nu} \]  
(6.30)

\[ Tr[\gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu] = 4 [\eta^{\kappa\lambda} \eta^{\mu\nu} - \eta^{\kappa\mu} \eta^{\lambda\nu} + \eta^{\kappa\nu} \eta^{\lambda\mu}] \]  
(6.31)

\[ Tr[\gamma^\kappa \gamma^\lambda \gamma^\rho \gamma^\mu \gamma_5] = -4 i \epsilon^{\kappa\lambda\rho\mu} \]  
(6.32)

where \( \epsilon^{\kappa\lambda\rho\mu} \) is completely antisymmetric and \( \epsilon^{0123} = +1 \).

\[ \gamma^\mu \gamma^\nu \gamma^\rho = -i \epsilon^{\mu\nu\rho\sigma} \gamma_\sigma \gamma_5 - \eta^{\mu\nu} \gamma^\rho - \eta^{\mu\rho} \gamma^\nu - \eta^{\nu\rho} \gamma^\mu \]  
(6.33)

\[ \gamma_\mu \gamma^\lambda \gamma^\nu = 2 \gamma^\lambda \]  
(6.34)

\[ \gamma_\mu \gamma^\kappa \gamma^\lambda \gamma^\nu = 4 \eta^{\kappa\lambda} \]  
(6.35)

\[ \gamma_\mu \gamma^\kappa \gamma^\lambda \gamma^\rho \gamma^\nu = 2 \gamma^\rho \gamma^\lambda \gamma^\kappa \]  
(6.36)

\[ \epsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma} = -2 i \gamma_5 \sigma^{\mu\nu} \]  
(6.37)

For example, we can evaluate

\[ \mathrm{Tr}(m - \gamma \cdot p) \gamma^\mu (m - \gamma \cdot p') \gamma^\nu = -4 m^2 \eta^{\mu\nu} + \mathrm{Tr} \gamma \cdot p \gamma^\mu \gamma \cdot p' \gamma^\nu \]

\[ = -4 m^2 \eta^{\mu\nu} + 4 (p^\mu p'^\nu - \eta^{\mu\nu} p \cdot p' + p' \cdot p^\mu) \]

\[ = -4 (m^2 + p \cdot p') \eta^{\mu\nu} + 4 (p^\mu p'^\nu + p' \cdot p^\mu) \]
Chapter 7

Quantum Fields in External Gauge Fields

7.1 Electromagnetic Fields

The coupling of a quantum field to an external electromagnetic field is dictated by the principle of gauge invariance. In classical electrodynamics, it is possible to avoid potentials and formulate all equations of motion in terms of the electric and magnetic fields $F_{\mu\nu}$. However, the potential $A_{\mu}(x)$ is indispensable to an economical description of the coupling of a quantum particle to electromagnetism. Fundamentally, this is because the Hamiltonian and Lagrangian play a much more central role in quantum dynamics than in classical dynamics, and the potential appears explicitly in the Hamiltonian and Lagrangian. (Recall that the Schrödinger equation involves the Hamiltonian explicitly.) The field strengths are related to the potential via

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \quad (7.1)$$

but it is clear that the potential is not given uniquely in terms of the field strength: If the potential is changed by a gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad (7.2)$$

the field strength $F_{\mu\nu}$ is unchanged. It is therefore important to introduce the potential into the Schrödinger equation in a way which preserves gauge invariance.

Let us first ask how gauge invariance is realized in classical particle electrodynamics. In order that the Euler-Lagrange equations reproduce the Lorentz force law, the scalar and vector potentials $(A^0, \mathbf{A})$ must enter the Lagrangian through the terms

$$-qA^0(x) + q\dot{x} \cdot \mathbf{A}(x). \quad (7.3)$$

Because of the term linear in velocity, the momentum conjugate to $\mathbf{x}$ becomes

$$\mathbf{p} = \mathbf{p}_{A=0} + q\mathbf{A}, \quad (7.4)$$
where $\mathbf{p}_{A=0}$ is the conjugate momentum with vanishing vector potential. Furthermore, when we form the Hamiltonian $H = \dot{x} \cdot \mathbf{p} - L$, the term linear in velocity cancels, so

$$H = H_0(\mathbf{p}_{A=0}, \mathbf{x}) + qA^0 = H_0(\mathbf{p} - qA, \mathbf{x}) + qA^0. \quad (7.5)$$

If we subject $A$ to a gauge transformation, the Lagrangian changes by the amount

$$q(\partial_0 \Lambda + \dot{x} \cdot \nabla \Lambda) = q \frac{d\Lambda}{dt}, \quad (7.6)$$

a total time derivative, so that the action $\int_{t_1}^{t_2} L$ changes by the amount $q\Lambda(\mathbf{x}(t_2), t_2) - q\Lambda(\mathbf{x}(t_1), t_1)$ and the added terms have no effect on the Euler-Lagrange equations. At first sight the Hamiltonian doesn’t look invariant, but notice that the transformation

$$\mathbf{p}' = \mathbf{p} - q \nabla \Lambda \quad \mathbf{x}' = \mathbf{x} \quad (7.7)$$

is a canonical transformation with generating function $W_2(\mathbf{x}, \mathbf{p}', t) = \mathbf{x} \cdot \mathbf{p}' + q\Lambda(\mathbf{x}, t)$. Furthermore if the generating function is time dependent the canonical transformation includes changing the Hamiltonian by $\partial_t W_2 = q\partial_t \Lambda$, so that after the canonical transformation the gauge transformed Hamiltonian is identical to the old one with the substitutions $\mathbf{x} \rightarrow \mathbf{x}'$, $\mathbf{p} \rightarrow \mathbf{p}'$.

The way $A_\mu$ enters the Schrödinger equation for a charged particle is now clear. The substitution $\mathbf{p} \rightarrow \mathbf{p} - qA$ corresponds in the Schrödinger equation to

$$\frac{1}{i} \nabla \rightarrow \frac{1}{i} \nabla - qA. \quad (7.8)$$

Furthermore the addition of $qA^0 = -qA_0$ to the Hamiltonian is prescribed by the substitution rule

$$\frac{1}{i} \frac{\partial}{\partial t} \rightarrow \frac{1}{i} \frac{\partial}{\partial t} - qA_0, \quad (7.9)$$

so both substitutions can be given the compact expression

$$\partial_\mu \rightarrow \partial_\mu - i qA_\mu \quad (7.10)$$

which is known as the minimal substitution rule. The gauge invariance of the Schrödinger equation is achieved by postulating in addition to (7.2) the change

$$\psi \rightarrow e^{iq\Lambda} \psi. \quad (7.11)$$

If we recall that in the semi-classical approximation $\psi \sim e^{iS}$, where $S$ is the classical action, we recognize that this change of the wave function under gauge transformation is the quantum analogue of the classical change in the action.

The gauge transformation makes an arbitrary, local, redefinition of the phase of the wave function. In fact, one could take the attitude that invariance under such local phase changes
is desirable from a physical point of view. (Since global phase changes are unobservable perhaps local ones should also be.) In that case one would be forced to introduce the electromagnetic field to realize the invariance! It is obvious that the Schrödinger equation is invariant under the combined changes (7.2) and (7.11). As a special case, the Dirac equation is invariant under the same gauge transformations. When we interpret the Dirac equation as a field equation, (7.11) is a transformation on fields as is (7.2), so the two are really on similar footing.

To sum up the above discussion, we display the Dirac equation in a potential $A_\mu$:

$$i\gamma \cdot (\partial - iqA)\psi = m\psi.$$  \hspace{1cm} (7.12)

Also the corresponding second quantized Hamiltonian is given by

$$H_A(t) = \int d^3 x \left( \psi^\dagger \left( \frac{i}{2} \mathbf{\alpha} \cdot \mathbf{\nabla} + \beta m \right) \psi - j^\mu A_\mu \right).$$ \hspace{1cm} (7.13)

Notice that a simple consequence of the Dirac equation is current conservation

$$\partial_\mu j^\mu = 0,$$ \hspace{1cm} (7.14)

even when $A \neq 0$. This can be understood as a consequence of the gauge invariance of quantum evolution. To see this we have to consider the unitary evolution operator $U(t, t_0)$ defined by

$$i\partial_t U(t, t_0) = H_S(t)U(t, t_0) \hspace{1cm} U(t_0, t_0) = I,$$ \hspace{1cm} (7.16)

where $H_S$ is the Schrödinger picture Hamiltonian. If we make a small change $\delta A_\mu$ in $A_\mu$, keeping the Schrödinger picture dynamical variables fixed, $U$ changes by the

\footnote{$U$ gives the unitary transformation between Heisenberg and Schrödinger pictures. Its analogue in classical mechanics is the generator $S(q, P, t)$ of the canonical transformation mapping the initial phase space variables $Q, P$ to those at time $t, q, p$. The analogue of the following equation for $U$ is the Hamilton-Jacobi equation for $S$,

$$\frac{\partial S}{\partial t} = -H(q, \frac{\partial S}{\partial q}, t).$$  \hspace{1cm} (7.15)}
\[ \delta U(t, t_0) = i \int_{t_0}^{t} dt' d^3x U(t, t') \delta A_\mu(x') j^\mu_S(x) U(t', t_0) \]  
(7.19)

\[ \delta U(t, t_0) = U(t, t_0) i \int_{t_0}^{t} dt' d^3x \delta A_\mu(x') j^\mu(x, t') \]  
(7.20)

with \( j^\mu(x, t') = U'^\dagger(t', t_0) j^\mu(x) U(t', t_0) \) the Heisenberg picture current operator\(^3\). An easy way to see this is to write \( U = T e^{-\int dt' H_S(t')} \). Alternatively, simply differentiate (7.20) with respect to time and show that it satisfies (5.3) to first order in \( \delta A \). Use is also made of the closure relation

\[ U(t, t') U(t', t_0) = U(t, t_0) \]  
(7.21)

which is a simple consequence of the differential equation satisfied by \( U \). Under a gauge transformation \( \delta A = \partial \Lambda \), so we see that

\[ \delta U = U i \int_{t_0}^{t} d^3x \partial_\mu \Lambda j^\mu(x). \]  
(7.22)

The requirement that \( U \) be invariant under gauge transformations that vanish at \( t_0, t \) is then that \( \partial_\mu j^\mu = 0 \).

### 7.2 Nonabelian Gauge Fields

The local phase transformation (7.11) on charged fields can be generalized (Yang-Mills). Suppose that \( \psi \) carries an internal index \( k \). Then in place of (7.11) we can consider

\[ \psi_k(x) \rightarrow \sum_l \Omega_{kl}(x) \psi_l(x) \]  
(7.23)

\(^2\)In the classical theory a change in the parameters of the Hamiltonian would give rise to a change in \( S \) satisfying

\[ \frac{\partial \delta S}{\partial t} = - \sum_k \frac{\partial \delta S}{\partial q_k} \frac{\partial H}{\partial p_k} - \delta H(q, \frac{\partial S}{\partial q}, t). \]  
(7.17)

Now if we put \( q = q(t) \) and \( \partial S/\partial q = p(t) \) the Hamilton-Jacobi equation just says that \( q(t), p(t) \) satisfy Hamilton’s equations so \( \partial H/\partial p = \dot{q} \) and we have

\[ \frac{d\delta S}{dt} = - \delta H(q(t), p(t), t) \]  
(7.18)

or \( \delta S = - \int_{t_0}^{t} dt' \delta H(q(t'), p(t'), t') \), the classical analogue of the following equation.

\(^3\)Equation (7.20) has a generalization to an arbitrary physical system: If one makes any small change \( \delta H_5 \) in the Schrödinger picture hamiltonian, with the Schrödinger picture operators fixed, the corresponding change in the evolution operator \( U(t, t_0) \) is given by \( \delta U = -i U \int_{t_0}^{t} dt' \delta H(t') \), where \( \delta H(t) \equiv U'^\dagger \delta H_5 U \) is the change in the Schrödinger hamiltonian transformed to the Heisenberg picture, which is the same as the change in the Heisenberg Hamiltonian \( H(t) \), keeping the Heisenberg dynamical variables fixed. The proof of this is exactly the same as that of Eq.(7.20.)
or, with suppressed indices
\[ \psi(x) \rightarrow \Omega(x)\psi(x) \]  
\[ (7.24) \]
with \( \Omega(x) \) in a unitary matrix representation of some continuous group \( G \). In this language (7.11) corresponds to the choice \( G = U(1) \), multiplication by a phase, an abelian group. If we require the dynamics to be invariant under the nonabelian local gauge transformations, we must introduce a nonabelian gauge field to absorb the noncovariance of
\[ \partial_\mu \psi \rightarrow \Omega(\partial_\mu + \Omega^{-1}\partial_\mu\Omega)\psi. \]  
\[ (7.25) \]
In analogy with the electromagnetic case, we need to introduce a matrix valued potential \( A_\mu(x) \) via the substitution
\[ \partial_\mu \rightarrow \partial_\mu - igA_\mu(x) \equiv D_\mu. \]  
\[ (7.26) \]
Then the requirement \( D_\mu \psi \rightarrow \Omega D_\mu \psi \) translates to
\[ A_\mu \rightarrow \Omega A_\mu \Omega^{-1} - \frac{i}{g}\partial_\mu\Omega\Omega^{-1}. \]  
\[ (7.27) \]
Clearly \( A \) takes values in the Lie algebra of the group \( G \). If \( \psi \) is a Dirac field, a gauge invariant dynamics is given by the field equation
\[ i\gamma \cdot D\psi = m\psi, \]  
\[ (7.28) \]
or the corresponding second quantized Hamiltonian
\[ H_A(t) = \int d^3 x (\bar{\psi}(1/\imath \alpha \cdot \nabla + \beta m)/\psi - g\bar{\psi}A_\mu \gamma^\mu \psi). \]  
\[ (7.29) \]
Just as in the electromagnetic case we can consider how the quantum evolution operator changes under a small change of \( A \), and identical steps lead to:
\[ \delta U(t, t_0) = U(t, t_0)i \int_{t_0}^t dt' d^3 x \bar{\psi} A_\mu \gamma^\mu \psi. \]  
\[ (7.30) \]
An infinitesimal gauge transformation \( \Omega = I + ig\epsilon G \) corresponds to
\[ \delta A_\mu = \epsilon(\partial_\mu G - ig[A_\mu, G]). \]  
\[ (7.31) \]
For \( G \) which vanish initially and finally the corresponding change in \( U \) is
\[ \delta U(t, t_0) = U(t, t_0)i \epsilon \int_{t_0}^t dt' d^3 x \]  
\[ (7.32) \]
\[ G_{ab}(-\partial_\mu (\bar{\psi}_a \gamma^\mu \psi_b) - ig(A_\mu)_{ca} \bar{\psi}_c \gamma^\mu \psi_b + ig(A_\mu)_{bc} \bar{\psi}_a \gamma^\mu \psi_c) \]  
\[ (7.33) \]
so gauge invariance implies the following generalization of current conservation (“covariant conservation”)
\[ D_\mu j^\mu \equiv \partial_\mu j^\mu - ig[A_\mu, j^\mu] = 0, \]  
\[ (7.34) \]
where the current is a matrix operator
\[ j^\mu(x)_{ba} \equiv \bar{\psi}_a(x)\gamma^\mu\psi_b(x). \]  
\[ (7.35) \]
7.3 External Gravitational Fields

According to the Principle of Equivalence, an external gravitational field is described by introducing a space-time dependent metric $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$ which then must enter the field equations in a generally covariant way. This prescription suffices for bosonic fields but new concepts must be brought in for fermionic fields. At this stage we shall confine our discussion to a real scalar field. The minimal generally covariant classical action is given by

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} (g^{\mu\nu}(x) \partial_{\mu} \phi \partial_{\nu} \phi + m^2 \phi^2). \tag{7.36}$$

To construct the Hamiltonian, we first calculate the conjugate momentum

$$\pi(x) \equiv \frac{\delta S}{\delta \dot{\phi}(x)} = -\sqrt{-g} g^{0\nu}(x) \partial_{\nu} \phi. \tag{7.37}$$

Then the Hamiltonian is

$$H(t) = \int d^3x (\pi(x) \dot{\phi}(x) + \frac{1}{2} \sqrt{-g}(g^{\mu\nu}(x) \partial_{\mu} \phi \partial_{\nu} \phi + m^2 \phi^2)) \tag{7.38}$$

$$= \int d^3x \left( -\frac{1}{2} g^{00} \sqrt{-g} + \frac{g^{0k}}{g^{00}} \pi \partial_k \phi + \frac{1}{2} \sqrt{-g} \left( g^{kl} - \frac{g^{0k} g^{0l}}{g^{00}} \right) \partial_k \phi \partial_l \phi \right. \right.$$  

$$+ \left. \frac{m^2}{2} \sqrt{-g} \phi^2 \right). \tag{7.39}$$

Just as with gauge fields we can ask how the evolution operator changes under a small change $\delta g_{\mu\nu}$ in the metric, $\delta U = -iU \int_{t_0}^t dt' \delta H(t')$, where $\delta H$ is computed holding $\phi, \pi$ fixed. The easiest way to do this is to evaluate the change in the Lagrangian $L$ at fixed $\phi, \dot{\phi}$. Since $L$ is related to $H$ by a Legendre transform, $\delta H = -\delta L$.

$$\delta L = -\frac{1}{2} \int d^3x \sqrt{-g} \delta g^{\mu\nu}(x) T_{\mu\nu}(x) \tag{7.40}$$

with $T_{\mu\nu}$ the energy momentum tensor

$$T_{\mu\nu} = \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} g_{\mu\nu}(g^{\sigma\rho} \partial_{\rho} \phi \partial_{\sigma} \phi + m^2 \phi^2). \tag{7.41}$$

Thus we have finally

$$\delta U = iU \frac{1}{2} \int_{t_0}^t d^4x' \sqrt{-g} \delta g_{\mu\nu}(x') T^{\mu\nu}(x'). \tag{7.42}$$

\footnote{Note that since $g^{\mu\nu}$ is the inverse matrix to $g_{\mu\nu}$, $\delta g^{\mu\nu} = -g^{\mu\rho} \delta g_{\rho\sigma} g^{\sigma\nu}$.}
Under the infinitesimal general coordinate transformation \( x^\mu = x'^\mu + \xi^\mu(x') \), the metric changes according to

\[
g_{\mu\nu}(x) = g'_{\mu\nu}(x') - \partial_\mu \xi^\rho \frac{\partial \xi^\rho}{\partial x' \nu} g_{\rho \nu} (7.43) \\
= g_{\mu\nu}(x') - D_\mu \xi_\nu - D_\nu \xi_\mu + \Gamma^\rho_\mu_\sigma \xi^\sigma g_{\rho \nu} + \Gamma^\rho_\nu_\sigma \xi^\sigma g_{\rho \mu} (7.44) \\
= g'_{\mu\nu}(x') - D_\mu \xi_\nu - D_\nu \xi_\mu + \xi^\rho \partial_\rho g_{\mu \nu} (7.45) \\
= g'_{\mu\nu}(x) - D_\mu \xi_\nu - D_\nu \xi_\mu (7.46)
\]

An infinitesimal change of integration variables is just a surface term:

\[
\int d^4 x' \mathcal{L}'(x') = \int d^4 x (1 - \partial_\rho \xi^\rho) (1 - \xi^\rho \partial_\rho) \mathcal{L}'(x) = \int d^4 x \mathcal{L}'(x) - \int d^4 x \partial_\rho (\xi^\rho \mathcal{L}'(x)) (7.47)
\]

so choosing \( \delta g_{\mu\nu} = -D_\mu \xi_\nu - D_\nu \xi_\mu \) should give \( \delta U = 0 \) for all \( \xi \) vanishing sufficiently rapidly at infinity, if the quantum field dynamics is invariant under general coordinate transformations. Thus general coordinate invariance implies that the energy momentum tensor is covariantly conserved:

\[
D_\mu T^{\mu\nu} = 0. (7.48)
\]

In the limit of flat space (no gravity) this condition reduces to ordinary energy-momentum conservation.
Chapter 8

Summing External Field Perturbations

As an important example, let us apply perturbation theory to the Dirac field in an external electromagnetic field. Since we shall work in interaction picture throughout, we shall not append the $I$ subscript to indicate interaction picture: that will be understood throughout this section. Then the persistence amplitude in the presence of $A$ is

$$
\langle \text{out}|\text{in}\rangle_A = \langle 0|Te^{i\int d^4x j^\mu(x)A_\mu(x)}|0\rangle
$$

(8.1)

$$
= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left( \frac{q}{2} \right)^n \int d^4x_1 \cdots d^4x_n A_{\mu_1} \cdots A_{\mu_n}
$$

(8.2)

$$
\langle 0|T[\bar{\psi}(x_1), \gamma^{\mu_1} \psi(x_1)] \cdots [\bar{\psi}(x_n), \gamma^{\mu_n} \psi(x_n)]|0\rangle.
$$

(8.3)

It is now a matter of applying Wick’s theorem to evaluate the $n^{th}$ term of the series.

We shall organize the calculation with Feynman diagrams. The Wick expansion expresses the time ordered product in terms of propagators completely characterized by two points. Each propagator is represented by a line connecting the two points, directed from the argument of $\bar{\psi}$ to that of $\psi$:

The lines terminate on vertices associated with the field $A$:

\[
S_F(x - y)_{ab}.
\]

\[
iq \int d^4xA_\mu(x)\gamma^\mu_{ab}.
\]
For each closed loop there is a trace over Dirac indices and a multiplicative factor of $-1$. This last factor is due to the anticommuting property of the Dirac field and arises because the order of fields in the product of currents that is contracted to form the loop differs from that of the contributing propagators by an odd number of interchanges.

Finally there is a combinatoric factor arising from a sometimes partial cancellation of the $1/n!$ multiplying the $n^{th}$ order term.

8.1 Connected Diagrams

Terms in the perturbation series described by disconnected diagrams factorize into a product of the values of each connected subdiagram. Thus all the useful information is contained in the subset of connected diagrams, and it is useful to know how the final answer is expressed in terms of connected diagrams only. Roughly speaking, the sum of all diagrams is simply the exponential of the sum of all connected diagrams. This statement applies to the expansion of

$$
\langle 0 | T e^{-i \int dt H'(t)} | 0 \rangle
$$

for a completely general Hamiltonian. The reason is purely combinatoric. Set

$$
G^c_k = \left\langle 0 | T \left( (-i \int dt H'(t))^k \right) | 0 \right\rangle_c
$$

where by the subscript $c$ we mean drop all contractions which produce any disconnected parts. Then the value of the sum of connected graphs at order $k$, $G^c_k/k!$. Now consider the terms at order $n = \sum_{k=1}^{\infty} kr_k$ in the perturbation series described by $r_k$ appearances of the connected graphs of order $k$. (Note that all but a finite number of the $r_k$ are 0.) The contractions leading to these terms can occur in $n! \prod_k [(k!)^{r_k} r_k!]$ ways. This is because there are $n!$ ways to assign the $n$ Hamiltonians to the $n$ vertices of all the connected subgraphs, but this overcounts the number of contractions by a factor of $k!$ for each of the connected subgraphs of order $k$, because different orders of the assignment to each subgraph do not give distinct contractions, and overcounts by a factor of $r_k!$ for each group of identical subgraphs for the same reason. Thus the value of the order $n$ terms is

$$
\frac{1}{n!} \sum_{kr_k=n} \frac{n!}{\prod_k [(k!)^{r_k} r_k!]} \prod_k (G^c_k)^{r_k} = \sum_{kr_k=n} \prod_k \left[ \frac{1}{r_k!} \left( \frac{G^c_k}{k!} \right)^{r_k} \right].
$$
Note that the factors in square brackets are 1 if all \( r_k = 0 \). Summing over all \( n \) simply relaxes the constraint on the summation over the \( r_k \), so we have finally

\[
\langle 0 | T e^{-i \int dt H'_e(t)} | 0 \rangle = \prod_{k=1}^{\infty} \left[ \sum_{r_k=0}^{\infty} \frac{1}{r_k!} \left( \frac{G^e_k}{k!} \right)^{r_k} \right] = \prod_{k=1}^{\infty} e^{G^e_k/k!} = e^{\sum_{k=1}^{\infty} G^e_k/k!},
\]

(8.7)

which is the desired result.

We now turn to the connected diagrams for an external electromagnetic field. At order \( n \) in the fields, the diagrams contributing to \( \ln \langle \text{out} | \text{in} \rangle \) are

\[
\langle \text{out} | \text{in} \rangle = -(i q)^n \int d^4 x_1 \cdots d^4 x_n \quad \text{Tr} [\gamma \cdot A(x_1) S_F(x_1 - x_2) \cdots \gamma \cdot A(x_n) S_F(x_n - x_1)].
\]

(8.8)

Since there are \( (n-1)! \) distinct diagrams with the same value (after the coordinate integrations) the net combinatoric factor is \( (n-1)!/n! = 1/n \). This factor can be interpreted as \( 1/S_G \) where \( S_G \) is the symmetry number of the graph \( \gamma \). It is also worth noting that the same formula applies to the case of nonabelian gauge field \( A_{\mu ab} \) with the understanding that the trace includes the trace over the internal indices as well as the spinor indices.

In fact, it is instructive to regard the coordinates as (continuous) indices, so \( S_{Fx1a,x2b} \) is a matrix and \( (\gamma \cdot A)_{x1,x2} \equiv \delta(x_1 - x_2) \gamma \cdot A \) is a matrix, so the term as a whole can be regarded as a grand trace

\[
-\frac{1}{n} \text{Tr} [i q \gamma \cdot A S_F]^n
\]

(8.9)

and the sum over all \( n \) is then recognized as the Taylor expansion for a logarithm:

\[
\ln \langle \text{out} | \text{in} \rangle = \text{Tr} [\ln (I - i q \gamma \cdot A S_F)].
\]

(8.10)

Making use of the identities \( \det A = \exp \{Tr \ln A\} \) and \( \det AB = \det A \det B \), which are fundamental properties of the determinant, and noting that in this matrix notation \( S_F = -i(m + (1/i) \gamma \cdot \partial)^{-1} \), we see that

\[
\langle \text{out} | \text{in} \rangle = \det(I - q \gamma \cdot A(m + \frac{1}{i} \gamma \cdot \partial)^{-1})
\]

\[
= \frac{\det(m + \frac{1}{i} \gamma \cdot \partial - q \gamma \cdot A)}{\det(m + \frac{1}{i} \gamma \cdot \partial)}
\]

\[
\equiv \frac{\det(m - i \gamma \cdot D)}{\det(m - i \gamma \cdot \partial)}
\]

(8.11) (8.12) (8.13)
where we have defined the covariant derivative operator as $D = \partial - iqA$. The denominator in (8.13) serves to normalize $\langle\text{out}|\text{in}\rangle$ to 1 at $A = 0$, which amounts to choosing the zero of energy to be the ground state energy of $H_0$. In practice these “explicit formulae” for $\langle\text{out}|\text{in}\rangle$ can not be evaluated exactly for general $A$, although for special external potentials such as those corresponding to constant field strength it is possible. Nonetheless they give the most efficient derivation of the perturbation series in powers of the external fields, and as we shall see give some general insight into the meaning of the amplitudes we are calculating.

8.2 Furry’s Theorem.

There appears to be a connected diagram (8.8) for every $n$. But for the electromagnetic case, only those with even $n$ are nonvanishing. To see why this is true use the trace property $\text{Tr}A^T = \text{Tr}A$ to show that

$$\text{Tr}[\gamma \cdot A(x_1)S_F(x_1 - x_2)\cdots \gamma \cdot A(x_n)S_F(x_n - x_1)] = \text{Tr}[S^T_F(x_n - x_1)\gamma^T \cdot A(x_n)\cdots S^T_F(x_1 - x_2)\gamma^T \cdot A(x_1)].$$

(8.14)

But $\gamma^\mu T = -(\gamma^0\gamma^2)^{-1}\gamma^\mu\gamma^0\gamma^2$. From this it follows that

$$S^T_F(x) = -i(\gamma^0\gamma^2)^{-1}\int \frac{d^4p}{(2\pi)^4}\frac{m + \gamma \cdot p}{m^2 + p^2}e^{ipx}\gamma^0\gamma^2$$

$$= (\gamma^0\gamma^2)^{-1}S_F(-x)\gamma^0\gamma^2$$

(8.15)

So we have

$$\text{Tr}[\gamma \cdot A(x_1)S_F(x_1 - x_2)\cdots \gamma \cdot A(x_n)S_F(x_n - x_1)] =$$

$$(-)^n\text{Tr}[S_F(x_1 - x_n)\gamma \cdot A(x_n)\cdots S_F(x_2 - x_1)\gamma \cdot A(x_1)].$$

(8.16)

After integrating over the $x$’s the only difference between the left and right sides is the labeling of dummy integration variables and the factor $(-)^n$ on the right. Thus for odd $n$ both sides must vanish, i.e. all connected diagrams for $\langle\text{out}|\text{in}\rangle$ with an odd number of $A$’s vanish. This is Furry’s theorem. A more basic way to understand the result is to note that the substitution $A \to -A$ can be undone by the charge conjugation transformation under which $j \to -j$. Thus $\langle\text{out}|\text{in}\rangle$ must be an even function of $A$.

The perturbation series can be similarly “summed” for the propagator in the presence of external gauge fields

$$\langle\text{out}|T\psi(x)\bar{\psi}(y)|\text{in}\rangle_A = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n$$

$$\langle 0|T[A \cdot j(x_1)\cdots A \cdot j(x_n)]\psi(x)\bar{\psi}(y)|0\rangle.$$ 

(8.18)

The connected subdiagrams are of two types: (1) In a given term there are any number of closed loop diagrams of the sort contributing to $\langle\text{out}|\text{in}\rangle_A$; (2) In each term there is exactly one subdiagram with a continuous line running from $y$ to $x$
Summing up all of the first type of diagram gives simply an overall factor of \( \langle \text{out}|\text{in} \rangle_A \) multiplying the value of each diagram of the second type. Apart from the net combinatoric factor the latter diagram has at order \( n \) the value

\[
= -(iq)^n \int d^4x_1 \cdots d^4x_n \quad S_F(x-x_1)\gamma \cdot A(x_1)S_F(x_1-x_2) \cdots \gamma \cdot A(x_n)S_F(x_n-y).
\]

(8.20)

In fact the combinatoric factor is just 1 since there are \( n! \) distinct contractions leading to this diagram (giving identical values since they only differ in the labeling of dummy integration variables), and that precisely cancels the \( 1/n! \) coming from the Dyson formula. In the matrix notation introduced above, we recognize the sum of all diagrams, including the disconnected closed loops, as

\[
\langle \text{out}|\text{in} \rangle_A \sum_{n=0}^{\infty} S_F(iq\gamma \cdot AS_F)^n = \langle \text{out}|\text{in} \rangle_A \frac{1}{S_F T - iq\gamma \cdot AS_F} = \langle \text{out}|\text{in} \rangle_A \frac{-i}{m - i\epsilon - i\gamma \cdot D}. \tag{8.21}
\]

(8.22)

In other words the right hand side is just proportional to a Green function for the differential operator \( m - i\gamma \cdot D \):

\[
\langle \text{out}|T\psi(x)\bar{\psi}(y)|\text{in} \rangle_A = \langle \text{out}|\text{in} \rangle_A S(x,y;A) \tag{8.23}
\]

where

\[
(m - i\gamma \cdot D)S(x,y;A) = -i\delta^4(x-y). \quad \tag{8.24}
\]
and the boundary condition is specified by giving an infinitesimal negative imaginary part to \(m\). Since \(A = 0\) in the distant past and future, this prescription corresponds to the requirement of only positive frequencies as \(t \to +\infty\) and negative frequencies as \(t \to -\infty\), which are manifest properties of \(\langle \text{out} | T\bar{\psi}(x)\bar{\psi}(y) | \text{in} \rangle_A\). The fact that this latter quantity is this Green function is also immediately seen by applying the differential operator and remembering the contribution from differentiating the step functions implicit in the time ordering symbol.

By making use of our result for \(\langle \text{out} | T\bar{\psi}(x)\bar{\psi}(y) | \text{in} \rangle_A\), we can give a much quicker derivation of the determinant formula for \(\langle \text{out} | \text{in} \rangle_A\). Looking back to (7.20) we see that if we make a change \(\delta A\) in \(A\), the change in \(\langle \text{out} | \text{in} \rangle\) is

\[
\delta \langle \text{out} | \text{in} \rangle_A = \langle \text{in} | \delta U(\infty, -\infty) | \text{in} \rangle = i \int d^4x \delta A_\mu(x) \langle \text{out} | j^\mu(x) | \text{in} \rangle_A.
\]

(8.25)

On the other hand the current matrix element can be related to the Green function of the previous paragraph

\[
\langle \text{out} | j^\mu(x) | \text{in} \rangle_A = \langle \text{out} | \frac{q}{2} [\bar{\psi}(x), \gamma^\mu \psi(x)] | \text{in} \rangle_A
\]

(8.26)

\[
= -\frac{q}{2} \text{Tr} \gamma^\mu \langle \text{out} | [\bar{\psi}(x), \bar{\psi}(x)] | \text{in} \rangle_A
\]

(8.27)

\[
= -q \text{Tr} \gamma^\mu \langle \text{out} | T[\psi(x)\bar{\psi}(x)] | \text{in} \rangle_A
\]

(8.28)

\[
= -q \text{Tr} \gamma^\mu S(x, x; A) \langle \text{out} | \text{in} \rangle_A.
\]

(8.29)

where we are interpreting \(\theta(0) = 1/2\). Using \(\delta \langle \text{out} | \text{in} \rangle / \langle \text{out} | \text{in} \rangle = \delta \ln \langle \text{out} | \text{in} \rangle\) we obtain

\[
\delta \ln \langle \text{out} | \text{in} \rangle_A = -iq \int d^4x \text{Tr} [\gamma \cdot \delta AS(x, x; A)]
\]

(8.30)

\[
= \text{Tr} [-q\gamma \cdot \delta A(m - i\gamma \cdot D)^{-1}]
\]

(8.31)

\[
= \delta \text{Tr} \ln[m - i\gamma \cdot D]
\]

(8.32)

which implies our previous formula up to a multiplicative \(A\) independent constant which is fixed by requiring \(\langle \text{out} | \text{in} \rangle_{A=0} = 1\). Actually this initial condition is somewhat artificial; it would be more natural to simply take \(\langle \text{out} | \text{in} \rangle_A = \det(m - i\gamma \cdot D)\). This corresponds to not removing the sea contribution to the energy in the case \(A = 0\).

It is instructive to see how the formal expression for the sea energy comes out of this evaluation. When \(A = 0\) the matrix element (in momentum basis) \((p^\prime | \ln(m - \gamma \cdot \partial)|p) = \delta(p^\prime - p) \ln(m + \gamma \cdot p)\), so when we take the trace over the continuous momentum indices, we set \(p^\prime = p\) and get an overall factor of \(\delta^4(0)\). To interpret this singular factor, think of the integral representation for \(\delta(p) = \int e^{ip\cdot x}d^4x/(2\pi)^4\). As \(p \to 0\) this is just \(VT/(2\pi)^4\) where \(V\) is the volume of space and \(T\) the duration of time. With this interpretation, we have

\[
\text{Tr} \ln(m - i\gamma \cdot \partial) = \frac{VT}{(2\pi)^4} \int d^4p \text{Tr} \ln(m + \gamma \cdot p)
\]

(8.33)

\[
= \frac{VT}{(2\pi)^4} \int d^4p \ 2 \ln(m^2 + p^2)
\]

(8.34)
where we have used the fact that the eigenvalues of $\gamma \cdot p$ are $+\sqrt{-p^2}$ twice and $-\sqrt{-p^2}$ twice. To compare this to the sea energy we obtained earlier, we need to interpret the integral over $p^0$. This is of course a divergent integral, but if we differentiate once with respect to $m^2$ to get the mass dependence, we get a convergent integral which can be evaluated by closing the contour in the upper half complex $p^0$ plane

$$
\int \frac{dp^0}{2\pi} \frac{1}{\omega(p)^2 - p^{i0} - i\epsilon} = \frac{i}{2\omega} = \frac{i}{dm^2} \omega(p).
$$

Thus

$$
\ln \langle \text{out}|\text{in} \rangle = 2iT \frac{V}{(2\pi)^4} \int d^3p [\omega(p) + C]
$$

where the constant is at least independent of the mass. Since $\langle \text{out}|\text{in} \rangle \sim e^{-iE_{\text{sea}}T}$ we see that we recover our previous result for $E_{\text{sea}}$ at least as far as the mass dependence is concerned.

The occurrence of complex energies in the Minkowski definition of $\langle \text{out}|\text{in} \rangle$ motivates the idea that these vacuum amplitudes should be defined fundamentally in Euclidean space and then continued back to Minkowski space as the application demands. In Euclidean space-time we put $T = -iT_E$ and the vacuum amplitude would have the behavior $\langle \text{out}|\text{in} \rangle \sim \exp(-E_{\text{sea}}T_E)$. If we repeat the calculation of $\text{Tr} \ln(m - i\gamma \cdot \partial)$ in Euclidean space we get $VT_E \int d^4p E^2 \ln(m^2 + p^2)/(2\pi)^4$ which is manifestly real albeit divergent, and we get the formula

$$
E_{\text{sea}} = -2\frac{V}{(2\pi)^4} \int d^4p E \ln(m^2 + p^2).
$$

This still has the mass dependence of our earlier result and the mass independent discrepancy is at least real. In effect, working in Euclidean space-time from the beginning discards undesirable complex contours from Wick rotations that fail to vanish only because of the poor high momentum (ultra-violet) behavior of the integrands of vacuum diagrams. Consequently, among modern field theorists Euclidean space-time is widely accepted as the best way to define vacuum amplitudes.

Finally, we consider the calculation of time ordered products of an arbitrary number of Dirac fields. Clearly the nonvanishing ones have an equal number of $\psi$’s and $\bar{\psi}$’s:

$$
\langle \text{out}|T[\psi(x_1) \cdots \psi(x_n)\bar{\psi}(y_n) \cdots \bar{\psi}(y_1)]|\text{in} \rangle.
$$

To each term there will correspond any number of connected closed loop diagrams and $n$ connected diagrams of exactly the type contributing to the $n = 2$ case. Each of these latter subdiagrams consists of a line from one of the $y_k$ to one of the $x_P$. The closed loops sum to an overall factor of $\langle \text{out}|\text{in} \rangle$. For each distinct pairing of the $x$’s with the $y$’s the other subdiagrams sum to a product of $n$ factors $S(x_P, y_k; A)$. In other words, there is a Wick expansion for time ordered products of Dirac fields in the presence of an external field exactly as in the $A = 0$ case. All one does is substitute $S_F(x - y) \rightarrow S(x, y; A)$ and multiply by an overall factor of $\langle \text{out}|\text{in} \rangle_A$. 

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Chapter 9

Scattering in External Fields

Our time dependent formalism is ideally suited for defining transition amplitudes\(^1\). Since external fields are turned off in the far future and distant past, particle states can be defined in exactly the same manner as the \emph{in} and \emph{out} vacua. We can expand the Heisenberg field operators at early(late) times in terms of annihilation and creation operators \(b_{\lambda}^{\text{in}}(p), d_{\lambda}^{\text{in}}(p)\) \((b_{\lambda}^{\text{out}}(p), d_{\lambda}^{\text{out}}(p))\), since they are free fields there. These are of course eigenoperators for \(H_0(-\infty)(H_0(+\infty))\) respectively. They are of course determined up to a phase by the labels \(\lambda, p\). We can fix the phases of the \(\text{out}\) operators in terms of the \(\text{in}\) operators by defining \(\Omega_{\text{out}} \equiv U^{-1}(\infty, -\infty)\Omega_{\text{in}} U(\infty, -\infty)\). \(^2\) Then incoming particle states are obtained by applying \(b_{\text{in}}^{\dagger}, d_{\text{in}}^{\dagger}\) to \(|\text{in}\rangle\) and outgoing particle states are obtained by applying \(b_{\text{out}}, d_{\text{out}}\) to \(|\text{out}\rangle\).

The transition amplitudes between multiparticle states at early times and multiparticle states at late times can be immediately transcribed to interaction picture:

\[
|\text{out}\rangle b_{1}\text{out} \cdots d_{N}\text{out} b_{M}^{\dagger} \cdots b_{1}^{\dagger}|\text{in}\rangle = |0, I\rangle b_{I} \cdots d_{I}U(\infty, -\infty)d_{I}^{\dagger} \cdots b_{I}^{\dagger}|0, I\rangle. \quad (9.1)
\]

The rules for expanding these amplitudes in perturbation theory are very similar to those for the outin matrix elements of time ordered products of Dirac field operators. The disconnected closed loops sum up to an overall factor of \(\langle \text{out}|\text{in}\rangle\). The creation and annihilation operators can either contract against each other or against one of the Dirac fields in \(U_I\):

\(^1\)This chapter brings together is a unified setting topics on scattering introduced in earlier chapters for the purpose of giving concrete examples of general principles.

\(^2\)Note that since \(U = U_0 U_I\) and \(U_0^{-1} \Omega_{\text{in}} U_0\) is just a numerical phase times \(\Omega_{\text{in}}\) this definition does ensure that \(\Omega_{\text{out}}\)is an eigenoperator of \(H_0(+\infty)\).
\[ \langle 0, I | b_\lambda(p) \bar{\psi}(x) | 0, I \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} \bar{u}_\lambda(p) e^{-ip \cdot x} \tag{9.2} \]

\[ \langle 0, I | d_\lambda(p) \psi(x) | 0, I \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} v_\lambda(p) e^{-ip \cdot x} \tag{9.3} \]

\[ \langle 0, I | \psi(x) b_\lambda^\dagger(p) | 0, I \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} u_\lambda(p) e^{ip \cdot x} \tag{9.4} \]

\[ \langle 0, I | \bar{\psi}(x) d_\lambda^\dagger(p) | 0, I \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} \bar{v}_\lambda(p) e^{ip \cdot x} \tag{9.5} \]

To illustrate how to use these rules consider the process in which a single particle at early time is scattered by the external potential to a single particle at late times.

\[ \langle \text{out} | b_{\lambda'\lambda}^\text{out}(q) b_{\lambda\lambda}^\text{init}(p) | \text{in} \rangle = \langle \text{out} | \text{in} \rangle_A \left[ \delta_{\lambda'\lambda} \delta^3(q - p) + \langle 0, I | b_{\lambda}(q) U_I b_{\lambda}^\dagger(p) | 0, I \rangle^c \right] \tag{9.6} \]

where the superscript \( c \) denotes the restriction to connected diagrams containing at least one vertex.

The first diagram has the value

\[ \frac{1}{(2\pi)^3 \sqrt{4\omega(q)\omega(p)}} \int d^4x e^{i(p-q) \cdot x} \bar{u}_{\lambda'}(q) iq \gamma \cdot A(x) u_\lambda(p). \tag{9.7} \]
The second diagram has two vertices and a factor of $S_F(y - x)$ between them. But notice that the second plus all the higher diagrams just amounts to replacing this $S_F$ by $S_F(y, x; A)$, so the sum of all the diagrams but the first has the value

$$\frac{1}{(2\pi)^3 \sqrt{4\omega(q)\omega(p)}} \int d^4x d^4y e^{i(p-x-q)y} \bar{u}_\lambda'(q)iq\gamma \cdot A(y)S_F(y, x; A)iq\gamma \cdot A(x)u_\lambda(p).$$

We put all this together in the form

$$\langle \text{out} | b_\lambda^{\text{out}}(q)b_\lambda^{\text{in}}(p) | \text{in} \rangle = \langle \text{out} | \text{in} \rangle_A \left[ \delta_{\lambda\lambda'}(q - p) + M(q, p; A) \right],$$

where

$$M(q, p; A) = \frac{1}{(2\pi)^3 \sqrt{4\omega(q)\omega(p)}} \int d^4x d^4y e^{i(p-x-q)y} \bar{u}_\lambda'(q)[iq\gamma \cdot A(x)\delta(x - y) + iq\gamma \cdot A(y)S_F(y, x; A)iq\gamma \cdot A(x)]u_\lambda(p).$$

We can put this last formula in a more suggestive form by defining the free Dirac plane wave functions

$$\psi_{p\lambda}^0(x) \equiv \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}}u_\lambda(p)e^{ip\cdot x}$$

and noting that

$$\psi_{p\lambda}(x) \equiv \psi_{p\lambda}^0(x) + \int d^4y S_F(x, y; A)iq\gamma \cdot A(y)\psi_{p\lambda}^0(y)$$

is a solution of the Dirac equation in the presence of $A$:

$$(m + \frac{1}{i}\gamma \cdot D)\psi_{p\lambda} = 0,$$

with the boundary condition that at early times the only positive frequency components are contained in the term $\psi_{p\lambda}^0(x)$. Thus we can write

$$M(q, p; A) = \int d^4x \psi_{q\lambda}^0(x)iq\gamma \cdot A(x)\psi_{p\lambda}(x),$$

which is reminiscent of the corresponding formula in nonrelativistic quantum mechanics for the scattering of a particle from an external potential.

In the case of a static potential, the time dependence of both $\psi_{p\lambda}^0(x)$ and $\psi_{p\lambda}(x)$ is given by a multiplicative phase $e^{-i\omega(p)t}$ and the time integral then provides an energy conserving delta function

$$M(q, p; A(x)) = 2\pi \delta(\omega(q) - \omega(p)) \int d^3x \psi_{q\lambda}^0(x)iq\gamma \cdot A(x)\psi_{p\lambda}(x).$$
Recall from basic scattering theory that if the scattering matrix for a particle from a static potential is written

\[ \langle q, out|p, in \rangle = \delta_{\lambda'\lambda}(q - p) - 2\pi i \delta(\omega(q) - \omega(p))T_{\lambda'\lambda}(q, p) \]  

(9.15)

then the differential scattering cross section is given by

\[ \frac{d\sigma}{d\Omega} = \frac{d^3 q}{d\Omega} \delta(\omega(q) - \omega(p)) \left( \frac{2\pi}{v} \right)^4 |T_{\lambda'\lambda}(q, p)|^2, \]  

(9.16)

\[ = q^2 dq \delta(\omega(q) - \omega(p)) \left( \frac{2\pi}{v} \right)^4 |T_{\lambda'\lambda}(q, p)|^2, \]  

(9.17)

\[ = p\omega(p) \left( \frac{2\pi}{v} \right)^4 |T_{\lambda'\lambda}(q, p)|^2, \]  

(9.18)

\[ = \omega(p)^2 (2\pi)^4 |T_{\lambda'\lambda}(q, p)|^2, \]  

(9.19)

where \( v \) is the speed of the incident particle. Comparing with our expression we find that for electron scattering from a static potential

\[ T_{\lambda'\lambda}(q, p) = i \int d^3 x \bar{\psi}_0 q \gamma \cdot \tilde{A}(q - p) u_\lambda(x). \]  

(9.20)

Note that in the absence of pair production, the factor \( \langle out|in \rangle \), the vacuum persistence amplitude, is a pure phase and doesn’t contribute in the absolute square of \( T \). In lowest order in \( A \) (the Born approximation) one simply replaces \( \psi \) by \( \psi^0 \).

Let us note some tricks that are useful in calculating cross sections for processes with unobserved final spins and unpolarized beams. When we calculate the absolute square of a spinor matrix element we can make use of the identities

\[ \sum_\lambda u_\lambda u_\lambda = m - \gamma \cdot p \]  

(9.21)

\[ \sum_\lambda v_\lambda v_\lambda = -m - \gamma \cdot p. \]  

(9.22)

For example these allow us to write

\[ \sum_{\lambda'\lambda} |\bar{u}_\lambda \gamma \cdot \tilde{A}(q - p) u_\lambda(p)|^2 \]  

\[ = \text{Tr}[\gamma \cdot \tilde{A}(m - \gamma \cdot p)\gamma \cdot \tilde{A}^*(m - \gamma \cdot q)] \]  

\[ = 4(-m^2 - p \cdot q)\tilde{A} \cdot \tilde{A}^* + 4(p \cdot \tilde{A}q \cdot \tilde{A}^* + q \cdot \tilde{A}p \cdot \tilde{A}^*) \]  

\[ = 2(p - q)^2 \tilde{A} \cdot \tilde{A}^* + 4(p \cdot \tilde{A}q \cdot \tilde{A}^* + q \cdot \tilde{A}p \cdot \tilde{A}^*) \]  

(9.23)

Physical quantities should be gauge invariant and cross sections are no exception. We can reveal the gauge invariance of this last formula by introducing \( \tilde{F}_{\mu\nu}(q) \equiv i(q_\mu \tilde{A}_\nu - q_\nu \tilde{A}_\mu) \) which is the Fourier transform of the gauge invariant field strength. Then

\[ \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} = 2(q^2 \tilde{A} \cdot \tilde{A}^* - q \cdot \tilde{A}q \cdot \tilde{A}^*). \]  

(9.24)
Recalling that the argument of $\tilde{A}$ is $q - p$ we then find that the squared spinor matrix element can be rewritten
\[
\sum_{\lambda\lambda'} \bar{u}_{\lambda'} \gamma \cdot \tilde{A}(q - p) u_{\lambda}(p)|^2 = \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + 2(p + q) \cdot \tilde{A} (p + q) \cdot \tilde{A}^*.
\] (9.25)

The first term is manifestly gauge invariant and the second is gauge invariant by virtue of the identity
\[
(q + p) \cdot (q - p) = q^2 - p^2 = -m^2 + m^2 = 0.
\] (9.26)

The differential cross section for unpolarized electron scattering in a general static potential with final spin unobserved is then
\[
\sum_{\lambda'} d\sigma^{\text{Born}}_{\lambda'} = \frac{1}{16\pi^2} \frac{1}{2} \sum_{\lambda} |\bar{u}_{\lambda'} q \gamma \cdot \tilde{A}(q - p) u_{\lambda}(p)|^2.
\] (9.27)

\[
= \frac{e^2}{16\pi^2} \left( \frac{1}{2} \tilde{F}_{\mu\rho} \tilde{F}^{\mu\rho} + (p + q) \cdot \tilde{A} (p + q) \cdot \tilde{A}^* \right).
\] (9.28)

\[
= \frac{\alpha}{4\pi} \left( \frac{1}{2} \tilde{F}_{\mu\rho} \tilde{F}^{\mu\rho} + |(p + q) \cdot \tilde{A}|^2. \right)
\] (9.29)

One can easily confirm that this agrees with our previous results for the special case of the Coulomb potential.

We expect that the transition amplitudes for antiparticles should be obtained from (9.9) by the substitution $A \rightarrow -A$. On the other hand the rules seem to give a different prescription:

\[
\langle \text{out}|d^{\text{out}}_{\lambda}(q)d^{\text{in}}_{\lambda}(p)|\text{in}\rangle = \langle \text{out}|\text{in}\rangle_A [\delta_{\lambda\lambda}\delta^3(q - p) + \tilde{\mathcal{M}}(q, p; A)],
\] (9.30)

with

\[
\tilde{\mathcal{M}}(q, p; A) = -\frac{1}{(2\pi)^3 \sqrt{4\omega(q)\omega(p)}} \int d^4x d^4y e^{i(p \cdot x - q \cdot y)}
\]

\[
\bar{v}_{\lambda}(p)[i q \gamma \cdot A(x) \delta(x - y) + i q \gamma \cdot A(x) S_F(x, y; A)i q \gamma \cdot A(y)]v_{\lambda}(q).
\] (9.31)

In fact it is not hard to show that $\tilde{\mathcal{M}}(q, p; A) = \mathcal{M}(q, p; -A)$ by inserting $v = i\gamma^2 u^*$ into (9.31) and transposing the matrix element, using $i\gamma^2 \gamma^0 \gamma^0 i\gamma^2 = -\gamma^\mu$ and $i\gamma^2 \gamma^0 S_T(x, y; A)\gamma^0 i\gamma^2 = -\gamma^\mu = S_F(y, x; -A)$. The latter fact can be seen either term by term in the expansion in $A$ or by examining the defining Green function equation for $S_F(x, y; A)$.

In addition to the scattering of electrons or positrons, an external field can induce electron positron pair production and also pair annihilation. Taking the first case, for example, we have

\[
\langle \text{out}|b^{\text{out}}_{\lambda}(q_1)d^{\text{out}}_{\lambda}(q_2)|\text{in}\rangle = \langle \text{out}|\text{in}\rangle_A [\mathcal{M}^{\text{PairCreate}}(q_1, q_2; A)],
\] (9.32)
with

$$M^{PairCreate}(\mathbf{q}_1, \mathbf{q}_2; A) = -\frac{1}{(2\pi)^3 \sqrt{4\omega(\mathbf{q}_1)\omega(\mathbf{q}_2)}} \int d^4x d^4y e^{-i(q_1 \cdot x + q_2 \cdot y)}$$  \hspace{1cm} (9.33)$$

$$\bar{u}_\lambda(\mathbf{q}_1)[iq\gamma \cdot A(x)\delta(x-y) + iq\gamma \cdot A(x)]v_\lambda(\mathbf{q}_2).$$

In this case, of course, there is no delta function term representing only vacuum persistence. Similarly for pair annihilation, we have

$$\langle out | \mathbf{b}_\lambda^{\dagger}(\mathbf{q}_1) \mathbf{d}_\lambda^{\dagger}(\mathbf{q}_2) | in \rangle = \langle out | in \rangle_A [M^{PairAnnih}(\mathbf{q}_1, \mathbf{q}_2; A)],$$  \hspace{1cm} (9.34)$$

with

$$M^{PairAnnih}(\mathbf{q}_1, \mathbf{q}_2; A) = \frac{1}{(2\pi)^3 \sqrt{4\omega(\mathbf{q}_1)\omega(\mathbf{q}_2)}} \int d^4x d^4y e^{-i(q_2 \cdot x + q_1 \cdot y)}$$  \hspace{1cm} (9.35)$$

$$\bar{v}_\lambda(\mathbf{q}_2)[iq\gamma \cdot A(x)\delta(x-y) + iq\gamma \cdot A(x)]S_F(x, y; A)u_\lambda(\mathbf{q}_1).$$

Notice the prominent appearance of the Green function $S_F(x, y; A)$ for the Dirac Equation with an external field in all of the four basic processes, electron scattering, positron scattering, pair production, and pair annihilation. Moreover, we have also seen how to express the vacuum persistence amplitude in terms of this same Green function. Thus we see that the solutions of the first quantized Dirac equation are of direct utility in finding the physical properties of the second quantized theory.

As a final note we show how $S_F$ even contains information about the bound states of an electron in a static external field. Actually it is better to deal not with an exactly static field, but with a field that is adiabatically switched on at some very early time $-T$, static for a very long time interval $\approx 2T$ and then adiabatically switched off at a very late time $T$. Then according to the adiabatic theorem, $|in\rangle$ is proportional to the ground state of the Hamiltonian $H(t)$ at all times, and $|out\rangle = |in\rangle \langle out|in\rangle$. In particular, $|in\rangle$ is the ground state $|G\rangle$ of $H_A$ the hamiltonian for the static potential under study, since $H(t) = H_A$ for $-T << t << T$. Now fix the times $T >> x^0 > y^0 >> -T$. Then we have

$$S_F(x, y) \langle out|in\rangle$$

$$= \langle out|\psi(x)\bar{\psi}(y)|in\rangle$$

$$= \langle out|in\rangle \langle G|\psi(x)\bar{\psi}(y)|G\rangle$$

$$= \langle out|in\rangle \sum_n \langle G|\psi(x)|n\rangle \langle n|\bar{\psi}(y)|G\rangle$$

$$= \langle out|in\rangle \sum_n e^{-i(E_n - E_G)(x^0 - y^0)} \langle G|\psi(x, 0)|n\rangle \langle n|\bar{\psi}(y, 0)|G\rangle$$  \hspace{1cm} (9.36)$$

From this last formula, we see that a harmonic analysis of $S_F$ in the time variable $t = x^0 - y^0$ yields the possible energy eigenvalues $E_n - E_G$ of the energy eigenstates $|n\rangle$. These states have the quantum numbers of a one electron state because they are created from the vacuum by $\bar{\psi}$. Note that the harmonic components for $x^0 < y^0$ have the interpretation as $-(E_n - E_G)$.
where $E_n$ are the energy levels of one positron states. Because $S_F$ satisfies the homogeneous Dirac equation for $t \neq 0$, the possible energy eigenvalues are solutions of the time independent Dirac equation

\[(m + \frac{1}{i} \gamma \cdot D + qA^0\gamma^0 - (E_n - E_G))\psi = 0, \quad t > 0\]  

(9.37)

or

\[(m + \frac{1}{i} \gamma \cdot D + qA^0\gamma^0 + (E_n - E_G))\psi = 0, \quad t < 0.\]  

(9.38)

Thus, again, the solution of the energy eigenvalue problem for the first quantized theory is directly applicable to that for the second quantized theory.

### 9.1 Relation to Time Ordered Products.

(This subsection can be skipped in a first reading.) It is useful to establish how the scattering amplitudes are related to matrix elements of time ordered products. First let us define

\[b^\dagger_{\lambda}(p, t) = \int d^3x \frac{1}{(2\pi)^{3/2}} \left(\frac{1}{\sqrt{2\omega(p)}}\right) e^{ip_x \bar{\psi}(x)\gamma^0\mu_{\lambda}(p)}\]  

(9.39)

\[d_{\lambda}(p, t) = \int d^3x \frac{1}{(2\pi)^{3/2}} \left(\frac{1}{\sqrt{2\omega(p)}}\right) e^{ip_x \bar{\nu}_{\lambda}(p)\gamma^0\psi(x)}\]  

(9.40)

\[b_{\lambda}(p, t) = \int d^3x \frac{1}{(2\pi)^{3/2}} \left(\frac{1}{\sqrt{2\omega(p)}}\right) e^{-ip_x \bar{\psi}(x)\gamma^0\mu_{\lambda}(p)}\]  

(9.41)

\[d^\dagger_{\lambda}(p, t) = \int d^3x \frac{1}{(2\pi)^{3/2}} \left(\frac{1}{\sqrt{2\omega(p)}}\right) e^{-ip_x \bar{\nu}_{\lambda}(p)\gamma^0\psi(x)}\]  

(9.42)

For free fields all these operators are constant in time and are just the creation and annihilation operators of the Dirac field. For a Dirac field in the presence of external fields which vanish at early and late times they are not constant but approach the “out” creation and annihilation operators at $t = \infty$ and the “in” operators at $t = -\infty$.

Now we shall make use of the following “reduction” trick:

\[F(t = +\infty)T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]F(t = -\infty)\]

\[= \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} T[F(t)\phi_1(x_1) \cdots \phi_N(x_N)]\]  

(9.43)

where $F(t)$ is any function of Heisenberg operators at time $t$, and the $\phi$'s are generic field operators. The sign choice is + for $F(t)$ and $T[\cdots]$ fermionic operators, but – in all other cases. In particular, $F$ can be any of the expressions (9.42) where the time appearing on the r.h.s. is set to $t$. This leads to a series of formulæ for the commutation of creation and
annihilation operators with time ordered products. For instance, take $F$ to be the r.h.s. of the expression for $b_\lambda(p)$.

\[
b^\text{out}_\lambda(p)T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]b^\text{in}_\lambda(p) = \frac{\bar{u}_\lambda(p)}{(2\pi)^{3/2} \sqrt{2\omega(p)}} \int d^4x \gamma^0 \frac{\partial}{\partial t} \left( e^{-ip \cdot x} T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)] \right) \]

\[
= \frac{\bar{u}_\lambda(p)}{(2\pi)^{3/2} \sqrt{2\omega(p)}} \int d^4xe^{-ip \cdot x} \left( \gamma^0 \frac{\partial}{\partial t} + i\gamma^0 \omega(p) \right) T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)]
\]

\[
= \frac{\bar{u}_\lambda(p)}{(2\pi)^{3/2} \sqrt{2\omega(p)}} \int d^4xe^{-ip \cdot x} i(m - i\gamma \cdot \partial) T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)]
\]

where in the last line we used

\[
\bar{u}_\lambda \gamma^0 \omega e^{-ip \cdot x} = \bar{u}(m + i\gamma \cdot \nabla) e^{-ip \cdot x}
\]

and then integrated by parts. Clearly this derivation can be repeated for each of the expressions in (9.42) leading to the reduction formulae:

\[
b^\text{out}_\lambda(p)T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]b^\text{in}_\lambda(p) = \frac{\bar{u}_\lambda(p)}{(2\pi)^{3/2} \sqrt{2\omega(p)}} \int d^4x e^{-ip \cdot x} \left( m - i\gamma \cdot \partial \right) T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)] (9.46)
\]

\[
b^\text{out}_\lambda(p)T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]b^\text{in}_\lambda(p) = \int d^4x T[\bar{\psi}(x)\phi_1(x_1) \cdots \phi_N(x_N)][(m - i\gamma \cdot \partial) e^{-ip \cdot x} \frac{u_\lambda(p)}{(2\pi)^{3/2} \sqrt{2\omega(p)}} (9.47)
\]

\[
d^\text{out}_\lambda(p)T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]d^\text{in}_\lambda(p) = \int d^4x T[\bar{\psi}(x)\phi_1(x_1) \cdots \phi_N(x_N)][(m - i\gamma \cdot \partial) e^{-ip \cdot x} \frac{v_\lambda(p)}{(2\pi)^{3/2} \sqrt{2\omega(p)}} (9.48)
\]

\[
d^\text{out}_\lambda(p)T[\phi_1(x_1) \cdots \phi_N(x_N)] \mp T[\phi_1(x_1) \cdots \phi_N(x_N)]d^\text{in}_\lambda(p) = \frac{\bar{v}_\lambda(p)}{(2\pi)^{3/2} \sqrt{2\omega(p)}} \int d^4xe^{ip \cdot x} \left( m - i\gamma \cdot \partial \right) T[\psi(x)\phi_1(x_1) \cdots \phi_N(x_N)] (9.49)
\]

The reduction formulae can be used to systematically reduce scattering amplitudes to expressions directly involving time ordered products.

We shall illustrate the procedure for the case of a particle scattering in an external field. The application to antiparticle scattering and pair production and annihilation will be left
to the reader:

\[
\langle \text{out}| \hat{b}^\dagger_{\lambda}^\text{out}(\mathbf{q}) \hat{b}^\text{in}_{\lambda}(\mathbf{p})|\text{in}\rangle = \langle \text{out}| \hat{b}^\dagger_{\lambda}^\text{in}(\mathbf{q}) \hat{b}^\text{in}_{\lambda}(\mathbf{p})|\text{in}\rangle \\
+ \frac{\bar{\alpha}_{\lambda}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} \int d^4x e^{-i\mathbf{q}\cdot \mathbf{x}} (m - i\gamma \cdot \partial) \langle \text{out}| \psi(x) \hat{b}^\dagger_{\lambda}(\mathbf{p})|\text{in}\rangle \\
= \langle \text{out}| \text{in}\rangle \delta_{\lambda\lambda'}(\mathbf{q} - \mathbf{p}) + \frac{\bar{\alpha}_{\lambda}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} \int d^4x d^4y \notag \\
e^{-i\mathbf{q}\cdot \mathbf{x}} (m - i\gamma \cdot \partial) \langle \text{out}| T[\bar{\psi}(y)\psi(x)]|\text{in}\rangle i(\mathbf{m} - i\gamma \cdot \frac{\partial}{\partial y}) e^{ip\cdot y} \notag \\
e^{-i\mathbf{q}\cdot \mathbf{x}} (m - i\gamma \cdot \partial) S_F(x, y; A)i(m + i\gamma \cdot \frac{\partial}{\partial y}) e^{ip\cdot y} \notag \\
\frac{\alpha_{\lambda}(\mathbf{p})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} \notag \\
\int d^4x d^4y \notag \\
\] (9.50)

This is identical to our original expression as can be seen by writing \( m - i\gamma \cdot \partial = m - i\gamma \cdot \notag \)
\( D + \gamma \cdot \partial \) so that

\[
i(m - i\gamma \cdot \partial) S_F(x, y; A)i(m + i\gamma \cdot \notag \)
\( \frac{\partial}{\partial y} ) = \delta(x - y)i(m + i\gamma \cdot \notag \)
\( \frac{\partial}{\partial y} ) + iq\gamma \cdot \notag \)
\( A S_F(x, y; A)i(m + i\gamma \cdot \notag \)
\( \frac{\partial}{\partial y} ) \). (9.53)

The first term on the r.h.s contributes a term \( \bar{u}(\mathbf{q})(m + \gamma \cdot \mathbf{q})u(\mathbf{p}) = 0 \) by the Dirac equation.

The second term can be simplified using the fact that \( S \) is also a Green function in its second argument:

\[
S_F(m + i\gamma \cdot \notag \)
\( \frac{\partial}{\partial y} - \gamma \cdot \partial \) \notag \)
\( A ) = -i\delta(x - y) \). (9.54)

Thus we have

\[
i(m - i\gamma \cdot \partial) S_F(x, y; A)i(m + i\gamma \cdot \notag \)
\( \frac{\partial}{\partial y} ) = i\gamma \cdot \notag \)
\( A \delta(x - y) + i\gamma \cdot \notag \)
\( A S_F(x, y; A)i\gamma \cdot \notag \) (9.55)

as desired.

Now let us consider a little more closely the meaning of the reduction formula

\[
\langle \text{out}| \hat{b}^\dagger_{\lambda}^\text{out}(\mathbf{q}) \hat{b}^\text{in}_{\lambda}(\mathbf{p})|\text{in}\rangle \notag \\
= \langle \text{out}| \text{in}\rangle \delta_{\lambda\lambda'}(\mathbf{q} - \mathbf{p}) + \frac{\bar{\alpha}_{\lambda}(\mathbf{q})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{q})}} \int d^4x d^4y \notag \\
e^{-i\mathbf{q}\cdot \mathbf{x}} (m - i\gamma \cdot \partial) S_F(x, y; A)i(m + i\gamma \cdot \notag \)
\( \frac{\partial}{\partial y} ) e^{ip\cdot y} \notag \\
\frac{\alpha_{\lambda}(\mathbf{p})}{(2\pi)^{3/2} \sqrt{2\omega(\mathbf{p})}} \notag \\
\int d^4x d^4y \notag \\
\] (9.56)

The factors \( m - i\gamma \cdot \partial \) and \( i(m + i\gamma \cdot \notag \)
\( \partial y ) \) look as though they should give zero because after integrating by parts, they become \( m + \gamma \cdot \mathbf{q} \) and \( m + \gamma \cdot \mathbf{p} \) respectively and these factors give zero next to the Dirac spinors. The error in this reasoning is of course that the
surface terms at $t = \pm \infty$ are not zero (being in fact the scattering amplitudes themselves!). To get a clearer idea of what is happening, suppose we continue $q^0, p^0$ a little away from their “on-shell” values of $\omega(q), \omega(p)$. Then the surface terms oscillate at infinity and are effectively zero (e.g. in any smooth wave packet). Then we can integrate by parts and the second term becomes

$$
\frac{\bar{u}_\lambda(q)}{(2\pi)^{3/2}\sqrt{2\omega(q)}} i (m + \gamma \cdot q) \int d^4xd^4ye^{-iqx+ipy} S_F(x, y; A) i(m + \gamma \cdot p) \frac{u_\lambda(p)}{(2\pi)^{3/2}\sqrt{2\omega(p)}}.
$$

(9.57)

where the r.h.s. defines $T(q, p)$, the Fourier transform of the Green function. The only way for this to be nonzero as $q^0, p^0$ approach there on-shell values is for $T$ to acquire poles in this limit. The residues of these poles are then related to the scattering amplitudes. This is the content of the reduction formula, but there is a rather direct way to see how these poles come about.

Consider first of all the region of integration $x^0 > T$ where $\psi$ is a free field, with $b^{out}, d^{out\dagger}$. Then this region contributes to $T(q, p)$ the bit

$$
\int_T^\infty dt e^{i(q^0-\omega(q))t} \frac{(2\pi)^{3/2}}{\sqrt{2\omega(q)}} \langle out | \sum_\lambda \bar{b}^{out}(q)u_\lambda(q) \bar{\psi}(y) | in \rangle
$$

(9.58)

For $q^0 = \omega$ the integrand is time independent so the integral over an infinite range gives a divergence. To study it, give $q^0$ a small positive imaginary part and do the integral:

$$
\frac{ie^{i(q^0-\omega(q))T}}{q^0 - \omega(q) + i\epsilon} \frac{(2\pi)^{3/2}}{\sqrt{2\omega(q)}} \langle out | \sum_\lambda \bar{b}^{out}(q)u_\lambda(q) \bar{\psi}(y) | in \rangle
$$

(9.59)

For the range of integration with $x^0 < -T$ $\psi$ contains $b^{in}, d^{int\dagger}$ and stands on the right so only the second operator contributes. The integral over this range is not singular for $q^0 \to \omega$.

Similar considerations apply to the integral over $y^0$. This time the region $y^0 < -T$ contributes a pole at $p^0 = \omega(p)$ involving $b^{int\dagger}$. There is also a pole from the region $y^0 > T$ involving $b^{out\dagger}$. This contribution doesn’t vanish because of the presence of $b^{out}$ from the first reduction; it just gives delta functions. Going through all these steps leads to

$$
T(q, p) \sim \frac{i}{q^0 - \omega(q) p^0 - \omega(p)} \left[ \frac{(2\pi)^3}{4\omega(q)\omega(p)} \right] \sum_{\lambda, \lambda'} u_{\lambda'}(q) \bar{u}_\lambda(p)
$$

(9.60)

$$
[\langle out | b^{out}_{\lambda'}(q) b^{int\dagger}_{\lambda}(p) | in \rangle - \langle out | in \rangle \delta_{\lambda \lambda'} \delta(q - p)], \quad q^0, p^0 \to \omega(q), \omega(p)
$$

which is of course exactly the behavior required to satisfy the reduction formula.
Chapter 10

Vacuum Polarization

Before leaving external field problems, it is interesting to consider the effect an external field has on the vacuum (negative energy sea in the Dirac case). We shall calculate the response of the vacuum to the application of a weak electromagnetic field. For definiteness consider the Dirac field described by

\[
H_A = H_0 - \int d^3x j_\mu(x,t) A_\mu(x,t).
\] (10.1)

Assume that \( A \to 0 \) as \( t \to -\infty \) and that the system starts out in the ground state of \( H_A(-\infty) = H_0(-\infty) \).

We should expect the field to induce charge and current densities in the vacuum. A simple measure of these induced currents is the expectation value of the Heisenberg picture current operator in the system state \( \langle in | j_\mu(x) | in \rangle \). We may express this matrix element in interaction picture and then develop it in an expansion in powers of \( A \). In the limit of very weak fields we can neglect all terms beyond those linear in \( A \):

\[
\langle in | j_\mu(x) | in \rangle = \langle in | U_t^{-1}(-\infty) f_\mu f_\nu | 0 \rangle \approx i \int d^4y \theta(t - t_y) \langle 0 | [j_\mu(x,t), j_\nu(y)] | 0, I \rangle A_\nu(y).
\] (10.3)

There is no term independent of \( A \) because the vacuum expectation value of the current vanishes in the absence of applied fields.

This is the linear response to an applied field and is characterized by the response function

\[
R^{\mu\nu}(x) \equiv i \theta(t) \langle 0 | [j_\mu(x,t), j_\nu(0)] | 0 \rangle.
\] (10.5)

where here and in the following we drop the subscripts \( I \) and it is understood that the currents are those of free fields. An important physical property of the response function is Einstein causality: \( R^{\mu\nu} \) vanishes for spacelike argument \( x^2 > 0 \) as follows from the fact that local operators commute at space-like separations. Thus application of an external field at the origin at \( t = 0 \) cannot evoke a response at \( x \) until enough time has elapsed for light to travel from the origin to \( x \). This property is not shared by the time ordered product.
10.1 Retarded Commutators from Time Ordered Products

The Wick expansion we have developed works best for time ordered products, so it is helpful that we can work out a relationship between the response function and the expectation value of the time-ordered product. This relationship is a general one that depends only on the time variable, so we suppress spatial and internal labels and consider two hermitian operators $O_1(t), O_2(t)$. We shall actually relate the Fourier transforms of the two quantities:

$$R(\omega) \equiv i \int dt e^{i\omega t} \theta(t) \langle [O_1(t), O_2(0)] | G \rangle$$  \hspace{1cm} (10.6)

$$T(\omega) \equiv i \int dt e^{i\omega t} \langle G | T[O_1(t)O_2(0)] | G \rangle$$  \hspace{1cm} (10.7)

where $|G\rangle$ is the Ground State of the system, assumed to be nondegenerate. Now using $\theta(t) = 1 - \theta(-t)$ we have

$$\theta(t)[O_1(t), O_2(0)] = T[O_1(t)O_2(0)] - O_2(0)O_1(t)$$  \hspace{1cm} (10.8)

so the difference between $R$ and $T$ involves

$$-i \int dt e^{i\omega t} \langle G | O_2(0)O_1(t) | G \rangle$$  \hspace{1cm} (10.9)

$$= -i \int dt e^{i\omega t} \sum_n \langle G | O_2(0) | n \rangle \langle n | O_1(0) | G \rangle e^{-i(E_G - E_n)t}$$  \hspace{1cm} (10.10)

$$= -2\pi i \delta(\omega) \langle G | O_2(0) | G \rangle \langle G | O_1(0) | G \rangle$$  \hspace{1cm} (10.11)

$$-i \int dt e^{i\omega t} \sum_{n \neq G} \langle G | O_2(0) | n \rangle \langle n | O_1(0) | G \rangle e^{-i(E_G - E_n)t}$$  \hspace{1cm} (10.12)

The important feature of this result is that by virtue of the fact that $E_G$ is the lowest energy eigenvalue, the r.h.s. vanishes for positive frequency $\omega > 0$. Thus in this case $R(\omega) = T(\omega)$.

Next we find a relation for negative frequency. For this case we relate $R$ to the anti-time-ordered product:

$$\theta(t)[O_1(t), O_2(0)] = O_1(t)O_2(0) - \bar{T}[O_1(t)O_2(0)].$$  \hspace{1cm} (10.13)

Now inserting a complete set of states allows us to conclude that the Fourier transform of the first term vanishes for negative frequency $\omega < 0$. Thus in this case we have $R(\omega) = -\bar{T}(\omega)$ where

$$\bar{T}(\omega) \equiv i \int dt e^{i\omega t} \langle G | \bar{T}[O_1(t)O_2(0)] | G \rangle = -T^*(-\omega).$$  \hspace{1cm} (10.14)

where we used the assumption that $O_1, O_2$ are hermitian.
In summary we have found that

\[ R(\omega) = \begin{cases} T(\omega) & \omega > 0 \\ T^*(\omega) & \omega < 0 \end{cases}. \] (10.15)

For \( \omega \) near zero we observe that the ground state contributes to \( T \) but not to \( R \). The contribution to \( T \) is

\[ T_G(\omega) = 2\pi i \delta(\omega) \langle G|O_2(0)|G\rangle \langle G|O_1(0)|G\rangle. \] (10.16)

If there is a gap separating \( E_G \) from the rest of the spectrum this is the only zero frequency discrepancy between \( R \) and \( T \).

Finally when we consider this relationship for field operators, it is natural to quote it for the spatial and temporal Fourier transform:

\[ R(p^0, p) = \begin{cases} T(p^0, p) & p^0 > 0 \\ T^*(-p^0, -p) & p^0 < 0 \end{cases}. \] (10.17)

Again the vacuum contributes to \( T \) (and not \( R \)) the amount

\[ T_G(p^0) = (2\pi)^4 \delta^4(p) \langle 0|O_1|0\rangle \langle 0|O_2|0\rangle. \] (10.18)

### 10.2 Calculation of Vacuum Polarization

If we Fourier transform the current induced by an external field, we obtain

\[ \tilde{J}^\mu(k) = \int d^4x e^{-ikx} \langle in|j^\mu(x)|in\rangle \] (10.19)

\[ = \int d^4x e^{-ikx} \int d^4y R^{\mu\nu}(x - y) A_\nu(y) + O(A^2) \] (10.20)

\[ = R^{\mu\nu}(k) A_\nu(k) + O(A^2) \] (10.21)

and we have just obtained the relation of \( R^{\mu\nu} \) to \( T^{\mu\nu} \).

To calculate \( T^{\mu\nu} \) we first apply the Wick expansion to the time ordered product of four Dirac fields contained in the two current amplitude. There are two distinct contractions corresponding to the diagrams:
The disconnected diagrams vanish because \( \langle 0 | j^\mu | 0 \rangle = 0 \) in the absence of external fields. (This is a simple consequence of charge conjugation invariance.) The unique connected diagram has the value
\[
- Q^2 \text{Tr}[\gamma^\mu S_F(x) \gamma^\nu S_F(-x)]
\]
where the minus sign comes from the single closed fermi loop. Inserting the known Fourier representation for \( S_F \) and carrying out the integration over \( x \) in the evaluation of \( T^{\mu \nu} \) leads to
\[
T^{\mu \nu}(k) = iQ^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left( \gamma^\mu \frac{m - p \cdot \gamma}{m^2 + p^2 - i\epsilon} \gamma^\nu \frac{m - (p - k) \cdot \gamma}{m^2 + (p - k)^2 - i\epsilon} \right). \tag{10.23}
\]

We immediately see from this expression that the integration over momentum is quadratically divergent at high momentum. The origin of this divergence is that \( S_F(x) \) behaves like \( 1/x^3 \) at small \( x \) which means that the two current amplitude behaves like \( 1/x^6 \) which means that its Fourier transform is ill-defined.

Before dealing with this divergence, let us simplify the integrand by first evaluating the trace
\[
N^{\mu \nu}(p, k) \equiv \text{Tr}(\gamma^\mu (m - p \cdot \gamma) \gamma^\nu (m - (p - k) \cdot \gamma)) = 8\eta^{\mu \nu} - 4(p^\mu k^\nu + p^\nu k^\mu) - 4\eta^{\mu \nu}(m^2 + p \cdot (p - k)) \tag{10.24}
\]
and secondly combining denominators using the Feynman trick
\[
\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1 - x)]^2} \tag{10.26}
\]
which is trivial to derive. Then
\[
T^{\mu \nu}(k) = iQ^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{N^{\mu \nu}(p, k)}{[m^2 + (p - kx)^2 - i\epsilon + x(1 - x)k^2]^2} \tag{10.27}
\]
where we have completed the square in the denominator.

Next we do a step which is not quite legitimate in view of the quadratic divergence, which is to change integration variables \( p \to p + kx \). After this the denominator depends only on \( p^2 \), so all terms in the numerator linear in \( p^\mu \) integrate to zero and can be dropped. Furthermore terms of the form \( p^\mu p^\nu \) can be replaced by \( p^2 \eta^{\mu \nu} / 4 \), since \( \int d^4p (p^2)p^\mu p^\nu \) must be proportional to \( \eta^{\mu \nu} \) and the proportionality constant is then determined to be \( \int d^4p (p^2)p^2 / 4 \) by comparing the trace of both sides. Thus we have the replacements
\[
N^{\mu \nu}(p + xk, k) \to 8\left( \frac{p^2}{4} \eta^{\mu \nu} + x^2 k^\mu k^\nu \right) - 8xk^\mu k^\nu - 4\eta^{\mu \nu}(m^2 + p^2 - x(1 - x)k^2)
\]
\[
- 2\eta^{\mu \nu}p^2 + 4x(1 - x)(k^2 \eta^{\mu \nu} - 2k^\mu k^\nu) - 4m^2 \eta^{\mu \nu} \tag{10.28}
\]
After all these steps so far we have reduced the integrals to
\[
T^{\mu \nu}(k) = iQ^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{-2\eta^{\mu \nu}p^2 + 4x(1 - x)(k^2 \eta^{\mu \nu} - 2k^\mu k^\nu) - 4m^2 \eta^{\mu \nu}}{[m^2 + p^2 - i\epsilon + x(1 - x)k^2]^2}. \tag{10.29}
\]
Next we wish to evaluate the $p$ integral. It is easiest to think about this evaluation after the Wick rotation to Euclidean momenta, so the integral is over 4 dimensional Euclidean space and the integrand is $O(4)$ invariant. Then the angular integrals can be done and the integral reduced to a one dimensional one. The rotation of the $p^0$ contour to the imaginary axis must avoid the singularities due to the vanishing of the denominator which occurs at

$$p^0 = \pm \sqrt{m^2 + \mathbf{p}^2 + x(1 - x)k^2 - i\epsilon}$$  \hspace{1cm} (10.30)$$

These poles remain in the fourth and second quadrant of the complex $p^0$ plane for all values of $k^2$. However they get infinitesimally close to the imaginary axis for $x(1 - x)k^2 \leq -m^2 - \mathbf{p}^2$ which we shall see is responsible for singular behavior in the result as a function of $k^2$. As long as we stick to $k^2 > -4m^2$, though, the poles stay well within their respective quadrants, and a counterclockwise contour rotation by 90 degrees encounters no singularities.

After the Wick rotation we change variables to $p^0 = ip^4$ so that $d^4p = id^4p_E$ and $p^2 = \mathbf{p}^2 + (p^4)^2$. Going to polar coordinates, $d^4p_E = p^3d\Omega$, we wish to evaluate the angular integrals $d\Omega$. A useful trick to do this in any number of dimensions is to integrate a Gaussian $e^{-\mathbf{p}^2}$ in both Cartesian and polar coordinates. In Cartesian coordinates in $D$ dimensions we get $\pi^{D/2}$. In polar coordinates it is

$$\Omega_D \int_0^\infty p^{D-1}dpe^{-p^2} = \Omega_D \frac{1}{2}\Gamma(D/2)$$  \hspace{1cm} (10.31)$$

Comparing we arrive at

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}.$$  \hspace{1cm} (10.32)$$

Notice that for $D = 1, 2, 3$ this gives the well known results $2, 2\pi, 4\pi$. For our case $D = 4$
and the result is $2\pi^2$. We also record here the useful formula

$$\int d^D p \frac{(p^2)^m}{(p^2 + A^2)^n} = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dp \frac{p^{D-1+2m}}{(p^2 + A^2)^n}$$

$$= \frac{A^{D+2m-2n} \pi^{D/2} \Gamma(m + D/2) \Gamma(n - m - D/2)}{\Gamma(D/2) \Gamma(n)}.$$  \hspace{1cm} (10.33)

The fact that the r.h.s. is a perfectly defined analytic function of $D, m, n$ allows for dimensional regularization as we shall see later.

Putting all this together and cutting off the $p$ integral at $\Lambda$ we have so far

$$T^\mu_\nu(k) = -\frac{Q^2}{8\pi^2} \int_0^\Lambda p^3 dp \int_0^1 \frac{dx}{[m^2 + p^2 - i\epsilon + x(1-x)k^2]^2} - 2\eta^\mu_\nu(p^2 + 2m^2) + 4x(1-x)(k^2\eta^\mu_\nu - 2k^\mu k^\nu)$$.  \hspace{1cm} (10.34)

The $p$ integrals are now elementary:

$$\int_0^\Lambda \frac{p^3 dp}{[p^2 + C]^2} = \frac{1}{2} \left[ \ln \frac{A^2 + C}{C} - 1 \right] + \frac{1}{2} \frac{C}{A^2 + C}$$

$$= \frac{1}{2} \left[ \ln \frac{A^2}{C} - 1 \right] + O(\Lambda^{-2})$$  \hspace{1cm} (10.35)

$$\int_0^\Lambda \frac{p^5 dp}{[p^2 + C]^2} = A^2 - C \ln \frac{A^2 + C}{C} - \frac{1}{2} \frac{A^4}{A^2 + C}$$

$$= \frac{1}{2} A^2 + \frac{1}{2} C - C \ln \frac{A^2}{C} + O(\Lambda^{-2}),$$  \hspace{1cm} (10.36)

$$\int_0^\Lambda \frac{p^7 dp}{[p^2 + C]^2} = \frac{1}{2} A^2 - C \ln \frac{A^2}{C} + O(\Lambda^{-2}),$$  \hspace{1cm} (10.37)

where for us $C = m^2 + x(1-x)k^2 - i\epsilon$. Putting these results into the expression for $T$ gives

$$T^\mu_\nu(k) = -\frac{Q^2}{8\pi^2} \int_0^1 dx \left[ -\eta^\mu_\nu(\Lambda^2 - C) - 2C(\ln \frac{\Lambda^2}{C} - 1) \right]$$

$$- 2\eta^\mu_\nu m^2 + 2x(1-x)(k^2\eta^\mu_\nu - 2k^\mu k^\nu) \left[ \ln \frac{\Lambda^2}{C} - 1 \right]$$

$$= -\frac{Q^2}{2\pi^2} \int_0^1 dx \cdot x(1-x)(k^2\eta^\mu_\nu - k^\mu k^\nu) \left( \ln \frac{\Lambda^2}{m^2 + x(1-x)k^2 - i\epsilon} - 1 \right)$$

$$+ \frac{Q^2}{2\pi^2} \eta^\mu_\nu \left( \Lambda^2 - m^2 - \frac{k^2}{6} \right).$$  \hspace{1cm} (10.39)

$$+ \frac{Q^2}{8\pi^2} \eta^\mu_\nu \left( \Lambda^2 - m^2 - \frac{k^2}{6} \right).$$  \hspace{1cm} (10.40)

We could do the last integral over $x$, but it is actually easier to see the properties of $T$ directly from the integral representation (10.40).

We have been casual about the way we cutoff the momentum integral, and now we can see a bad consequence of this: a violation of gauge invariance. In Fourier components a gauge transformation on the external field has the form $A_\mu(k) \rightarrow A_\mu + k_\mu \lambda(k)$. The induced current was given by $R^\mu_\nu(k)\lambda_\nu(k)$, so gauge invariance would imply $R^\mu_\nu(k)k_\nu = 0$, and in view of the relation between $R$ and $T$, $T^\mu_\nu(k)k_\nu = 0$. Clearly, the last line of (10.40) fails to
satisfy this condition. The reason for this error can be traced to insufficient care with the manner in which we regularized the divergent integral. It is fortunate that the momentum dependence of the erroneous terms is a simple polynomial. This is in fact a characteristic of all such errors induced by ultraviolet divergences: Differentiating the integral a finite number of times with respect to the external momenta renders it convergent, so a finite number of derivatives must kill the mistake. In this case it would require three derivatives to kill the mistake. We shall later discuss gauge invariant regularization procedures that prevent such mistakes from occurring, provided of course that the theory can be consistently quantized. But for now we shall be satisfied with simply adjusting the polynomial dependence of our results to be consistent with gauge invariance. The nonpolynomial part of $T$ can of course not be removed by such an adjustment. Making this adjustment, and at the same time absorbing the $-1$ in a rescaling of the cutoff, we then obtain

$$T_{G1}^{\mu\nu}(k) = (k^\mu k^\nu - k^2 \eta^{\mu\nu})T(k^2)$$

(10.41)

$$T(k^2) = \frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Lambda^2 e^{-\gamma}}{m^2 + x(1-x)k^2 - i\epsilon}.$$  (10.42)

It is important to appreciate that gauge invariant regularization does not cure the problem of ultraviolet divergences, although it does reduce its severity. Our initial expression for $T$ was quadratically divergent, but we have seen that gauge invariance effectively reduces the divergence to a logarithmic one. We shall see that this last divergence, although present in the quantities we are calculating, disappears after expressing the answer in terms of physically measurable parameters. Note also that our polynomial adjustment of $T^{\mu\nu}$ to make it gauge invariant allows an undetermined constant $\gamma$. However this ambiguity is linked to the cutoff dependence, and will disappear along with the latter in physical quantities.

### 10.3 The Physics of Vacuum Polarization

Our result for $T^{\mu\nu}$ can now be used to give us the response function

$$R_{G1}^{\mu\nu}(k) = (k^\mu k^\nu - k^2 \eta^{\mu\nu})R(k^2)$$

(10.43)

where

$$R(k^2) = \begin{cases} T(k^2) & k^0 > 0 \\
T^*(k^2) & k^0 < 0 \end{cases}.  \quad (10.44)$$

Note that since $T$ is real for $k^2 > -4m^2$, the two cases merge for that range of momentum. Since we have incorporated gauge invariance in our answer, we are free to fix a convenient gauge for discussing the physical interpretation of our result. Let us choose Lorentz gauge, $k^\mu A_\mu = 0$. Then the Fourier transform of the current induced by the external field is simply

$$\langle j_\mu(k) \rangle = -k^2 R(k^2) A_\mu(k).$$

(10.45)
We must now recognize that the induced currents will produce induced fields via Maxwell’s equations. As long as the external field is sufficiently weak the induced currents and the induced field will also be weak, and it will be consistent to assert that the total current is given by the response function times the total field.

\[ \langle j_\mu(k) \rangle_{TOTAL} = -k^2 R(k^2)(\tilde{A}_\mu^e(k) + \tilde{A}_\mu^{IND}(k)). \] (10.46)

It is this total current that we should use in Maxwell’s equations to calculate \( A^{IND} \)

\[ k^2 \tilde{A}_\mu^{IND}(k) = \langle j_\mu(k) \rangle_{TOTAL} = -k^2 R(k^2)(\tilde{A}_\mu^e(k) + \tilde{A}_\mu^{IND}(k)). \] (10.47)

This gives a self-consistent equation for \( A^{IND} \) in terms of the external field.

\[ \tilde{A}_\mu^{IND}(k) = -\frac{R(k^2)}{1 + R(k^2)} \tilde{A}_\mu^e(k). \] (10.48)

Finally, if we add the induced field to the external field, we obtain the total field

\[ \tilde{A}_\mu^{TOT}(k) = \frac{1}{1 + R(k^2)} \tilde{A}_\mu^e(k). \] (10.49)

In summary we have calculated the total electromagnetic field that arises in the “medium” of the Dirac sea in the presence of an externally applied field. The externally applied fields are what are traditionally called the \( \vec{D} \) and \( \vec{H} \) fields. (Recall that the sources of these fields are the external charge and current density respectively.) On the other hand the total fields are traditionally given the name \( \vec{E} \) and \( \vec{B} \). The dielectric “constant” of the medium is defined by \( \vec{D} = \epsilon \vec{E} \) and the magnetic permeability by \( \vec{B} = \mu \vec{H} \). Thus we can interpret our calculation by attributing a \( k \) dependent dielectricity and magnetic permeability to the Dirac sea

\[ \epsilon(k^2) = 1/\mu(k^2) = 1 + R(k^2). \] (10.50)

Since \( R \) is positive for static fields (\( k^0 = 0 \)), the vacuum is a polarizable diamagnetic medium. The fact that \( \epsilon = 1/\mu \) means that the velocity of light is unaltered by the medium (i.e. the medium preserves Poincaré invariance). The effectiveness of the medium in screening external fields is reduced at shorter wavelength (\( R \) decreases as \( k^2 \) increases).

### 10.4 Charge Renormalization

We now come to the resolution of the logarithmic divergence that remains in our expression for \( R(k^2) \). We begin by asking how we measure charge. We seem to have particles and antiparticles of charge \( \pm Q \) in the theory. However this is not the measured charge, the charge we could define by

\[ e^2_{ph} \equiv \lim_{R \to \infty} 4\pi R^2 F(R) \] where \( R \) is the spatial separation between two such charged particles and \( F \) is the force exerted by one on the other. In other words, \( Q \) represents the external or “bare” charge, which acts as source to the \( \vec{D} \) field. The Fourier
component of the latter field is just $-iQk/k^2$. The measured force is given by $Q\vec{E}$ the Fourier component of which is $-iQ^2k/[k^2(1 + R(k^2))]$. The long distance part of the force is controlled by the Fourier components with $k \approx 0$ which are clearly those of a Coulomb force with effective charge squared of

$$e^2 = \frac{Q^2}{1 + R(0)}.$$  \hspace{1cm} (10.51)

Putting $k = 0$ in our expression for $R$ gives us

$$e^2 = \frac{Q^2}{1 + (Q^2/12\pi^2)\ln(\Lambda^2/m^2)}.$$ \hspace{1cm} (10.52)

It is $e$ and not the parameter $Q$ that we measure in experiments, all of which are performed within the “medium” represented by the vacuum. The fine structure constant is $\alpha = e^2/4\pi \approx 1/137$.

If $e$ is the measured charge, then the measured electric field should be related to measured force by

$$\vec{F} = e\vec{E}_{\text{meas}} = Q\vec{E} = \frac{Q}{1 + R(k^2)} \vec{D}$$

$$= \frac{e(1 + R(0))}{1 + R(0) + [R(k^2) - R(0)]/\sqrt{1 + R(0)}} \vec{D}$$

$$= \frac{e}{1 + [R(k^2) - R(0)]/(1 + R(0))} \frac{\vec{D}}{\sqrt{1 + R(0)}}$$ \hspace{1cm} (10.55)

Now since $\vec{D}$ is simply proportional to $Q$, $\vec{D}_{\text{meas}} \equiv \vec{D}/\sqrt{1 + R(0)}$ is what we can call the measured $\vec{D}$ field since it has $Q$ replaced by $e$. Thus we have the following relationship between measured fields

$$\vec{E}_{\text{meas}} = \frac{1}{1 + [R(k^2) - R(0)]/(1 + R(0))} \vec{D}_{\text{meas}}.$$ \hspace{1cm} (10.56)

The measured dielectric constant is accordingly

$$\epsilon(k^2) = 1 + \frac{R(k^2) - R(0)}{1 + R(0)} \approx 1 + \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{m^2}{m^2 + x(1-x)k^2}$$ \hspace{1cm} (10.57)

and we see that the cutoff dependence has disappeared when we express measured quantities in terms of measured parameters. This is what is known as Renormalizability, and is a feature of a wide class of quantum field theories. Such theories encounter infinities in intermediate stages of a given calculation, but the measurable quantities always come out finite.

After renormalization, the dielectric constant is fixed to be 1 at $k = 0$, *i.e.* the medium is effectively absent then. Before, we argued that the effects of the medium should be reduced at large $k$. By going to large $k$ we should begin to see more and more of the bare charge.
We can phrase this by defining a $k$ dependent coupling by $\alpha(k^2) = e^2/4\pi\epsilon(k^2)$. Then as $k^2$ increases from zero, corresponding to shorter distances, $\alpha$ increases until it blows up at some finite value of $k^2$. It’s clear that this will happen because the ln starts out at zero goes negative and behaves monotonically without bound. The pole occurs at a value of $k^2 \sim m^2 \exp(12\pi^2/e^2)$. We have seen how poles in amplitudes are associated with particles. Unfortunately this pole occurs at space-like momentum, i.e. imaginary mass, a tachyon. This is the physically unacceptable Landau Ghost. If it were really present it would signify an inconsistency of electrodynamics. Fortunately, the whole issue is completely open since we have made approximations in our calculation that amount to weak coupling perturbation theory, and as we have seen the effective coupling gets strong at values of $k^2$ much less than the ghost mass, thus invalidating perturbation theory.

One way to understand why this approximation has led to this problem is to return to the bare expressions

$$e^2 = \frac{1}{1/Q^2 + (1/12\pi^2) \ln(\Lambda^2/m^2)}.$$  \hfill (10.58)

Now in the renormalization procedure, we attempt to take $\Lambda \to \infty$ holding $\epsilon$ fixed. But this is only possible if $Q$ is imaginary, which would mean we started with a Hamiltonian which was not hermitian. In order to escape this conclusion, the relation between bare and measured coupling would have to be fundamentally altered by higher order corrections. This is a logical possibility, but many field theorists including Landau doubted that the problem would go away. At this point it is appropriate to mention that for some quantum field theories the sign in front of the logarithmic divergence is opposite to that in QED:

$$g^2 = \frac{1}{1/g_0^2 - b\ln(\Lambda^2/m^2)}, \quad b > 0.$$  \hfill (10.59)

In this case one can take $\Lambda \to \infty$ with $g$ fixed and $g_0$ real. This phenomenon is known as asymptotic freedom since the bare coupling goes to 0 through real values as $\Lambda \to \infty$. These quantum field theories can be renormalized consistently in the weak coupling limit as long as one restricts one’s attention to very large momenta. The other side of the coin is that they become strong coupling theories at low momenta, and so weak coupling approximations are useless for studying their particle spectrum.

### 10.4.1 Pair production

Next we turn to the interpretation of the singularity in $T$ or $R$ as $k^2 \to -4m^2$. At this point the logarithm has a branch point, leading to a discontinuity depending on whether one goes past the branch point in the upper or lower half plane:

$$T_{\pm}(k^2) = \frac{Q^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Lambda^2}{m^2 + x(1-x)k^2}$$

$$\pm i\frac{Q^2}{2\pi} \int_0^1 dx x(1-x) \theta(-x(1-x)k^2 - m^2).$$  \hfill (10.60)
The integral in the second term contributes only if $k^2 < -4m^2$ when the range of $x$ contributing is

$$\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{m^2}{k^2}} < x < \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{k^2}}. \quad (10.61)$$

The integral over that range is elementary and yields

$$T_{\pm}(k^2) = \frac{Q^2}{2\pi^2} \int_0^1 dx x (1-x) \ln \frac{\Lambda^2}{|m^2 + x(1-x)k^2|}$$

$$+ i \frac{Q^2}{2\pi^3} \frac{1}{3} \theta(-k^2 - 4m^2) \sqrt{\frac{1}{4} + \frac{m^2}{k^2}} \left(1 - \frac{2m^2}{k^2}\right). \quad (10.62)$$

The $-i\epsilon$ prescription tells us to choose the lower half plane continuation, i.e. the lower (+) sign is to be taken.

$$T(k^2) = \frac{Q^2}{2\pi^2} \int_0^1 dx x (1-x) \ln \frac{\Lambda^2}{|m^2 + x(1-x)k^2|}$$

$$+ i \frac{\alpha}{3} \theta(-k^2 - 4m^2) \sqrt{1 + \frac{4m^2}{k^2}} \left(1 - \frac{2m^2}{k^2}\right). \quad (10.63)$$

$$R(k^2) = \frac{Q^2}{2\pi^2} \int_0^1 dx x (1-x) \ln \frac{\Lambda^2}{|m^2 + x(1-x)k^2|}$$

$$+ i \frac{\alpha_0}{3} \epsilon(k^0) \theta(-k^2 - 4m^2) \sqrt{1 + \frac{4m^2}{k^2}} \left(1 - \frac{2m^2}{k^2}\right), \quad (10.64)$$

where we have used the definition of the bare fine structure constant $\alpha_0 = Q^2/4\pi$ and the relation between $R$ and $T$.

We can associate the appearance of an imaginary part of $T$ with pair production which becomes energetically possible for energies larger than $2m$. To see the connection recall the variational equation satisfied by the vacuum persistence amplitude

$$\delta \ln \langle \text{out}|\text{in} \rangle = i \int d^4 x \frac{\langle \text{out}|j^\mu(x)|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle} \delta A^\mu(x) \quad (10.65)$$

$$\approx - \int d^4 x d^4 y \langle 0, I|T[j^\mu_I(x)j^\nu_I(y)]|0, I\rangle A^\nu(y) \delta A^\mu(x) \quad (10.66)$$

where we used first order perturbation theory in the external field to approximate the current matrix element. The results of our calculation gave

$$i \langle 0, I|T[j^\mu_I(x)j^\nu_I(y)]|0, I\rangle = \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-y)}(k^\mu k^\nu - k^2 \eta^\mu\eta^\nu)T(k^2), \quad (10.67)$$

so we find, in weak field approximation,

$$\langle \text{out}|\text{in} \rangle \approx \exp \left\{ i \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} T(k^2) \tilde{A}_\mu(-k)(k^\mu k^\nu - k^2 \eta^\mu\eta^\nu) \tilde{A}_\nu(k) \right\}. \quad (10.68)$$
It is illuminating to express this in terms of the Fourier components of the field strengths

\begin{align}
\tilde{F}_{\mu\nu}(k) &= i(k_\mu \tilde{A}_\nu(k) - k_\nu \tilde{A}_\mu(k)) \\
\tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k) &= -2(k_\mu k_\nu - k^2 \eta^{\mu\nu})\tilde{A}_\mu(-k)\tilde{A}_\nu(k) \\
\langle \text{out}|\text{in} \rangle &\approx \exp \left\{ -\frac{i}{4} \int \frac{d^4k}{(2\pi)^4} T(k^2)\tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k) \right\}.
\end{align}

As long as \( T \) is real, this is a pure phase and \( |\langle \text{out}|\text{in} \rangle| = 1 \). But when \( T \) acquires an imaginary part as it does for \( k^2 < -4m^2 \) this is no longer true. Specifically,

\[ |\langle \text{out}|\text{in} \rangle|^2 \approx \exp \left\{ \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \text{Im} T(k^2)\tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k) \right\}, \tag{10.72} \]

which is the probability that there is no pair creation. The probability of pair creation is \( 1 - |\langle \text{out}|\text{in} \rangle|^2 \) and is approximately given by

\[ P_{\text{pair}} \approx -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \text{Im} T(k^2)\tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k) \tag{10.73} \]

\[ \approx -\int \frac{d^4k}{(2\pi)^4} \frac{\alpha}{6} \theta(-k^2 - 4m^2) \sqrt{1 + \frac{4m^2}{k^2}} \left( 1 - \frac{2m^2}{k^2} \right) \tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k), \]

where we have replaced the bare fine structure constant \( \alpha_0 = \alpha(1 + R(0)) \rightarrow \alpha \) which is correct to the order we are calculating\(^1\).

Note that one could also find this result by calculating \( |\langle \text{out}|b^{\text{out}}d^{\text{out}}|\text{in} \rangle|^2 \) directly in perturbation theory and summing over all final states. It is of course important that this probability be positive (and \( |\langle \text{out}|\text{in} \rangle| < 1 \)). Indeed, \( \tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k) \) is negative for \( k \) timelike, because then there is a Lorentz frame where \( k = 0 \) which implies \( B = 0 \) so \( \tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k) = -2|\tilde{E}|^2 \). Since it is an invariant, it must be negative in all frames.

When the fields have support only where \( T \) is real, the amplitude \( \langle \text{out}|\text{in} \rangle \), even though it is a pure phase, gives information about the energy of the system in the presence of external fields. To get this connection consider a static field with adiabatic switching off at early and late times:

\[ F_{\mu\nu}(x) = F_{\mu\nu}(x)\epsilon(t). \tag{10.74} \]

We take \( \epsilon \) to be a symmetric function of \( t \) with central value 1 and gradual fall off to zero at times roughly \( \pm T \). The exact shape is unimportant: we only need that \( \epsilon(k^0) \) peaked at 0 with a width of order \( 1/T \). Clearly

\[ \int dk^0 \epsilon(k^0)^2 = 2\pi \int dt \epsilon(t)\epsilon(-t) \approx 4\pi T. \tag{10.75} \]

\(^1\)Strictly speaking, this calculation of \( \langle \text{out}|\text{in} \rangle \) is for a fixed external field for which \( QA \) is fixed and finite. Renormalization applies only when the induced fields are included. In that case the \( \tilde{F} \) appearing in these formulas are the total fields, and for them \( Q\tilde{F} \) is indeed held fixed as the cutoff is removed. That is, \( \sqrt{1 + R(0)\tilde{F}} \) is the measured field strength.
Thus in the limit $T \to \infty$, $\tilde{\epsilon}^2$ can be approximated by $4\pi T \delta(k^0)$. Thus in this limit our approximate formula for $\langle \text{out}|\text{in} \rangle$ reads

$$\langle \text{out}|\text{in} \rangle \approx \exp \left\{ -2iT \int \frac{d^3k}{(2\pi)^3} \frac{1}{4} \tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k)T(k^2) \right\}. \quad (10.76)$$

The coefficient of $-2iT$ is just the energy of the Dirac system in the presence of static external fields. Since $T$ is positive for spacelike $k$, and

$$\frac{1}{4} \tilde{F}_{\mu\nu}(-k)\tilde{F}^{\mu\nu}(k) = \frac{1}{2} [ |\tilde{B}|^2 - |\tilde{E}|^2 ] \quad (10.77)$$

we see that the energy increases under the addition of a magnetic field and decreases with an electric field in accord with our conclusion that the vacuum is a diamagnetic dielectric medium.

### 10.5 Superconductivity and the Higgs Mechanism

There is one physical phenomenon which can occur in quantum gauge field theories, although not for $QED$ in four dimensions. It can happen in some theories that $R(k^2)$ possesses a pole at zero:

$$R(k^2) \sim \frac{K}{k^2}, \quad k^2 \to 0. \quad (10.78)$$

Note that this infrared singularity does not occur in our previous calculation even for $m = 0$. But if it does occur, then

$$\tilde{A}(k)_{TOT} = \frac{1}{1 + R(k^2)} \tilde{A}_e \sim \frac{k^2}{K} \tilde{A}_e \sim 0, \quad k^2 \to 0. \quad (10.79)$$

Thus $A_{TOT}$ is screened at long wavelengths by the induced currents. In particular, for a static Coulomb potential $\tilde{A}^0 \sim \delta(k^0)/k^2$, the singularity at vanishing $k$ disappears so in coordinate space the potential falls off faster than any power. Because of Lorentz covariance this screening is effective for both electric and magnetic fields. The screening of the magnetic fields means there is a Meissner effect, i.e. the vacuum in this situation is a relativistic superconductor. The vanishing of the total field at $k^2 = 0$ means that the vacuum cannot support massless photons. On the other hand there most likely is a negative value of $k^2$ call it $-M^2$ for which $1 + R(-M^2) = 0$. For such values of $k$, $A_{TOT}$ can be nonzero even for vanishing external field. These waves correspond to particles of mass $M$. Thus this phenomenon, sometimes called the Higgs mechanism, gives the photon a mass without violating gauge invariance. A nonrelativistic version of this effect was long known for superconductors. It was first discussed in the context of relativistic quantum field theory by Schwinger for QED in two space-time dimensions. Four dimensional versions were first discussed by Higgs, Englert and Brout, and Guralnik, Hagen and Kibble. In spite of the long list of discoverers, it seems that Higgs’ name has stuck. The Higgs mechanism is at the heart of the electroweak unified gauge theory, because it is responsible for the masses of the $W$ and $Z$ bosons.

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10.6 Correction to the Photon Propagator

Although our discussion of vacuum polarization assumed classical vector potentials, we can adapt it to the one loop correction to the photon propagator in QED. For this purpose, it is the $T^{\mu\nu}$ form that is relevant. Let’s anticipate the Feynman gauge choice for the free photon propagator $-i\eta_{\mu\nu}/(k^2 - i\epsilon)$, which we break up into transverse and longitudinal pieces:

$$D_{\mu\nu}(k) = -i\eta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} - \frac{i}{k^4}k_{\mu}k_{\nu}$$

(10.80)

Then the one loop correction is
Chapter 11

Path History Quantization

One of the principal drawbacks of the canonical operator formulation of quantum mechanics is that it obscures symmetries that bring in time in an essential way. Lorentz boosts are of this type, so the operator approach inevitably hides the full symmetries of relativity. This is of course also true in the Hamilton equation form of classical mechanics. In classical mechanics one can work with the Lagrangian and Action Principle which keep dynamical symmetries like Poincaré transparent. The path integral approach to quantum mechanics is the quantum analogue of this alternative and, as we shall see, is a much more convenient formulation of quantum field theory than the operator approach. Even in the operator approach we have seen the advantage of expressing results in terms of finite time evolutions, since it is these that reflect the true symmetries of the system. The central object in the path integral approach then is not a state but an amplitude for the evolution of one state into another.

To keep notation simple, we shall suppress indices in dealing with a general quantum mechanical system the coordinates of which are collectively denoted \( q \) and the conjugate momenta of which are \( p \). Then we seek an alternative scheme for calculating, for example, the amplitude

\[
\langle q''|U(t_1, t_2)|q'\rangle. \tag{11.1}
\]

\( U \) is of course a very complicated operator for finite time differences, but as \( t_1 \to t_2 \), it is simply related to the Hamiltonian.

Thus we are led to break up the time interval into infinitesimal pieces \( t_1 - t_2 = (N + 1)a \) and employ the closure property of \( U \) to write it as a product of \( N + 1 \) infinitesimal evolutions.

\[
\langle q''|U(t_1, t_2)|q'\rangle = \langle q''|U(t_1, t_1 - a)U(t_1 - a, t_1 - 2a) \cdots U(t_2 + a, t_2)|q'\rangle \tag{11.2}
\]

\[
= \int \prod_{k=1}^{N} dq_k \langle q''|U(t_1, t_1 - a)|q_N\rangle \langle q_N|U(t_1 - a, t_1 - 2a)|q_{N-1}\rangle \cdots \langle q_1|U(t_2 + a, t_2)|q'\rangle. \tag{11.3}
\]
Next, we write, assuming $H_{S}$ is constant over a time interval $a$,

$$\langle q_{k}|U(t_{2} + ka, t_{2} + (k - 1)a)|q_{k-1}\rangle$$

$$\approx \int dp_{k-1}\langle q_{k}|e^{-iaH(t_{2} + (k-1/2)a)/2}|p_{k-1}\rangle \langle p_{k-1}|e^{-iaH(t_{2} + (k-1/2)a)/2}|q_{k-1}\rangle$$

$$\equiv \int dp_{k-1}\exp\left\{-\frac{i}{2\hbar}a(\mathcal{H}_{k}(q_{k}, p_{k-1}, ia) + \mathcal{H}_{k}(q_{k-1}, p_{k-1}, -ia^{*}))\right\}$$

$$\langle q_{k}|p_{k-1}\rangle \langle p_{k-1}|q_{k-1}\rangle$$

$$= \int \frac{dp_{k-1}}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}[p_{k-1}(q_{k} - q_{k-1}) - \frac{a}{2}(\mathcal{H}_{k}(q_{k}, p_{k-1}, ia) + \mathcal{H}_{k}(q_{k-1}, p_{k-1}, -ia^{*}))]\right\}$$

where for the moment $\mathcal{H}_{k}$ is defined by these equations. Putting everything together we obtain

$$\langle q''|U(t_{1}, t_{2})|q'\rangle = \int \prod_{k=1}^{N} \frac{dq_{k}dp_{k}}{2\pi\hbar} \frac{dp_{0}}{2\pi\hbar}\exp\left\{\frac{i}{\hbar} \sum_{k=1}^{N+1}[p_{k-1}(q_{k} - q_{k-1}) \right.$$}

$$- \frac{a}{2}(\mathcal{H}_{k}(q_{k}, p_{k-1}, ia) + \mathcal{H}_{k}(q_{k-1}, p_{k-1}, -ia^{*}))\right\},$$

where $q_{N+1} \equiv q''$ and $q_{0} \equiv q'$. Apart from the assumption that external fields are constant over the time interval $a$ this formula is exact. But it is not useful until we get a simple approximation for $\mathcal{H}_{k}$. For $a \to 0$ we should be able to approximate

$$\langle q_{k}|e^{-iaH(t_{2} + (k-1/2)a)/2}|p_{k-1}\rangle \approx \langle q_{k}|(1 - i\frac{a}{2}H(t_{2} + (k - 1/2)a))|p_{k-1}\rangle$$

$$\approx (1 - i\frac{a}{2}H_{k}^{W}(q_{k}, p_{k-1})) \langle q_{k}|p_{k-1}\rangle$$

$$\approx \exp\{-i\frac{a}{2}H_{k}^{W}(q_{k}, p_{k-1})\} \langle q_{k}|p_{k-1}\rangle$$

where $H_{k}^{W}(q, p)$ is the operator $H(t_{2} + (k - 1/2)a)$ rewritten through use of the canonical commutation relations with all $p$'s on the right and all $q$'s on the left. After this is done $q$ can then be replaced with the eigenvalue $q_{k}$ and $p$ with the eigenvalue $p_{k-1}$. In the limit $a \to 0$ with $t_{1} - t_{2} = (N + 1)a$ fixed it should be valid to replace $\mathcal{H}_{k}(q_{k}, p_{k-1})$ with $H_{k}^{W}(q_{k}, p_{k-1})$. We define the quantity

$$H_{k}^{Q}(q_{k}, q_{k-1}, p_{k-1}) = \frac{1}{2}(H_{k}^{W}(q_{k}, p_{k-1}) + H_{k}^{W}(q_{k-1}, p_{k-1}))$$

which appears in the path integral. Up to the reordering terms $H_{k}^{W}$ is just the classical Hamiltonian for the system. In the common case where the Hamiltonian is a function of $p$'s plus a function of $q$'s it is nothing more nor less than the classical Hamiltonian. But notice that the object appearing in the path integral is (11.9) which depends on the two coordinates.
describing the initial and final states of the basic unit of propagation even in the case where there are no reordering terms, when it is simply the average of the classical Hamiltonian over the two coordinates. In this continuum limit we can think of the sum in the exponent in (11.5) as an integral

$$i \frac{\hbar}{\hbar} \int_{t_1}^{t_2} dt [p(t) \dot{q}(t) - \frac{1}{2}(H^W(q_>(t), p(t), t) + H^W(q_<(t), p(t), t)^*)].$$  \hspace{1cm} (11.10)

The coefficient of $i/\hbar$ is just the classical action $\int dt L$ plus terms of order $\hbar^2$ expressed as a Legendre transform of the Hamiltonian. In the continuum limit, the number of integration variables tends to infinity and the limit gives the definition of the path integral representation of $\langle q_f | U(t_1, t_2) | q_i \rangle$.

We summarize by suppressing indices and writing the path integral formula as

$$\langle q'' | U(t_2, t_1) | q' \rangle = \int Dq(t) Dp(t) \exp \left\{ \int_{t_1}^{t_2} dt [p \dot{q} - H(p, q)] \right\},$$  \hspace{1cm} (11.11)

where the integral is over all $p(t)$ and all $q(t)$ satisfying $q(t_1) = q'$, $q(t_2) = q''$. If $H$ is quadratic in the $p$'s, the $p$ integral can be performed leading to

$$\langle q'' | U(t_2, t_1) | q' \rangle = K \int Dq(t) \exp \left\{ \int_{t_1}^{t_2} dt L(q, \dot{q}) \right\}.$$  \hspace{1cm} (11.12)

### 11.1 The Classical Limit

For a general Hamiltonian, this is as far as one can go without further approximations. One such approximation one can always try is the limit $\hbar \rightarrow 0$, the classical limit. Such a limit is dominated by the functions $q(t), p(t)$ for which the coefficient of $i/\hbar$ is stationary. Since, in this limit, this coefficient is just the classical action, the stationarity conditions are simply the classical Hamilton equations: $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$. Thus the path history version of the quantum principle is that for $\hbar \neq 0$ transition amplitudes are computed by evaluating $e^{i(\text{Action})/\hbar}$ for all possible histories and averaging this expression over all such histories. The classical limit is understood as the situation in which this average is dominated by solutions of the classical equations of motion.

### 11.2 Imaginary Time.

In working with the path integral it is technically advantageous to work with actually damped integrands rather than the oscillating integrand occurring in the quantum path integral. This can be achieved with the Wick rotation $it = \tau$ where real positive $t$ is rotated to real positive $\tau$. Considering the basic unit of the path integral, the matrix element of the operator, $e^{-iaH/\hbar}$, we see that this rotation is mathematically justified when $H$ is an operator bounded below, i.e. its eigenvalue spectrum is bounded below. It obviously should not be attempted if $H$ has
eigenvalues down to $-\infty$. Fortunately, most reasonable physical systems have this property, and for these the Wick rotated path integral is the superior one to work with, especially for applications outside of perturbation theory. For a constant Hamiltonian (no external fields) this path integral calculates $\langle q_f | e^{-\beta H} | q_i \rangle$ where $\beta = (\tau_1 - \tau_2)/\hbar$. If we identify $q_f = q_i = q$ and integrate over $q$, it calculates $\text{Tr} e^{-\beta H}$, the statistical mechanical partition function for temperature $1/\beta$. In this way the Wick rotated path integral is related to a quantity of direct physical interest in another context. In the limit $\beta \to \infty$ (low temperature) one obtains information about the energy levels and degeneracies of the system. However, for applications to quantum mechanics it is necessary to continue back to real time at the end of the calculation of physical transition amplitudes.

Technically the Wick rotation amounts to replacing $ia$ by $\delta > 0$ in (11.5). Thus one obtains, after approximating $\mathcal{H}_k$ by $H^W_k$,

$$
\langle q'' | U(-i\tau_1, -i\tau_2) | q' \rangle \approx \int \prod_{k=1}^{N} \frac{dq_k dp_k dp_0}{2\pi\hbar} \exp\left\{ \frac{1}{\hbar} \sum_{k=1}^{N+1} [ip_k(q_k - q_{k-1}) - \frac{\delta}{2}(H^W_k(q_k, p_{k-1}) + H^W_k(q_{k-1}, p_{k-1}))]\right\}. \tag{11.13}
$$

The Wick rotated version of the quantum action (11.10) is of course

$$
\frac{1}{\hbar} \int_{\tau_1}^{\tau_2} d\tau [i p(\tau) \dot{q}(\tau) - \frac{1}{2}(H^W(q_>(\tau), p(\tau), \tau) + H^W*(q_<(\tau), p(\tau), \tau))]. \tag{11.14}
$$

Here we have identified $q(t) \equiv q(\tau)$ and the complex conjugation in $H^W*$ ignores the $i$'s coming from the Wick rotation.

### 11.3 Matrix Elements of Time Ordered Products.

For $T > t_1 > \cdots > t_n > -T$

$$
\langle q_f | U(T, t_1) q_1 U(t_1, t_2) q_2 \cdots q_n U(t_n, -T) | q_i \rangle = \langle q_f | U(T, -T) T [q_1(t_1) q_2(t_2) \cdots q_n(t_n)] | q_i \rangle \tag{11.15}
$$

where the Heisenberg picture operators $q_k(t_k) \equiv U^{-1}(t_k, -T) q_k U(t_k, -T)$ are defined so that Heisenberg and Schrödinger pictures agree at $t = -T$. Working with the l.h.s. of this relation we can insert a complete set of coordinate basis states between each pair of $U$'s and then replace each operator $q_k$ by its eigenvalue and each matrix element of $U$ by its path integral representation. The integrals over the basis labels $q_k$ then simply extend the sum over piecewise histories $q_i \to q_n \to \cdots \to q_1 \to q_f$ to the sum over all histories $q_i \to q_f$. Thus we arrive at

$$
\langle q_f | U(T, -T) T [q_1(t_1) q_2(t_2) \cdots q_n(t_n)] | q_i \rangle = \int \mathcal{D}q \mathcal{D}p [\tau_1] \cdots [\tau_n] \exp\left\{ \frac{1}{\hbar} \int_{-T}^{T} d\tau [i p(\tau) \dot{q}(\tau) - H^Q(q(\tau), p(\tau), \tau)]\right\}. \tag{11.16}
$$
where we have used the Wick rotated version with $iT = T$. Written in this way with the
time ordering symbol on the l.h.s. this formula is valid for any time ordering.

In field theory we are really interested in the vacuum (ground state) expectation value of
time ordered products. These can be obtained by taking the limit $T \to \infty$. Then, inserting
energy eigenstates at the left and right, all states but the ground states are exponentially
suppressed so we have

$$\langle q_f|U(\infty, -\infty)T[q_1(t_1)q_2(t_2)\cdots q_n(t_n)]|q_i\rangle$$

$$= \langle q_f|0\rangle \langle 0|U(\infty, -\infty)T[q_1(t_1)q_2(t_2)\cdots q_n(t_n)]|0\rangle \langle 0|q_i\rangle$$

$$= \langle q_f|0\rangle \langle 0|q_i\rangle \langle out|T[q_1(t_1)q_2(t_2)\cdots q_n(t_n)]|in\rangle. \quad (11.17)$$

The wave functions that multiply the desired result can be obtained from

$$\langle q_f|U_{Ext=0}(\infty, -\infty)|q_i\rangle = e^{-2i\infty E_G} \langle q_f|0\rangle \langle 0|q_i\rangle, \quad (11.18)$$

which of course has its own path integral representation. Thus by division we obtain, defining
energy so that $E_G = 0$,

$$\langle out|T[q_1(t_1)q_2(t_2)\cdots q_n(t_n)]|in\rangle$$

$$= \int DqDp \ p(\tau) \cdots q(\tau) \exp\left\{ \frac{1}{\hbar} \int_{-\infty}^{\infty} d\tau [i\hbar p(\tau) \dot{q}(\tau) - H^Q(q(\tau), p(\tau), \tau)] \right\}$$

$$\int DqDp \ \exp\left\{ \frac{1}{2} \int_{-\infty}^{\infty} d\tau [i\hbar p(\tau) \dot{q}(\tau) - H^Q_{Ext=0}(q(\tau), p(\tau))] \right\}. \quad (11.19)$$

### 11.4 Coordinate Space Path Integral.

A strong motivation for using the path integral formulation for quantum field theory is that
it makes possible a more symmetrical treatment of space and time. This symmetry is even
more striking after the Wick rotation when the Lorentz group $O(3, 1)$ becomes simply $O(4)$
the group of rotations in four dimensions. To achieve the full force of this benefit though we
would like to be able to use the configuration space action $\int dtL(q, \dot{q}, t)$ rather than the phase
space one. We can do this provided it is possible to “integrate out” the conjugate momenta
$p$. This is generally possible in quantum field theory because field theoretic Hamiltonians
typically only depend on the conjugate momenta $p$. Thus the integral over the $p$ is gaussian and can be explicitly performed. In the language
of quantum mechanics, the field theoretic Hamiltonian is always of the form $p^2/2 + V(q) +
[f(q)p + pf(q)]/2 = p^2/2 + V(q) + f(q)p - i\hbar f'(q)/2$. In this case we can integrate out the $p$’s even before the continuum limit which converts ordinary integrals to path integrals:

$$\int \frac{dp_{k-1}}{\sqrt{2\pi\hbar}} \exp\left\{ \frac{1}{\hbar} [ip_{k-1}(q_k - q_{k-1}) - \delta \left( \frac{1}{2} p_{k-1}^2 + \frac{1}{2} (V(q_k) + V(q_{k-1})) \right) + p_{k-1}(f(q_k) + f(q_{k-1}) - i\hbar \left( f'(q_k) - f'(q_{k-1})) \right)] \right\}. \quad (11.20)$$

$$= \frac{1}{\delta^{1/2}} \exp\left\{ -\frac{1}{\hbar} \left( q_k - q_{k-1} + \frac{i\hbar}{2} \left( f(q_k) + f(q_{k-1}) \right) \right)^2 \right\} \cdot \frac{\delta}{2} (V(q_k) + V(q_{k-1}) - i\hbar \left( f'(q_k) - f'(q_{k-1}) \right)). \quad (11.21)$$

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So that the path integral expression (11.13) becomes

\[
\langle q'' | U | q' \rangle \approx \left( \frac{1}{2\pi \hbar \delta} \right)^{(N+1)/2} \int \prod_{k=1}^{N} dq_k \exp \left\{ -\frac{1}{\hbar} \sum_{k=1}^{N+1} \left[ \frac{(q_k - q_{k-1} + \frac{i\delta}{2}(f(q_k) + f(q_{k-1}))^2}{2\delta} \\
+ \frac{\delta}{2}(V(q_k) + V(q_{k-1}) - \frac{i\hbar}{2}(f'(q_k) - f'(q_{k-1})) \right] \right\} \int Dq \exp \left\{ -\frac{1}{\hbar} \int_{\tau_1}^{\tau_2} d\tau \left[ \frac{1}{2} \left( \frac{dq}{d\tau} + \frac{i}{2}(f(q_k) + f(q_{k-1})) \right)^2 + V(q(\tau)) \right] \right\}.
\]

(11.22)

Note that the (divergent) prefactor is necessary to obtain the same result for the evolution amplitude as with the usual operator formalism, and it naturally appears when we start the path history formulation in phase space. However, notice also that the dependence on the evolution time \((N+1)\delta\) is exactly of the form that would come from adding a constant \(- (1/2\delta) \ln(2\pi \hbar \delta)\) to the overall energy of the system. Thus since only energy differences are measurable, the physics is insensitive to the presence of this factor. In quantum field theory we could lump this contribution into the zero point vacuum energy we are supposed to subtract in any case. We recognize the coefficient of \(1/\hbar\) in the exponent as the (imaginary time) classical action for the system. It is this configuration space path integral that gives quantum field theory its neatest expression. Then the classical action has the form \(\int d^4x \mathcal{L}(\phi, \partial_\mu \phi)\) with \(\mathcal{L}\) a Lorentz scalar field. The configuration variables of quantum field theory are the fields \(\phi(x, t)\) one for each point in space-time. To define the sum over histories of fields one therefore needs a lattice in space-time. In the Wick-rotated version this could be taken, for example, to be a hypercubic lattice in 4 dimensional Euclidean space. The path integral defined via such a lattice can be taken as the definition of quantum field theory. We shall find that perturbation theory can be developed directly from the continuum path integrals, essentially because one can avoid the actual evaluation of the integrals by various tricks.

### 11.5 Gaussian Integrals

When evaluating ground state averages of physical quantities in perturbation theory, one can manage to avoid ever having to do a functional integral. This is because the free field functional integral will cancel between numerator and denominator after extracting the source dependence. However there are cases where one needs to know the numerator (or denominator) separately, for example, when one uses the path integral representation of the partition function. Since the free field integral is simply gaussian, we can in fact calculate it.
We start by noting that the general multivariable gaussian ordinary integral is given by

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} \left( \frac{dx_i}{\sqrt{2\pi}} \right) e^{-\frac{1}{2} \sum_{km} x_k M_{km} x_m} = \prod_{i} m_i^{-1/2} = \det^{-1/2} M,$$

(11.23)

where $m_i$ are the eigenvalues of the real symmetric matrix $M$. If we always define Euclidean functional integrals in terms of a lattice, this result can be directly applied. Then after taking the continuum limit, we can write for the neutral scalar field

$$\int \mathcal{D}\phi \exp \left\{ -\int d^4x \left[ \frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 \right] \right\} \equiv \det^{-1/2}[m^2 - \partial^2].$$

(11.24)

A charged scalar field can be decomposed $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ in terms of two real scalar fields so the corresponding formula is

$$\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ -\int d^4x_E \left[ (\partial \phi)^* \partial \phi + m^2 \phi^* \phi \right] \right\} \equiv \det^{-1}[m^2 - \partial^2].$$

(11.25)

We have already encountered determinants of differential operators in our study of external field problems, for example, the $\text{out in}$ matrix element for a charged scalar field in the presence of an external gauge field is proportional to $\det^{-1}[m^2 - (\partial - iQA)^2]$. Thus we can immediately write the path history representation for this matrix element:

$$\langle \text{out} | \text{in} \rangle_A = \frac{\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ -\int d^4x_E \left[ (\partial + iQA)\phi^*(\partial - iQA)\phi + m^2 \phi^* \phi \right] \right\}}{\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ -\int d^4x_E \left[ \partial \phi^* \cdot \partial \phi + m^2 \phi^* \phi \right] \right\}}.$$  

(11.26)

Since this is what must be inserted into the gauge field path integral to couple gauge fields to charged fields, this completes the process for converting to path integration language the qft of scalar fields interacting with gauge fields. To get $\text{out in}$ matrix elements of time ordered products of fields we use the generating functional

$$\langle \text{out} | T[e^{i \int d^4x (J^* \phi + \phi^* J)}] | \text{in} \rangle \quad \text{(11.27)}$$

$$= \frac{\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ -\int d^4x_E \left[ (\partial + iQA)\phi^*(\partial - iQA)\phi + m^2 \phi^* \phi - J^* \phi - \phi^* J \right] \right\}}{\int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left\{ -\int d^4x_E \left[ \partial \phi^* \cdot \partial \phi + m^2 \phi^* \phi \right] \right\}} \quad \text{(11.28)}$$

We have written the functional integrals in Euclidean space, which we indicate with the $E$ subscript. To express the results in terms of Minkowski space, just use $d^4x_E = id^4x$. Remember that $\Delta_F(x, y; A)$ is precisely the continuation back to Minkowski space of the
Euclidean Green function $\Delta_E(x_E, y_E; A)$. To confirm the last equality one simply completes the square and changes variables in the by now familiar way. Doing this in Euclidean space leads to the exponent $\int d^4x_E d^4y_E J^*(x) \Delta_E(x_E, y_E; A) J(y)$, which, when continued back to Minkowski space, gives the result shown.

We know from our experience with the operator formulation that the boundary conditions that fix $\Delta_F$ in Minkowski space are the requirements that it contains only positive frequencies at very late times and only negative frequencies at very early times. In Euclidean space, the boundary condition on $\Delta_E$ is simply that it vanish at infinity. Since $\Delta_F, \Delta_E$ are analytic continuations of one another, either statement of the boundary conditions is satisfactory. From the path integral point of view it is more natural to choose the Euclidean version.
Chapter 12
Path Integrals for Gauge Fields

The path integral for gauge theories presents special problems because of the gauge invariance of the action. We shall establish the proper formulation for Quantum Electrodynamics by first obtaining the path integral in Coulomb gauge $\nabla \cdot A = 0$, following the general procedure sketched in this chapter. Once that has been done, we can discuss within the new path integral formalism more general gauges, including covariant ones.

We have already discussed the quantization of the free EM field in Coulomb gauge. In that case Gauss’ law $\nabla \cdot E = 0$ implied that $A_0 = 0$. In the presence of a current $j^\mu$, this conclusion is modified:

$$\nabla \cdot E = -\nabla^2 A_0 = j^0, \quad \nabla \cdot A = 0 \quad (12.1)$$

$$A_0(x,t) = \int d^3 y \frac{j^0(y,t)}{4\pi|x-y|} \quad (12.2)$$

Then the contribution of the electric field to the Hamiltonian is

$$\frac{1}{2} \int d^3x E^2 = \frac{1}{2} \int d^3x \left[ E_T^2 + (\nabla A^0)^2 \right] = \frac{1}{2} \int d^3x \left[ E_T^2 - A^0\nabla^2 A^0 \right]$$

$$= \frac{1}{2} \int d^3x \left[ E_T^2 - A^0 j^0 \right]$$

$$= \frac{1}{2} \int d^3x E_T^2 + \frac{1}{2} \int d^3x \frac{j^0(x,t)j^0(y,t)}{4\pi|x-y|} \quad (12.3)$$

The Hamiltonian operator for charged fields interacting with the quantized e.m. field $A$ in Coulomb gauge is

$$H_{eff} = \int d^3x \left( \frac{1}{2} \Pi_T^2 + \frac{1}{2} \partial_k A_T \cdot \partial_k A_T - A_T \cdot J_e \right)$$

$$+ \int (J^0_e + j^0) \left( -\frac{1}{2\nabla^2} \right) (J^0_e + j^0) + H_{fields}(A)|_{A_0=0}, \quad (12.4)$$

Here we have added an external current $j^\mu \rightarrow j^\mu + J^\mu_e$, and we understand all operators to be in Schrödinger picture. We also used the fact that $A^0$ appears linearly in both the Dirac field
and charged scalar field in the form $\int d^3x A^0 j^0$. For the Dirac field $j^0 = Q \bar{\psi} \gamma^0 \psi$ manifestly has no $A$ dependence, whereas for the scalar field $j^0 = iQ(\pi \phi - \pi^\dagger \phi^\dagger)$ has no $A$ dependence after $\dot{\phi}$ has been expressed in terms of $\pi^\dagger$.

We see that this is a case where the conjugate momentum appears only quadratically so we can immediately transcribe an infinitesimal evolution between eigenstates of $A$ near imaginary time $\tau$.

\begin{equation}
\langle A''|e^{-d\tau H_{eff}}|A'\rangle \approx \delta(\nabla \cdot A'') \exp\left\{-d\tau \int d^3x \left[\frac{1}{2}(\dot{A})^2(\tau) + \frac{1}{2}(\partial A)^2(\tau) - A(\tau) \cdot J_e\right]\right\}
\end{equation}

\begin{equation}
-\frac{1}{2\sqrt{2}} \left( j^0_e + j^0 \right) \left( j^0_e + j^0 \right) - d\tau H_{fields}(A(\tau)|_{A_0=0}),
\end{equation}

where the terms involving $A(\tau)$ with no time derivative are averaged over $A(\tau) = A''$, $A'$ and $\dot{A}(\tau)$ symbolizes the continuum limit of $(A'' - A')/d\tau$. The delta function multiplying the r.h.s. symbolizes the condition $\nabla \cdot A = 0$. It is necessary if we wish to formally integrate over all three components of $A$. Notice that for the moment we are only changing the e.m. field into path integral language, leaving the charged fields as operators.

Next we employ a widely used trick for simplifying the term describing the instantaneous Coulomb interaction. This involves introducing an auxiliary variable $A^4(x, t)$ and writing the identity

\begin{equation}
\exp\left\{-d\tau \int (j^0_e + j^0) \left( -\frac{1}{2\sqrt{2}} \right) (j^0_e + j^0)\right\} = \frac{\int DA^4(\tau) \exp\left\{-d\tau \int d^3x \left[\frac{1}{2}(\nabla A^4)^2 - A^4(j^0_e + j^4)\right]\right\}}{\int DA^4(\tau) \exp\left\{-d\tau \int d^3x \frac{1}{2}(\nabla A^4)^2\right\}}
\end{equation}

where $j^4 = ij^0$ and $J^4_e = ij^0_e$ as appropriate after the Wick rotation. This identity is proved by completing the square by a shift of integration variable $A^4 \to A^4 - (1/\nabla^2)j^4$. Here we use the fact that ultimately $d\tau \to 0$ to neglect any commutators that might arise due to the fact that $j^0$ is really an operator. (This latter approximation is in exactly the same spirit as the replacement of $H_k$ by $H^W_k$ in our general derivation of the path integral.) The variable $A^4$ enters now exactly as the imaginary time component of the vector potential would have entered before it was eliminated by solving the Gauss’ law constraint. For example, using $\nabla \cdot A = 0$ we can write

\begin{equation}
\int d^3x \left[\frac{1}{2} \dot{A}^2 + \frac{1}{2}(\nabla A_k)^2 - A \cdot J_e\right] + \frac{1}{2}(\nabla A^4)^2 - A^4 J^4_e = \int d^3x \left[\frac{1}{4} F_{\mu \nu} F_{\mu \nu} - A^4 J^4_e\right].
\end{equation}

Furthermore $H_{fields}|_{A^4=0} - \int d^3x A^4(\tau)j^4 = H_{fields}$ with $A^4 \neq 0$ playing the role of the imaginary time component of the vector potential.
Composing many infinitesimal evolutions, we arrive at the path integral representation

$$
\langle A_f | U(\infty, -\infty) | A_i \rangle = \frac{\int D A_\mu \delta(\nabla \cdot A) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_\mu^i)} \langle f | T e^{-\int d\tau H_{\text{fields}}(A_\mu(\tau))} | i \rangle}{\int D A_\mu e^{-\int d\tau (\nabla A^2)/2}}.
$$

(12.8)

(12.9)

As discussed before, by taking the evolution over an infinite time interval we effectively project onto the ground states (vacuum) at early and late times. We can therefore pick convenient initial and final states, e.g. we can let them be eigenstates of \(A\) with vanishing eigenvalue and ground states of \(H_{\text{fields}}(A_\mu(\tau = \pm\infty), \mp\infty)\) respectively. We then obtain the following general formula for the \(\text{out in}\) matrix element of time ordered products of fields

$$
\langle \text{out} | T[\psi(y_1) \cdots \psi(y_m) \bar{\psi}(z_m) \cdots \bar{\psi}(z_1)] | \text{in} \rangle_{J_e, A_\mu} = \frac{\int D A_\mu \delta(\nabla \cdot A) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_\mu^i)} \langle \text{out} | T[\psi(y_1) \cdots \bar{\psi}(z_1)] | \text{in} \rangle_{A(\tau), A_\mu(\tau)}}{\int D A_\mu \delta(\nabla \cdot A) e^{-\int d^4x F_{\mu\nu} F_{\mu\nu}/4} \langle \text{out} | \text{in} \rangle_{A(\tau)}}.
$$

(12.10)

An important observation is that the set of operators in the time ordered product on the l.h.s. can be expanded to include any number of vector potentials by functionally differentiating w.r.t. the external current \(J_\mu^i\). This is clear from the pure exponential dependence on \(J_e\) on the r.h.s. Thus in addition to describing the presence of real external sources, the \(J_e\) dependence provides a generating function for all correlation functions of any number of vector potentials in the source free case: simply set \(J_e = 0\) after differentiating the appropriate number of times w.r.t. \(J_e\). To illustrate this point, consider the path integral for the free e.m. field (no charged fields).

$$
\langle \text{out} | \text{in} \rangle_{J_e} = \frac{\int D A_\mu \delta(\nabla \cdot A) e^{-\int d^4x (F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_\mu^i)}}{\int D A_\mu \delta(\nabla \cdot A) e^{-i \int d^4x F_{\mu\nu} F_{\mu\nu}/4}}
$$

(12.11)

$$
\rightarrow \frac{\int D A_\mu \delta(\nabla \cdot A) e^{-i \int d^4x F_{\mu\nu} F_{\mu\nu}/4 - A_\mu J_\mu^i}}{\int D A_\mu \delta(\nabla \cdot A) e^{-i \int d^4x F_{\mu\nu} F_{\mu\nu}/4}};
$$

(12.12)

where the last line gives the Minkowski version. We can extract the dependence on \(J_e\) by shifting the integration variable in the numerator of the last line in \(A\) is cancelled. It must of course also be restricted by the Coulomb gauge condition \(\nabla \cdot C = 0\). After the shift the coefficient of the linear term in \(A\) becomes after an integration by parts

$$
\partial_\mu (\partial_\mu C_\mu - \partial_\nu C_\mu) + J_\nu = 0.
$$

(12.13)

This equation is only consistent if the external current is conserved \(\partial_\mu J_\mu = 0\) which we are assuming. For \(\nu = 4\), remembering the Coulomb gauge condition, this equation determines

$$
C^4 = (-1/\nabla^2) J^4.
$$

After using current conservation to write \(\partial_\mu C^4 = (1/\nabla^2) \nabla \cdot J^4\) the spatial components are determined to be \(C = (-1/\partial^2)(J - \nabla(1/\nabla^2) \nabla \cdot J)\). The question of boundary conditions is settled in Euclidean space by requiring that \(C\) vanish in all four directions at
infinity. As we have discussed this prescription becomes the familiar \( i e \) one when continued back to Minkowski space. Inserting these results into the path integral we find

\[
\langle \text{out} | \text{in} \rangle_{J_e} = e^{-\int d^4x (E_\mu^C F_\mu^C/4 - C_\mu J_e^\mu)}
\]

\[
= e^{\frac{1}{2} \int d^4x d^4y J_e^\mu(x) D_{\mu\nu}(x-y) J_e^\nu(y)}
\]

\[
\rightarrow = e^{(1/2) \int d^4x d^4y J_e^\mu(x) D_F^\mu(x-y) J_e^\nu(y)}.
\]

where the last line is the Minkowski version. Comparison shows

\[
D_{jk}(x) = \int \frac{d^4p}{(2\pi)^4} e^{ix \cdot p} \frac{\delta_{jk} - p^j p^k / p^2}{p^2}
\]

\[
D_{44}(x) = \int \frac{d^4p}{(2\pi)^4} e^{ix \cdot p} \frac{1}{p^2} = \delta(t) \frac{1}{4\pi |x|}
\]

\[
D_{4k} = D_{k4} = 0.
\]

and

\[
D_F^{jk}(x) = -i \int \frac{d^4p}{(2\pi)^4} e^{ix \cdot p} \frac{\delta_{jk} - p^j p^k / p^2}{p^2 - i\epsilon}
\]

\[
D_F^{00}(x) = i \int \frac{d^4p}{(2\pi)^4} e^{ix \cdot p} \frac{1}{p^2} = \delta(t) \frac{i}{4\pi |x|}
\]

\[
D_F^{0k} = D_F^{k0} = 0.
\]

To get correlation functions of any number of vector potentials we differentiate \( \langle \text{out} | \text{in} \rangle_{J_e} \) with respect to the current any number of times. Because of current conservation we may only differentiate w.r.t. three components of \( J_\mu \) say the two transverse components \( J_T \) and \( J^4 \). But that is sufficient since the longitudinal component of \( A \) is zero by the gauge condition. If we set \( J_e = 0 \) after differentiating, it is easy to check that the Wick expansion follows with the two point function

\[
\langle A_{\mu}(x) A_{\nu}(y) \rangle = D_{\mu\nu}(x-y).
\]

Note that with path integrals it is more precise to speak of correlation functions, which are functional averages of some number of fields, than of the vacuum expectations of time ordered products of field operators. They are of course numerically equal to each other.

### 12.1 General Gauges

The path integral formulation of gauge theories is particularly suited to the discussion of general gauges. To keep the formalism covariant we would like to be able to replace the non-covariant Coulomb gauge condition with a covariant one such as \( \partial_\mu A_\mu = 0 \). The procedure for gauge fixing that follows is due to Fadeev and Popov, and is quite general. The method
starts by selecting some general gauge fixing condition $F(A, \partial A) = 0$. This condition should have the property that for any value of the gauge potential it is possible to find a gauge transformation to a potential for which $F = 0$, and further that if $F(A) = 0$ then $F(A_\Omega) \neq 0$ with $\Omega$ any nontrivial gauge transformation which vanishes at infinity. We don’t require this property for more general gauge transformations because that would rule out Coulomb gauge which seems to be perfectly adequate. The path integrand in such a gauge should contain a factor of $\delta(F(A))$.

The F-P procedure is to define a functional $\Delta_F(A)$ by the requirement

$$1 = \Delta_F(A) \int D\Omega \delta(F(A_\Omega)) \tag{12.24}$$

where $A_\Omega$ is the transformation of $A$ by the gauge group element $\Omega(x)$, and the measure $D\Omega$ is gauge invariant. By this definition $\Delta_F$ is clearly gauge invariant. Now insert this representation for 1 in the “unfixed” gauge field path integrand. Next change functional integration variables so that $A_\Omega \to A$. Here we implicitly assume that the unfixed measure $DA_{\mu}$ is invariant under changes of variables which are gauge transformations\(^1\). Then the infinite volume of the gauge group $\int D\Omega$ comes out as a common factor in both the numerator and denominator of the functional average and so cancels. We are left with the factors

$$\Delta_F(A) \delta(F(A)) \tag{12.25}$$

in the functional integrand. The delta function fixes the gauge and the factor $\Delta_A$ is in general needed to guarantee that different choices for $F$ yield the same answer for gauge invariant quantities. (It is only for functional averages of gauge invariant quantities that the rest of the integrand stays invariant under the variable change that removes $\Omega$ from the delta function.)

Our Coulomb gauge path integral did not include a factor of $\Delta_{Coul}(A)$ so we need to confirm that it is inconsequential for the abelian case. For this we have to compute

$$\frac{1}{\Delta_{Coul}} = \int D\Lambda \delta(\nabla \cdot A + \nabla^2 \Lambda) = \frac{1}{\det(-\nabla^2)}. \tag{12.26}$$

We see that $\Delta_{Coul}$ is independent of $A$ so that it will cancel between the numerator and denominator of functional averages. Thus our failure to put it in gives no differences in physical quantities. Thus all gauge choices are formally equivalent to Coulomb gauge provided the Fadeev-Popov factor $\Delta_F(A)$ is included along with the gauge fixing delta functional.

In practice $\Delta_F(A)$ always multiplies $\delta(F(A))$ so the former is only needed for $A$ satisfying the gauge condition. For such $A$, $\delta(F(A_\Omega))$ only contributes for infinitesimal $\Omega$:

$$F(A(x) + \delta A(x)) \approx \int d^4y \frac{\delta F(A(x))}{\delta A_{\mu ab}(y)} \delta A_{\mu ab}(y). \tag{12.27}$$

\(^1\)In a completely general context this assumption might clash with the more basic translational invariance of the measure. In such a case there would be a Jacobian accompanying the variable change. This is not needed for the usual abelian and nonabelian gauge theories however.
Denoting the infinitesimal generators of the gauge group by $G(z)$, $\delta A$ is linear in the matrix elements of $G$: $\delta A_{\mu ab}(y) = \int d^4 z L_{\mu ab,cd}(y, z; A) G_{cd}(z)$ where $L$ is a linear differential operator depending on $A$ in general. For example, the infinitesimal nonabelian gauge transformation reads $\delta A = \partial G - ig[A, G]$. Thus $\Delta_F = \det(\delta F, \delta A)$ where the determinant is that of a linear differential operator which is also a matrix in the internal group space with matrix elements labeled $(ab, ef)$:

$$\int d^4 y \frac{\delta F_{ab}(A(x))}{\delta A_{\mu cd}(y)} L_{\mu cd,ef}(y, z; A).$$  (12.28)

An important class of covariant gauges consists of the Lorentz gauges $\partial \cdot A = f$ where $f$ is some fixed function. The F-P determinant for this gauge is $\det(-\partial^2)$ in the abelian case of QED. Notice that it is independent of both $f$ and $A$. In the nonabelian case, it would depend upon $A$ of course, but is still independent of $f$. So for QED the F-P determinant cancels between numerator and denominator. and it is safe to ignore it. Functional averages of gauge invariant quantities will be independent of $f$. We can exploit this to get rid of the functional delta function in the path integrand by averaging over $f$ with a gaussian weight function $e^{-\int d^4 x f^2(x)/2\alpha}$. This just multiplies the numerator and denominator by the same constant so it won’t alter gauge invariant quantities. The net effect of this is to remove the delta function and instead add a non gauge invariant term $\int d^4 x (\partial \cdot A)^2(x)/2\alpha$ to the Euclidean action. For $\alpha = 1$ (Feynman Gauge) the effect of this term is to simplify the kinetic term for $A$:

$$\int d^4 x \left(\frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \frac{1}{2\alpha} (\partial \cdot A)^2\right) = \int d^4 x \left(\frac{1}{2} \partial_{\mu} A_{\nu} \partial_{\mu} A_{\nu} + \frac{1 - \alpha}{2\alpha} (\partial \cdot A)^2\right)
\sim \int d^4 x \frac{1}{2} \partial_{\mu} A_{\nu} \partial_{\mu} A_{\nu}, \quad \alpha \rightarrow 1$$  (12.29)

Just as with the Coulomb gauge, the photon propagator for the “$\alpha$” gauges is obtained by evaluating the path integral for the gauge field in the presence of an external source. The source dependence is easily obtained by shifting $A \rightarrow A + C$ with $C$ satisfying

$$\partial^2 C_{\mu} + \frac{1 - \alpha}{\alpha} \partial \partial \cdot C = -J_{\mu}$$  (12.30)

which is solved first for $\partial \cdot C = -\alpha (\partial^{-2}) \partial \cdot J$ and then

$$C_{\mu} = -\frac{1}{\partial^2} (J_{\mu} - (1 - \alpha) \frac{\partial_{\mu} \partial_{\nu} J_\nu}{\partial^2}).$$  (12.31)

2There are some applications where it is nonetheless important to keep it. For example, it contributes a constant to the zero point energy which just subtracts the spurious contribution of the time-like and longitudinal components of the vector potential. Also when one computes the finite temperature partition function in a covariant gauge, the F-P factor removes two photon degrees of freedom so the total number is $4 - 2 = 2$. Without the F-P factor the Stefan-Boltzmann law would be off by a factor of 2!
Thus the propagator is

\[ D^\alpha_{\mu\nu}(x) = \int \frac{d^4p E}{(2\pi)^4} e^{ip\cdot x} \frac{\delta_{\mu\nu} - (1 - \alpha)p_\mu p_\nu/p^2}{p^2}. \quad (12.32) \]

In the continuation to Minkowski space \( \delta_{\mu\nu} \) is replaced by \( \eta_{\mu\nu} \), \( p^2 \) by \( p^2 - i\epsilon \), and there is an additional factor of \(-i\):

\[ D^{\alpha}_{\mu\nu}(x) = -i \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{\delta_{\mu\nu} - (1 - \alpha)p_\mu p_\nu/(p^2 - i\epsilon)}{p^2 - i\epsilon}. \quad (12.33) \]

We note the great simplification for Feynman gauge (\( \alpha = 1 \)). The case \( \alpha = 0 \), known as Landau gauge is effectively the gauge \( \partial \cdot A = 0 \) because the coefficient of the gauge breaking term in the action blows up damping out all contributions to the integral not satisfying this condition. We see that in this case the divergence of the propagator vanishes, \( \partial_\mu D^0_{\mu\nu} = 0 \). The fundamental reason that QED is independent of \( \alpha \) is that \( A \) always couples to a conserved current so that the terms involving \( \alpha \) in the propagator decouple.

To get a bit more insight into the role of current conservation consider a photon propagator attached to two conserved vertices in momentum space

\[ A_{mp} \sim \frac{R_{\mu}R^{2\mu}}{p^2} = \frac{R^1 \cdot R^2 - R^1_0 R^2_0}{p^2}. \quad (12.34) \]

It would seem that this coupling corresponds to the propagation of four photon states, 3 “space” components and 1 time component, the last one coupling with the “wrong” sign. These wrong sign states are sometimes called “ghosts.” But by current conservation \( R^\mu_0 = p \cdot R^a/p^0 \). In the limit of physical photon momentum, i.e. for which \( p^2 = 0 \) the residue of the pole is just

\[ R^1 \cdot R^2 = \frac{p \cdot R^1 p \cdot R^2}{p^2}. \quad (12.35) \]

which is to say only the transverse states (perpendicular to \( p \)) truly propagate.
Chapter 13

Path Integrals for Anticommuting Quantum Fields

To convert the persistence amplitude for the Dirac field in the presence of external gauge fields to a path integral formalism, we must be able to produce a factor of \( \text{det}\{m - i\gamma \cdot D\} \) in the numerator unlike the denominator as gaussian integrals tend to produce. This requires the introduction of “anti-commuting” numbers or \( a \) numbers in contrast to ordinary \( c \) numbers. Any two \( a \) numbers \( e, f \) satisfy \( ef + fe = 0 \). In particular the square of an \( a \) number vanishes! Thus the most general function of a single \( a \) number is the linear one \( c_1 + c_2a \). The theory of functions of \( a \) numbers is quite trivial. A function of \( N a \) numbers is at most linear in each variable, but that involves terms with up to \( N \) factors. With \( N \to \infty \) one can of course have any number of factors as long as each factor is a different \( a \) number.

How do we integrate over \( a \) numbers? To define this we define \( \int da f(a) \) to be a linear operation that assigns a unique \( c \) number to each function \( f \). We also require the fundamental translation property \( \int da (a + e) = \int da a \). But there are only two linearly independent functions of \( a \) namely 1 and \( a \) itself. So we only need to specify \( \int da1 \) and \( \int daa \). The translation property for the second of these holds only if \( \int da1 = 0 \). Thus we only need to specify \( \int daa \) to be some fixed \( c \) number, and then integration is completely defined! By definition we take \( \int daa = 1 \). Then the integral of the arbitrary function \( c_1 + c_2a \) is simply \( c_2 \), the coefficient of the linear term in \( a \).

Now consider integration of a gaussian over \( 2M a \) numbers, \( a_k, \bar{a}_k \):

\[
\int da_1 d\bar{a}_1 \cdots da_M d\bar{a}_M e^{\bar{a}^{T} Ca} = \frac{1}{M!} \int da_1 d\bar{a}_1 \cdots \int da_M d\bar{a}_M (\bar{a}^{T} Ca)^M \tag{13.1}
\]

\[
= \int da_1 d\bar{a}_1 \cdots \int da_M d\bar{a}_M a_{1k_1} a_{2k_2} \cdots a_{Mk_M} = \text{det } C. \tag{13.2}
\]

This shows that integrating gaussians over anticommuting numbers yields determinants with positive powers. Next consider the case where the exponent of the gaussian is a bilinear in
$M$ a numbers $b_k$: $\frac{1}{2} b^T A b$ where $A$ is an antisymmetric matrix. It can be reduced to the previous case by considering its square:

$$
\left( \int db_1 \cdots db_M e^{b^T A b/2} \right)^2 = \int db_1 \cdots db_M db_1' \cdots db_M' e^{(b^T A b + b'^T A b')/2}
$$

$$
= (-)^{\frac{M^2}{2}} \int da_1 da_1' \cdots da_M da_M' e^{\bar{a}^T A a}
$$

$$
= (-)^{\frac{M^2}{2}} \det A
$$

(13.5)

where the change of variables $a = (b + ib')/\sqrt{2}$, $\bar{a} = (b - ib')/\sqrt{2}$ has been used. The phase out front is never relevant, since when $M$ is odd, both sides vanish. We conclude that

$$
\int db_1 \cdots db_M e^{b^T A b/2} = \det^{1/2} A.
$$

(13.6)

Since the l.h.s. is a polynomial in the matrix elements of $A$ we have proved an interesting corollary that the determinant of an antisymmetric matrix is the square of a polynomial in the matrix elements. That polynomial is sometimes known as the Pfaffian, $Pf(A) = \det^{1/2} A$ for antisymmetric matrices $A$.

Now we can repeat our discussion of bosonic functional integrals for the fermionic case:

$$
\langle \text{out} | \text{in} \rangle_A = \frac{\det(m - i\gamma \cdot D)}{\det(m - i\gamma \cdot \partial)}
$$

(13.7)

$$
= \int \mathcal{D}\psi \mathcal{D}\tilde{\psi} \exp \left\{ -i \int d^4 x (\bar{\psi}(m - i\gamma \cdot D)\psi) \right\}
$$

$$
\int \mathcal{D}\psi \mathcal{D}\tilde{\psi} \exp \left\{ -i \int d^4 x (\bar{\psi}(m - i\gamma \cdot \partial)\psi) \right\}.
$$

(13.8)

Introducing anticommuting sources, $\eta, \bar{\eta}$ in the combination $i \int d^4 x [\bar{\eta}\psi + \bar{\psi}\eta]$ in the exponent and competing the square, one can easily see that the source dependence is just a factor

$$
e^{i \int \bar{\eta}(m - i\gamma \cdot D)^{-1}\eta} = e^{\int d^4 x d^4 y i \bar{\eta}(x) S_F(x, y; A) i\eta(y)}.
$$

(13.9)

By differentiating with respect to the sources, one can show that this expression is just the generating function for the outin matrix element of time ordered products of fields:

$$
\langle \text{out} | \text{Te}^{\int d^4 x [\bar{\eta}\psi + \bar{\psi}\eta]} | \text{in} \rangle
$$

$$
= \int \mathcal{D}\psi \mathcal{D}\tilde{\psi} \exp \left\{ -i \int d^4 x \left[ (\bar{\psi}(m - i\gamma \cdot D)\psi - \bar{\eta}\psi - \bar{\psi}\eta) \right] \right\}
$$

$$
\int \mathcal{D}\psi \mathcal{D}\tilde{\psi} \exp \left\{ -i \int d^4 x (\bar{\psi}(m - i\gamma \cdot \partial)\psi) \right\}
$$

$$
= \frac{\det(m - i\gamma \cdot D)}{\det(m - i\gamma \cdot \partial)} \exp \left\{ \int d^4 x d^4 y i \bar{\eta}(x) S_F(x, y; A) i\eta(y) \right\}.
$$

(13.10)
Chapter 14

Perturbation Theory for $\phi^3$ Scalar Field Theory

Let us now apply what we have learned to the scalar field theory with Lagrangian density

$$\mathcal{L} = -\frac{1}{2} (\partial \phi)^2 - \frac{m_0^2}{2} \phi^2 - \frac{g}{3!} \phi^3$$

(14.1)

One’s first thought is to take the perturbation in interaction picture to be

$$H'_I(t) = \left(\frac{g}{3!}\right) \int d^3x \phi^3(x,t).$$

We shall find that, because the interactions don’t turn off at early and late times, this is too glib. But let’s see where it leads. We have no external fields, so the time dependent perturbation theory reads

$$\langle G | T[\phi(x_1) \cdots \phi(x_n)] | G \rangle = \frac{\langle 0, I | T \exp \left\{ -\frac{ig}{3!} \int d^4x \phi^3_I(x) \right\} \phi_I(x_1) \cdots \phi_I(x_n) | 0, I \rangle}{\langle 0, I | T \exp \left\{ -\frac{ig}{3!} \int d^4x \phi^3_I(x) \right\} | 0, I \rangle}.$$  

(14.2)

Her $|G\rangle$ is the exact ground state.

14.1 The Vacuum

The simplest application of the Dyson formula is the case without any fields in the time ordered product:

$$\langle G | I | G \rangle = \langle G | G \rangle = \frac{\langle 0, I | T \exp \left\{ -\frac{ig}{3!} \int d^4x \phi^3_I(x) \right\} | 0, I \rangle}{\langle 0, I | T \exp \left\{ -\frac{ig}{3!} \int d^4x \phi^3_I(x) \right\} | 0, I \rangle} = 1.$$  

(14.3)

Which is consistent but pretty trivial. The numerator and denominator are separately complicated expressions but complications all cancel. Let’s look more closely at the structure of the numerator:

$$\text{Num} = 1 + \frac{1}{2} \left(\frac{-ig}{6^2}\right) \int d^4x d^4y \langle 0, I | T \phi^3_I(x) \phi^3_I(y) | 0, I \rangle.$$  

(14.4)
Applying the Wick expansion to the second term can be visualized as two Feynman diagrams. These diagrams are characterized by having no external lines and are sometimes called vacuum bubbles. The bottom line for them is that they are always cancelled, so we never have to deal with them.

### 14.2 One-point function

The next simplest thing to consider is the VEV of the scalar field, \( \langle G|\phi(x)|G \rangle = \langle G|\phi(0)|G \rangle \) by translation invariance. It is just a constant number, call it \( v \). What if we start trying to calculate it in perturbation theory. Applying the Dyson formula to first order gives

\[
v = \langle G|\phi(0)|G \rangle \approx -\frac{ig}{3!} \int d^4 x \langle G|T\phi(0)\phi^3(x)|G \rangle + \cdots \tag{14.5}\]

The Wick expansion shows this as a “tadpole” diagram. The actual value of the number \( v \) is not particularly significant. We chose \( H' \) so that at zeroth order in perturbation theory \( \langle G|\phi(0)|G \rangle_0 = 0 \). But then the perturbation makes it non zero. As we continue perturbation theory, it is much more convenient to write \( \phi = v + \hat{\phi} \) where \( \langle G|\hat{\phi}(0)|G \rangle_0 = 0 \) exactly to all orders in perturbation theory. In terms of \( \hat{\phi} \) the Lagrangian reads

\[
L = -\frac{1}{2}(\partial \hat{\phi})^2 - \frac{m_0^2}{2}(v + \hat{\phi})^2 - \frac{g}{3!}(v + \hat{\phi})^3
\]

\[
= -\frac{1}{2}(\partial \hat{\phi})^2 - \frac{m^2}{2}\hat{\phi}^2 - \left[ -\frac{m^2}{2}\hat{\phi}^2 + \frac{m_0^2}{2}(v + \hat{\phi})^2 + \frac{g}{3!}(v + \hat{\phi})^3 \right] \tag{14.6}
\]

where \( m \) represents the true physical mass of the scalar particle, in general different from the “bare” input mass \( m_0 \). The correct procedure is to take all the terms we have enclosed in square brackets to define the perturbation \( H' \).

\[
H' = -\frac{m^2}{2}\hat{\phi}^2 + \frac{m_0^2}{2}(v + \hat{\phi})^2 + \frac{g}{3!}(v + \hat{\phi})^3
\]

\[
\equiv \frac{g}{3!}\hat{\phi}^3 + \frac{\delta_m}{2}\hat{\phi}^2 + \delta_v \hat{\phi} + \frac{m_0^2}{2}v^2 + \frac{g}{3!}v^3 \tag{14.7}
\]

The last two terms are constants which will cancel between numerator and denominator. The values of \( \delta_m \) and \( \delta_v \) are chosen at the end of the calculation so that \( \langle G|\hat{\phi}|G \rangle = 0 \) and so that \( m \) stays the physical mass. In practice we never need to find \( \delta_v \) because its only effect is to set to zero all “tadpole” diagrams and subdiagrams. Thus all we have to do is delete all tadpoles and then forget about \( \delta_v \).

### 14.3 Two Point Function and the Physical Mass

We have seen that the Fourier transform of the free field two point function has a pole at \( p^2 = -m^2 \). We now argue that this feature holds generally for two point functions even in the interacting case.
Let’s first see how this happens in perturbation theory to second order

\[
\int d^4 x e^{-i p \cdot x} \langle G | T \phi(x) \phi(0) | G \rangle = \frac{-i}{q^2 + m_0^2} \\
+ \frac{1}{2} \frac{(-i g)^2}{3!} \int d^4 x e^{-i p \cdot x} \langle 0, I | T \phi_I(x) \phi_I(0) \int d^4 y d^4 z \phi_I^2(y) \phi_I^2(z) | 0, I \rangle
\]

(14.8)

In the second term, if both \( \phi \)'s hit the same Hamiltonian we get a tadpole correction to the propagator which is simply proportional to \((q^2 + m_0^2)^{-2}\). If each \( \phi \) hits a different Hamiltonian we are left with

\[
\frac{-i}{q^2 + m_0^2} + \frac{1}{2} \frac{(-i g)^2}{3!} (\frac{-i}{q^2 + m_0^2})^2 \left[ \int d^4 y e^{-i q \cdot (y-z)} \langle 0, I | T \phi_I(y - z) \phi_I(0) | 0, I \rangle \right]^2 \\
\equiv \frac{-i}{q^2 + m_0^2} \left[ 1 + \frac{-i}{q^2 + m_0^2} (-i) \Pi(q^2) \right] \approx \frac{-i}{q^2 + m_0^2 + \Pi(q^2)}
\]

(14.9)

We see that the pole location has shifted to \( q^2 = -m^2 \) where \( m \) satisfies

\[
-m^2 + m_0^2 + \Pi(-m^2) = 0.
\]

(14.10)

We now see that it would be a good idea to use the true mass \( m \) in the zeroth order propagator and tune the \( \delta_m \) counterterm so that corrections don’t change the pole location, i.e. so that \( \Pi(-m^2) = 0 \) to all orders in perturbation theory.

Now, we give the general argument:

\[
\int d^4 x e^{-i p \cdot x} \langle G | T \phi(x) \phi(0) | G \rangle
\]

(14.11)

We lose no generality in taking one point to be 0 because of translation invariance. Fix \( t > 0 \) and insert a complete set of energy momentum eigenstates between the two fields:

\[
\int_0^\infty dt \int d^3 x e^{-i p \cdot x} \sum_n \int d^3 p \langle G | \phi(x) | n, \vec{p} \rangle \langle n, \vec{p} | \phi(0) | G \rangle \\
= \int_0^\infty dt \int d^3 x e^{-i p \cdot x} \sum_n \int d^3 p e^{i \vec{p} \cdot \vec{x}} | \langle n, \vec{p} | \phi(0) | G \rangle |^2
\]

(14.12)

\[
= \sum_n (2\pi)^3 \int_0^\infty dt e^{i(q_0 - E_n)t} | \langle n, \vec{q} | \phi(0) | G \rangle |^2
\]

(14.13)

\[
= \sum_n \frac{-i(2\pi)^3}{i(q_0 - E_n + i\epsilon)} | \langle n, \vec{q} | \phi(0) | G \rangle |^2
\]

(14.14)

If \( |n, \vec{q} \rangle \) is a single particle state, \( E_n = \sqrt{\vec{q}^2 + m_n^2} \) and the pole at \( q^2 = -m_n^2 \) is explicit. If it is a multi-particle state the singularity is smeared and becomes a branch point. The residue of the pole depends on the matrix element \( \langle n, \vec{q} | \phi(0) | G \rangle \equiv \sqrt{Z}/[(2\pi)^{3/2} \sqrt{\omega(q)}] \) for a single
particle state by Lorentz invariance. \( Z < 1 \) will not be unity as in the free particle case and the single particle pole is

\[
\frac{-i}{p^2 + m_0^2 - i\epsilon} \rightarrow \frac{-iZ}{p^2 + m^2 - i\epsilon}
\] (14.15)

This change in residue \( Z \) is called “wave-function” renormalization.

The argument we have just given shows that \( \phi(x)|g\) has a single particle component, as well as multi-particle components. It would be nice to use \( \phi(x) \) as a creation operator, but since it also creates multi-particle states, we need somehow arrange these unwanted components to disappear. Taking the spatial Fourier transform \( \int d^3x e^{i\vec{q}\cdot\vec{x}} \phi(x) \) only guarantees that it creates momentum \( \vec{q} \) but not that it creates only a single particle state. If we could select the component with frequency \( e^{i\omega(\vec{q})t} \), with \( \omega(\vec{q}) = \sqrt{\vec{q}^2 + m^2} \). A temporal Fourier transform \( \int_{-T}^T dte^{-i\omega(t)} \int d^3x e^{i\vec{q}\cdot\vec{x}} \phi(x) \) would do this to accuracy \( 1/2T \). Since the single particle energy is separated by a gap \( O(m) \) from the vacuum and multi-particle energies this is good enough as long as \( 2mT \ll 1 \). In summary

\[
\frac{1}{2T} \int_{t_0-T}^{t_0+T} dt e^{-i\omega(\vec{q})t} \int \frac{d^3x}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \phi(x)|G\rangle \equiv \int F(t-t_0) dt e^{-i\omega(\vec{q})t} \int \frac{d^3x}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \phi(x)|G\rangle
\]

\[
\equiv A(\vec{q})|G\rangle
\] (14.16)

is a pure single particle state, with mass \( m \) and momentum \( \vec{q} \), of the exact interacting theory. Taking out the factors as shown gives \( A(t_0) \) the normalization

\[
\langle G|A(\vec{q})A(\vec{q})|G\rangle \approx \int d^3p \langle G|A(\vec{q})|\vec{p}\rangle \langle \vec{p}|A(\vec{q})|G\rangle
\]

\[
\approx \delta(\vec{q} - \vec{q}' - \vec{p}) \langle \vec{q}|(\phi(0)|G\rangle^2
\] (14.17)

It is constructed from a Heisenberg picture field operator smeared in the neighborhood of time \( t_0 \).

We can make a two particle state by applying a second \( A^\dagger \). However, it only has a clean two particle interpretation when the two particles are in wave packets space-like separated from each other. In that case the two \( A^\dagger \)'s commute and represent two independent particles. However, if they are aimed to scatter, the packets will eventually overlap and scattering can occur.
14.3.1 Reduction Formulae

Let $-\infty < t_1, t_2, \ldots, t_n < \infty$. Call $f(\vec{x}, t) = \int d^3 q f(\vec{q}) e^{i \vec{q} \cdot \vec{x} - i \omega(\vec{q}) t} / (2\pi)^3$. Then it is easy to show

$$T \{ A_a(\infty) \Omega_1(x_1) \ldots \Omega_n(x_n) \} - T \{ \Omega_1(x_1) \ldots \Omega_n(x_n) A_a(-\infty) \}$$

$$= \int dt_0 \frac{d}{dt_0} T \{ A_a(t_0) \Omega_1(x_1) \ldots \Omega_n(x_n) \}$$

$$= \int dt_0 \int d^4 x F(t - t_0) \frac{\partial}{\partial t} f^*_a(x, t) T \{ \phi(x) \Omega_1(x_1) \ldots \Omega_n(x_n) \}$$

$$= \int dt_0 \int d^4 x F(t - t_0) f^*_a(x, t) \left( i \omega_a (\pm i \nabla) + \frac{\partial}{\partial t} \right) T \{ \phi(x) \Omega_1(x_1) \ldots \Omega_n(x_n) \}$$

$$T \{ A_a^\dagger(\infty) \Omega_1(x_1) \ldots \Omega_n(x_n) \} - T \{ \Omega_1(x_1) \ldots \Omega_n(x_n) A_a^\dagger(-\infty) \}$$

$$= \int dt_0 \int d^4 x F(t - t_0) \frac{\partial}{\partial t} f_a(x, t) T \{ \phi(x) \Omega_1(x_1) \ldots \Omega_n(x_n) \}$$

$$= \int dt_0 \int d^4 x F(t - t_0) f_a(x, t) \left( -i \omega_a (\mp i \nabla) + \frac{\partial}{\partial t} \right) T \{ \phi(x) \Omega_1(x_1) \ldots \Omega_n(x_n) \}.$$

In these reduction formulae we have used the fact that by construction $f_a$ satisfies a Schrödinger equation with hamiltonian $\omega_a(-i \nabla) = \sqrt{m_a^2 - \nabla^2}$. Spatial integration by parts then allows the spatial derivatives to be transferred to $\phi(x)$.

The reduction formulas can be used to relate scattering amplitudes represented as, e.g.

$$\langle G | A(\infty, \vec{q}_1) A(\infty, \vec{q}_2) A^\dagger(-\infty, \vec{p}_1) A^\dagger(-\infty, \vec{p}_2) | G \rangle$$

(14.24)

to Fourier transforms of

$$\langle G | T(\phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4)) | G \rangle \equiv \int \prod_k \frac{d^4 q_k}{(2\pi)^4} \exp \left\{ i \sum_k q_k \cdot x_k \right\} T(q_1, q_2, q_3, q_4)$$

The scattering amplitude is proportional to $T$ in the on-shell limit $q_k^2 \to -m_k^2$. Then applying $\partial / \partial t_k \pm i \omega(\mp i \nabla_k)$ is equivalent to multiplying $T$ by $-iq^0 \pm i \omega(\vec{q})$, which vanishes in one of the on-shell limits when $q^0 \to \pm \omega(\vec{q})$. There will be one such on-shell zero for each external particle. Thus there is only scattering if the F.T. has a mass-shell pole for each particle. The reduction formula gives the precise connection including all normalization factors. But it is quicker to get all the right factors by just considering Feynman diagrams. The required
poles come from complete propagators attached to each external line. Lorentz invariance
determines all factors up to overall $Z$ factors. Therefore we can use lowest order perturbation
theory to determine everything. The rule is to “amputate” all external legs and replace each
propagator with $\sqrt{Z} e^{\pm i q \cdot x}/(2\pi)^{3/2}/\sqrt{2\omega}$.

In summary, by choosing the perturbation carefully we set things up so that $\langle G|\hat{\phi}|G \rangle = 0$
to all orders in perturbation theory (this just amounts to dropping all tadpole corrections!),
and so that $\Pi(-m^2) = 0$ to all orders in perturbation theory. Then near mass shell $\Pi(q^2) \approx
(q^2 + m^2)(Z^{-1} - 1)$, and we see how the residue shifts. Since we don’t know $m^2$ and $v$
in advance we have to keep adjusting $\delta_m$ and $\delta_v$ as we proceed.

14.4 Feynman Diagrams

The Wick expansion is very efficiently analyzed via Feynman diagrams. These are a collections
of lines, one for each propagator in the Wick expansion, connected together at vertices
determined by the interaction Lagrangian. In the case at hand, $\mathcal{L}' = -\frac{g}{3!} \phi^3 - \frac{\delta_m}{2}\phi^2 - \delta_v \phi$.

1 List Feynman rules in $x$ space for $\phi^3$.

2 Motivate and translate rules to $p$ space.

3 Connectedness: sufficient to focus on connected diagram– general term is product of
connected pieces.

4 Example: Sum of all vacuum graphs = $\exp$ of sum of all connected vacuum graphs.

5 Lowest order four point function.
Chapter 15

Operator Quantization of the Electromagnetic Field

[Note: This chapter describes the operator alternative to the path integral description of QED. It may be skipped on a first reading.]

Treated classically the e.m. field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfies Maxwell’s equations

$$\partial_\nu F^{\mu\nu} = J^\mu, \quad (15.1)$$

which imply current conservation $\partial_\mu J^\mu = 0$ for consistency. So far we have developed the theory of Dirac and scalar quantum fields interacting with fixed external e.m. fields. For such systems the current is of course an operator, so Maxwell’s equations imply that the e.m. field must also be a quantum operator, which inserted into the Dirac equation produces a nonlinear quantum field equation.

15.1 Quantized Electromagnetic Field Interacting with a Conserved Current

The first step in developing the quantum theory for $A_\mu$ is to understand the canonical structure of Maxwell’s equations. One first notes that the equations follow from stationarity of the action

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} + \int d^4x A_\mu J^\mu, \quad (15.2)$$

which is gauge invariant provided $\partial_\mu J^\mu = 0$. It is straightforward to find the momentum conjugate to $A_\mu$

$$\Pi^\mu = F_0^\mu = \partial_0 A^\mu - \partial^\mu A_0. \quad (15.3)$$

The spatial components of $\Pi$ are just those of minus the electric field strength, but $\Pi^0 = 0$. This last fact poses a difficulty for quantization since it is inconsistent with nonvanishing
canonical commutation relations. We know how to assign operator properties to \( \vec{A} \) and \( \vec{\Pi} \), but not to \( A_0 \). Before facing this difficulty, we construct the canonical Hamiltonian

\[
H_{\text{CAN}} = \int d^3x [\vec{\Pi} \cdot \vec{A} - \mathcal{L}] \tag{15.4}
\]

\[
= \int d^3x \left[ \frac{1}{2} \vec{\Pi}^2 + \frac{1}{2} (\nabla \times \vec{A})^2 - \vec{\Pi} \cdot \vec{J} + \nabla A_0 \cdot \vec{\Pi} - A_0 J^0 \right]. \tag{15.5}
\]

Notice that the troublesome variable \( A_0 \) appears only linearly and in the last two terms. After an integration by parts the coefficient is just \( - (\nabla \cdot \vec{\Pi} + J^0) \), which would vanish if we could use Gauss’ Law.

Classically, we could certainly enforce Gauss’ Law and then the canonically uncertain \( A_0 \) would disappear from the dynamics. Attempting to enforce Gauss’ Law as a quantum operator equation would contradict the canonical commutation relations for \( \vec{A}, \vec{\Pi} \)

\[
[A_k(\vec{x}, t), \Pi^m(\vec{y}, t)] = i \delta^m_k \delta(\vec{x} - \vec{y}), \tag{15.6}
\]

so we postpone discussion of this point. If we leave the operator character of \( A_0 \) unspecified, it is clearly important to know how quantum evolution will be affected if we make a change in \( A_0 \). Let the evolution operator for a given \( A_0 \) be \( U_{A_0}(t, -\infty) \). Then by a familiar argument

\[
\delta U_{A_0}(t, -\infty) = i U_{A_0}(t, -\infty) \int d^4x \delta A_0 (\nabla \cdot \vec{\Pi} + J^0) \tag{15.7}
\]

where the operators multiplying \( U_{A_0} \) are in Heisenberg picture. The Heisenberg equations for \( \vec{A}, \vec{\Pi} \) imply

\[
\frac{\partial}{\partial t} (\nabla \cdot \vec{\Pi} + J^0) = \frac{\partial J^0}{\partial t} + \nabla \cdot \vec{J} \int d^3x [\nabla \cdot \vec{\Pi}, A_0](\nabla \cdot \vec{\Pi} + J^0) \tag{15.8}
\]

\[
= i \int d^3x [\nabla \cdot \vec{\Pi}, A_0](\nabla \cdot \vec{\Pi} + J^0) \tag{15.9}
\]

Although we are not free to impose Gauss’ Law as an operator equation, we can restrict our incoming states to satisfy

\[
(\nabla \cdot \vec{\Pi} + J^0)|_{t = -\infty} |in\rangle = 0. \tag{15.10}
\]

It then follows from (15.9) that \((\nabla \cdot \vec{\Pi} + J^0)|in\rangle = 0\) for all time. By making this restriction we therefore arrange that

\[
\delta U_{A_0} |in\rangle = 0 \tag{15.11}
\]

for arbitrary operator changes in \( A_0 \), that is, the quantum evolution is independent of how we treat \( A_0 \). This is the quantum analog of what is true in the classical treatment, and suffices to resolve the difficulties. By imposing Gauss’ Law on initial states, we are free to make a choice for \( A_0 \) which simplifies the dynamical problem.
For example, suppose we want to “solve” the constraint by “imposing Coulomb gauge.” We do this by writing
\[ \vec{\Pi} = \vec{\Pi}_T + \frac{1}{\nabla^2} \vec{\nabla} \nabla \cdot \vec{\Pi} \] (15.12)
\[ = \vec{\Pi}_T - \frac{1}{\nabla^2} \vec{\nabla} J^0 + \frac{1}{\nabla^2} \vec{\nabla} (\nabla \cdot \vec{\Pi} + J^0) \] (15.13)
so that \( \nabla \cdot \vec{\Pi}_T = 0 \). Inserting (15.13) into the term in the Hamiltonian containing \( \Pi \) and judiciously integrating by parts, we obtain
\[ \int d^3x \, \vec{\Pi}^2 = \int d^3x \, \vec{\Pi}_T^2 + \int J^0 \left( -\frac{1}{\nabla^2} \right) J^0 \] (15.14)
\[ + \int \left( 2 \frac{1}{\nabla^2} J^0 - \frac{1}{\nabla^2} (\nabla \cdot \vec{\Pi} + J^0) \right) (\nabla \cdot \vec{\Pi} + J^0) + \int \left[ \frac{1}{\nabla^2} J^0, \nabla \cdot \vec{\Pi} + J^0 \right]. \]
The last term, which arises because we reordered operators so that \( \nabla \cdot \vec{\Pi} + J^0 \) stands on the right wherever it appears, formally vanishes because the fields entering \( J \) are canonically independent of \( \Pi \) and also \( J^0 \) commutes with itself at equal times. The next to last term has a factor of \( \nabla \cdot \vec{\Pi} + J^0 \) on the right. By choosing
\[ A_0 = \frac{1}{\nabla^2} J^0 - \frac{1}{2\nabla^2} (\nabla \cdot \vec{\Pi} + J^0), \] (15.15)
this term is cancelled by the term linear in \( A_0 \) and the Hamiltonian then simplifies to
\[ H_{COUL} = \int d^3x \, \vec{\Pi}_T^2 + \int J^0 \left( -\frac{1}{\nabla^2} \right) J^0 + \int \left( \frac{1}{2} (\nabla \times \vec{A})^2 - \vec{A} \cdot \vec{J} \right). \] (15.16)
Since only \( \Pi_T \) now enters the Hamiltonian, it is appropriate to make a similar decomposition of \( \vec{A} = \vec{A}_T + \vec{\nabla}(1/\nabla^2) \nabla \cdot \vec{A} \). The gradient term does not contribute to the curl of \( A \) and an integration by parts allows that term in \( H_{COUL} \) to be simplified as
\[ \int (\nabla \times \vec{A})^2 = \int (\nabla \times \vec{A}_T)^2 = \int \vec{A}_T \cdot (\nabla \times (\nabla \times \vec{A}_T)) \] (15.17)
\[ = \int \vec{A}_T \cdot (-\nabla^2) \vec{A}_T = \int \nabla_k \vec{A}_T \cdot \nabla_k \vec{A}_T. \] (15.18)
The longitudinal component of \( \vec{A} \) appears in \( H_{COUL} \) through the coupling to \( \vec{J} \). To interpret this term we have to consider a little more closely the inner product structure on the space of states. The states which satisfy the Gauss’ Law constraint have the character of momentum eigenstates with a fixed eigenvalue. Such states would of course have infinite norm. If we restrict our \( in \) states to satisfy Gauss’ Law, and we want our \( out \) states to have a finite inner product with these \( in \) states, it follows that the \( out \) states should not be taken to satisfy the constraint. In fact, the natural dual space to momentum eigenstates are position eigenstates, with \( \langle q | p \rangle \propto e^{iqp} \). Thus we should choose our \( out \) states to be eigenstates of
\[ \nabla \cdot \vec{A}, \text{ which is the conjugate variable to } \nabla \cdot \vec{\Pi}. \text{ Since } \nabla \cdot A \text{ commutes with the Hamiltonian, as a Heisenberg picture operator it will be independent of time. Thus, if we choose our out states to be eigenstates of } \nabla \cdot \vec{A}(\vec{x}, +\infty), \text{ with eigenvalue } A_L(\vec{x}) \text{ they will also be eigenstates of } \nabla \cdot \vec{A}(\vec{x}, -\infty) \text{ with the same eigenvalue.} \]

We can always choose phases, as in the standard Schrödinger representation, so that \[ i\langle A', \text{out}\mid \vec{\Pi} = (\delta/\delta A')\langle A', \text{out}\rangle. \] Then the dependence of persistence amplitudes on the choice of \( A_L \) is determined by

\[
i\delta \langle A_L, \text{out}\mid in \rangle = \int d^3 x \frac{\delta A_L(\vec{x})}{\Delta^2} \langle A_L, \text{out}\rangle \left( -\frac{1}{\Delta^2} \right) \nabla \cdot \vec{\Pi}(\vec{x}, +\infty) \langle in \rangle \tag{15.19}
\]

\[
= \int d^3 x \frac{\delta A_L(\vec{x})}{\Delta^2} \langle A_L, \text{out}\rangle \left( +\frac{1}{\Delta^2} \right) J^0(\vec{x}, +\infty) \langle in \rangle. \tag{15.20}
\]

Since \( J^0(\vec{x}, t) \) is just the infinitesimal generator of local phase changes on the charged Heisenberg fields at time \( t \), we see that an infinitesimal change in \( A_L \) can be compensated by a local gauge transformation on the charged fields at \( t = +\infty \), which determine the definition of the out states. The principle of gauge invariance includes the statement that state vectors differing by such gauge changes describe the same physical state, since they will give identical predictions for all gauge invariant observations. Thus we are free to fix \( A_L \) to be any convenient function. Coulomb or radiation gauge corresponds to the choice \( A_L = 0 \).

We have generally defined \( \langle \text{out}\rangle \) by the \( t \to \infty \) limit of \( \langle t\rangle = \langle t\rangle |U(t, -\infty)\rangle. \) Our choice of \( \langle A_L, \text{out}\rangle \) is implemented by the replacement \( \langle t\rangle \to \langle A_L, in\rangle \) where the latter is the eigenstate of \( \nabla \cdot A(\vec{x}, -\infty) \) with eigenvalue \( A_L \). Then the corresponding \( \langle A_L, t\rangle \) is the corresponding eigenstate of \( \nabla \cdot A(\vec{x}, t) \). Since \( \langle A_L, t\rangle \) satisfies \( i(\partial/\partial t)\langle A_L, t\rangle = \langle A_L, t\rangle |HCOUL(t)\rangle \), we see that replacing \( \nabla \cdot A \) in \( HCOUL \) by \( A_L \) will yield the same \( \langle A_L, out\rangle \). In particular the radiation gauge choice \( A_L = 0 \) leads to the effective Hamiltonian

\[
H_{eff} = \int d^3 x \bar{\Pi}^2_T + \int J^0 \left( -\frac{1}{\Delta^2} \right) J^0 + \int \left( \frac{1}{2} \partial_k \vec{A}_T \cdot \partial_k \vec{A}_T - \vec{A}_T \cdot \vec{J} \right). \tag{15.21}
\]

One may use \( H_{eff} \) to compute any physical quantity. It only contains the transverse components of \( \vec{A} \) and \( \bar{\Pi} \). From the canonical commutation relations for these two quantities one can easily evaluate those for the transverse components

\[
[A_{Tk}(\vec{x}), \Pi_{Tm}(\vec{y})] = i \left( \delta_{km} - \frac{\nabla_k \nabla_m}{\Delta^2} \right) \delta(\vec{x} - \vec{y}). \tag{15.22}
\]

The operator acting on the delta function simply reflects the fact that the l.h.s. has vanishing divergence because of the transversality of the operators.

We have obtained the effective Coulomb gauge Hamiltonian (15.21) by reduction from a gauge independent quantization procedure. A much quicker route to the same answer is to fix the gauge before quantization by setting \( \nabla \cdot A = 0 \) from the beginning. Then the Gauss’ law constraint can be “solved” by setting

\[
A^0(\vec{x}, t) = -\frac{1}{\Delta^2} J^0(\vec{x}, t) = \int d^3 y \frac{J^0(\vec{y}, t)}{4\pi|\vec{x} - \vec{y}|}. \tag{15.23}
\]
The longitudinal component of $\vec{\Pi}$ is also eliminated because $\nabla \cdot \vec{\Pi} = \nabla^2 A^0$. One passes to quantum mechanics by promoting only $\vec{A}$ and $\Pi_T$ to operators. (If the currents are operators, $A^0$ is an operator by virtue of the constraint, but it is not independent.) Then the transverse projector must appear on the r.h.s. of the canonical commutation relations.

### 15.1.1 Polarization and Helicity of Photons.

An explicit realization of the commutation relations can be given in terms of creation and annihilation operators as follows:

$$A_{Tk}(\vec{x}, 0) = \int \frac{d^3 k}{\sqrt{(2\pi)^3 2|k|}} [a_k(\vec{k})e^{i\vec{k} \cdot \vec{x}} + a_k^+(\vec{k})e^{-i\vec{k} \cdot \vec{x}}]$$

$$\Pi_{Tk}(\vec{x}, 0) = -i \int \frac{d^3 k}{\sqrt{(2\pi)^3 2}} [a_k(\vec{k})e^{i\vec{k} \cdot \vec{x}} - a_k^+(\vec{k})e^{-i\vec{k} \cdot \vec{x}}]$$

with

$$[a_k(\vec{k}), a_m^+(\vec{q})] = (\delta_{km} - \frac{k_k k_m}{k^2})\delta(\vec{k} - \vec{q}).$$

Inserting these into $H_{eff}$ gives

$$H_{eff} = \int d^3 k |\vec{k}| \vec{a}^+(\vec{k}) \cdot \vec{a}(\vec{k}) - \int d^3 x \vec{A}_T \cdot \vec{J} + E_0,$$

where $E_0$ is the usual (infinite) zero point energy of the oscillators which will be dropped from now on. This formula shows us immediately that for $\vec{J} = 0$, the quantum e.m. field is interpretable as a system of massless bosons (photons). The vacuum $|0\rangle$ is defined by $a_k(\vec{k})|0\rangle = 0$ and the $n$ photon state is represented by

$$a_{m_1}^+(\vec{q}_1)a_{m_2}^+(\vec{q}_2)\cdots a_{m_n}^+(\vec{q}_n)|0\rangle.$$

Because of transversality there are two photon states for each momentum. These two polarization states will next be shown to correspond to the two helicities $\pm 1$ of the photon.

First for fixed $\vec{k}$ let us introduce two (in general complex) basis vectors $\vec{\epsilon}_a$, $a = 1, 2$ for the plane perpendicular to $\vec{k}$, satisfying $\vec{k} \cdot \vec{\epsilon}_a = 0$ and the orthonormality and completeness relations

$$\vec{\epsilon}_a \cdot \vec{\epsilon}_b = \delta_{ab},$$

$$\sum_a \vec{\epsilon}^m_a \cdot \vec{\epsilon}^{ns}_a = \delta_{mn} - \frac{k^m k^n}{k^2}.$$

We can then introduce two independent sets of creation and annihilation operators via

$$\vec{\alpha}(\vec{k}) = \sum_a \vec{\epsilon}_a a_a(\vec{k}).$$
We shall relate the multiplicity associated with the index $a$ to the spin of the photon. First recall the classical expression for the angular momentum carried by the e.m. field,

$$\vec{J} = \int d^3x \vec{x} \times (\vec{E} \times \vec{B})$$

$$= \int d^3x \sum_k E_k (\vec{x} \times \vec{\nabla}) A_k - \int d^3x \vec{x} \times (\vec{E} \cdot \vec{\nabla}) \vec{A}. \quad (15.32)$$

We can recognize the first term in the last line as the “orbital” angular momentum, which will not contribute to the helicity of a one photon state. This is because acting on a one photon state the $\vec{\nabla}$ is replaced by $\vec{k}$ and because of the cross product the term will be perpendicular to $\vec{k}$. The second term, after an integration by parts becomes

$$\vec{S} = \int d^3x \vec{E} \times \vec{A}$$

$$= -i \int d^3k \vec{a}^\dagger(\vec{k}) \times \vec{a}(\vec{k}). \quad (15.33)$$

Applying $\vec{S}$ to a one photon state $a^\dagger_a(\vec{k})|0\rangle$, yields

$$\vec{S} a^\dagger_a(\vec{k})|0\rangle = i \sum_b (\vec{\epsilon}_a \times \vec{\epsilon}_b^*) a^\dagger_b(\vec{k})|0\rangle. \quad (15.34)$$

Thus we see that the $2 \times 2$ matrix $S_{ab} = i\vec{\epsilon}_a \times \vec{\epsilon}_b^*$ acts as a spin matrix on the index of the creation operator. To get the helicity interpretation, consider the case of $\vec{k} = k\hat{z}$. Then the helicity matrix is

$$S^3_{ab} = i(\epsilon_1^a \epsilon_2^b - \epsilon_2^a \epsilon_1^b). \quad (15.35)$$

This matrix is $\text{diag}\{1, -1\}$ with the choices

$$\vec{\epsilon}_1 = (1, i, 0)/\sqrt{2} \quad \vec{\epsilon}_2 = (1, -i, 0)/\sqrt{2}, \quad (15.36)$$

so with this choice of polarization vectors, $a^\dagger_1$ creates a photon with helicity +1 and $a^\dagger_2$ creates a photon with helicity −1. This establishes that the photon is a spin one particle. There is no zero helicity state for the photon: this is consistent with Poincaré invariance because the photon is massless.

The polarization vector enters scattering amplitudes multilinearly, with a factor of $\epsilon$ for each incoming photon and a factor $\epsilon^*$ for each outgoing photon. Its four-vector index forms a Minkowski scalar product with that of the vertex coupling the gauge potential to the charged fields. According to gauge invariance, this vertex satisfies current conservation: its scalar product with the momentum entering it gives zero. Thus changing each polarization vector by an amount proportional to its four-momentum leaves the scattering amplitude unaltered. In Coulomb gauge the polarization vector is of the form $\epsilon = (\epsilon, 0)$ with $k \cdot \epsilon = 0$, so $k^\mu \epsilon^\mu = 0$. But $k^\mu k^\mu = 0$ since the photon is massless. Thus we can characterize the polarization vector completely by the covariant condition $k^\mu \epsilon^\mu = 0$. Any further specification, e.g. $\epsilon^0 = 0$, is merely a gauge choice which can be made at will and exploited to simplify detailed calculations. This is particularly advantageous in the calculation of Compton scattering for polarized photons.
15.2 Charged Fields Interacting with the Quantized Electromagnetic Field

We have seen that when the e.m. field interacts with a conserved current, the time component of the current enters the Hamiltonian only through the coefficient in the term linear in $A_0$ so that the coefficient of $A_0$ is just the Gauss’ Law constraint with nonzero charge density. This feature is quite general, even when the e.m. field couples to dynamical charged fields. This is obvious in the case of the Dirac field because the Dirac field Hamiltonian is linear in $A_\mu$. The $A$ dependence of the scalar field Hamiltonian includes quadratic pieces in the spatial components of the potential, but nonetheless $A_0$ still enters only linearly after the Hamiltonian is expressed solely in terms of coordinates and momenta. This means that the elimination of $A_0$ in the passage to Coulomb gauge proceeds exactly as in the previous section.

Thus, the Hamiltonian for e.m. field plus charged fields in Coulomb gauge will quite generally be of the form

$$H_{eff} = \int d^3x \left( \frac{1}{2} \Pi_T^2 + \frac{1}{2} \partial_k A_T \cdot \partial_k A_T - A_T \cdot J_e \right) + \int (J_e^0 + j^0) \left( -\frac{1}{2} \nabla^2 \right) \left( J_e^0 + j^0 \right) + H_{fields}\big|_{A_0=0}, \quad (15.39)$$

where $J_e^\mu$ is an optional external current, $j^\mu$ is the current operator for the fields, and the subscript $fields$ refers to the Hamiltonian operator for any dynamical fields in the system. For the Dirac field

$$j^0 = Q[\bar{\psi}, \gamma^0 \psi] \quad (15.40)$$

$$H_{fields} = \int d^3x \bar{\psi} \left( \frac{1}{i} \gamma \cdot \partial + m - Q \gamma \cdot A \right) \psi, \quad (15.41)$$

and for the scalar field,

$$j^0 = -iQ(\pi \phi - \phi^\dagger \phi^\dagger) \quad (15.42)$$

$$H_{fields} = \int d^3x (\pi \phi^\dagger \phi + (\nabla + iQA) \phi^\dagger \cdot (\nabla - iQA) \phi). \quad (15.43)$$

One obstacle to formulating an efficient perturbation theory for systems with interacting quantum fields is that one can’t “turn off” the interactions at early and late times as is possible with externally applied fields. Thus out and in states are eigenstates of complicated interacting Hamiltonians. One approach to this difficulty is to artificially make the coupling constants time dependent and force them to vanish at early and late times. Another approach, which we shall favor, is to relax the requirement that the initial and final states be eigenstates of the Hamiltonian with vanishing external fields. Then one calculates in first instance a quantity that is not of immediate interest, but which can be simply related to such quantities.
A quantity of more or less direct physical interest is the vacuum expectation value of the time ordered product of several quantum fields. More generally the output matrix element of such a time ordered product is relevant if time varying external fields are present. So let us consider how to obtain this quantity in perturbation theory by first calculating with general initial and final states. Using the evolution operator and assuming \( t_1 > t_2 > \cdots > t_n \), we therefore consider

\[
\langle f| U(\infty, -\infty) T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|i \rangle = \langle f| U(\infty, t_1) \Omega_{S_1} U(t_1, t_2) \cdots U(t_{n-1}, t_n) \Omega_{S_n} U(t_n, -\infty)|i \rangle.
\]  

(15.44)

Choose the time \( T \) so that all external fields vanish for times earlier than \(-T\) and later than \( T \). Then

\[
U(t_n, -\infty)|i\rangle = U(t_n, -T) e^{-i(\infty-T)H_S}|i\rangle = U(t_n, -T) e^{-i(\infty-T)E_G} \sum_r e^{-i(\infty-T)(E_r-E_G)}|r\rangle \langle r|i\rangle.
\]  

(15.45)

(15.46)

We would now like to argue that the infinite oscillations wash out all contributions but the (assumed nondegenerate\(^1\)) ground state. In a field theory this is quite plausible since the excited states correspond to particles so the sum over \( r \) is really an integral over a range of continuous energies. But even without this smearing, we can make the washing out rigorous by calculating with imaginary time: \( it = \beta > 0 \). Then \( i\infty \) is really +\( \infty \) and all excited states are damped exponentially. Massless particle states could introduce a subtlety here, but the part of phase space that is not exponentially damped is infinitesimal: this has the effect of changing exponential damping to a power law damping. If we buy this argument, then we can assert quite generally that \( U(t_n, -\infty)|i\rangle = U(t_n, -\infty)|0\rangle \langle 0|i \rangle \) and similarly \( \langle f| U(\infty, t_1) = \langle f|0\rangle \langle 0| U(\infty, t_1) \).

Since we take (as usual) the Heisenberg and Schrödinger pictures to coincide at \( t = -\infty \), then \( |in\rangle = |0\rangle \) and \( \langle out \rangle = \langle 0|U(\infty, -\infty) \). Thus we have obtained the relation

\[
\langle f| U(\infty, -\infty) T[\Omega_1(t_1) \cdots \Omega_n(t_n)]|i \rangle = \langle f|0\rangle \langle 0|i \rangle \langle out|in \rangle.
\]  

(15.47)

In other words calculating with any initial and final states that have finite overlap\(^2\) with the true ground state gives us a constant times the desired matrix element. We can easily evaluate the multiplicative constant by considering by the same reasoning

\[
\langle f| U(\infty, -\infty)|i \rangle = \langle f|0\rangle \langle 0|i \rangle \langle out|in \rangle \rightarrow e^{-2i\infty E_G} \langle f|0\rangle \langle 0|i \rangle \text{ External Fields = 0.}
\]  

(15.48)

(15.49)

---

\(^1\)There are interesting cases of degenerate vacua, when there is “spontaneous symmetry breakdown.” In such cases the choice of initial and final states determines which of the degenerate vacua is picked out.

\(^2\)The infinite number of degrees of freedom in quantum field theory requires care here: the overlap between different states in a theory with \( n \) degrees of freedom is typically \( f^n \) with \( f < 1 \). Since \( n = \infty \), we should expect \( \langle f|0\rangle \langle 0|i \rangle \sim e^{-\infty} \). In field theory \( n = \infty \) because the volume of space is infinite and because space is continuous. Thus strict application of the above relation should be done in the presence of both an infrared and ultraviolet cutoff, which can then be removed after extracting the desired amplitude.
Putting this into our relation we obtain

$$\langle \text{out} | T[\Omega_1(t_1) \cdots \Omega_n(t_n)] | \text{in} \rangle = e^{-2i\infty E_G} \frac{\langle f | U(\infty, -\infty) T[\Omega_1(t_1) \cdots \Omega_n(t_n)] | i \rangle}{\langle f | U(\infty, -\infty)_{Ext=0} | t \rangle}.$$  

(15.50)

where the subscript on \(U\) in the denominator denotes vanishing external fields. In field theory applications \(E_G\) is the energy of the vacuum, which is zero if we measure all energies relative to that of the vacuum. In the absence of gravity all physical quantities depend only on energy differences, so we lose nothing by doing this. Gravity couples directly to the energy density and therefore is sensitive to the energy as opposed to energy differences, but then \(E_G\) only appears in the combination \(\Lambda \equiv E_G + \Lambda_0\), with \(\Lambda_0\) the “bare” cosmological constant cosmological constant. Replacing \(\Lambda_0\) by \(\Lambda\) in effect sets \(E_G = 0\).

The formula (15.50) is a convenient starting point for developing perturbation theory. Any breakup

$$H_S(t) = H_0(t) + H'(t)$$  

(15.51)

determines an interaction picture defined by

$$\Omega_I(t) = U_0^{-1}(t, -\infty) \Omega_S U_0(t, -\infty) = U_I^{-1}(t, -\infty) \Omega(t) U_I(t, -\infty),$$  

(15.52)

where

$$i\dot{U} = H_S(t) U = U H(t)$$  

(15.53)

$$i\dot{U}_0 = U_0 H_{0I}(t)$$  

(15.54)

$$i\dot{U}_I = H_I(t) U_I$$  

(15.55)

and all \(U\)’s are the identity at \(t = -\infty\). Then the evolution operator satisfies

$$U(t_1, t_2) = U_0(t_1, -\infty) U_I(t_1, t_2) U_0(t_2, -\infty).$$  

(15.56)

Plugging these relations into (5.114) then gives

$$\langle \text{out} | T[\Omega_1(t_1) \cdots \Omega_n(t_n)] | \text{in} \rangle = e^{-2i\infty E_G} \frac{\langle f | U_0(\infty, -\infty) T[U_I(\infty, -\infty) \Omega_{I1}(t_1) \cdots \Omega_{nI}(t_n)] | i \rangle}{\langle f | U_0(\infty, -\infty)_{Ext=0} U_I(\infty, -\infty)_{Ext=0} | i \rangle}.$$  

(15.57)

This formula is completely general: we have even allowed \(H_0\) to contain time varying external fields, which is hardly ever done in practice. Since all operators in this formula are in interaction picture, it is most convenient to choose \(|i\rangle, |f\rangle\) to have simple properties with respect to \(H_{0I}(+\infty)\). Let us call the ground state of this operator \(|in, 0\rangle\). Then \langle \text{in}, 0 | U_0(\infty, -\infty) \rangle is the ground state of \(H_{0I}(+\infty)\) and therefore deserves the name \langle \text{out}, 0 \rangle. When all external fields vanish, \(H_{0I}\) is time independent and we call its ground state \(|0, I\rangle \equiv |in, 0\rangle\) and
its ground state energy $E_0$. Then $\langle in, 0|U_0(\infty, -\infty)_{Ext=0} = e^{-2i\infty E_0}\langle 0, I|$. Thus choosing $|i\rangle = |f\rangle = |0, I\rangle = |in, 0\rangle$ we obtain the useful formula

$$
\langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle = \frac{e^{-2i\infty(E_G-E_0)}\langle out, 0|T[U_I(\infty, -\infty)\Omega_{1I}(t_1)\cdots\Omega_{nI}(t_n)]|in, 0\rangle}{\langle 0, I|U_I(\infty, -\infty)_{Ext=0}|0, I\rangle}.
$$

(15.58)

In the usual case where we do not include external fields in $H_0$, the formula simplifies further

$$
\langle out|T[\Omega_1(t_1)\cdots\Omega_n(t_n)]|in\rangle = e^{-2i\infty E_G}\frac{\langle 0, I|T[U_I(\infty, -\infty)\Omega_{1I}(t_1)\cdots\Omega_{nI}(t_n)]|0, I\rangle}{\langle 0, I|U_I(\infty, -\infty)_{Ext=0}|0, I\rangle}.
$$

(15.59)

Using the Wick expansion one can describe the perturbation series for the numerators and denominators of these formulas using Feynman diagrams. The diagrams contributing to the denominator are all those completely disconnected from either external fields or from the points assigned to the operators in the numerator. The numerator contains this same sum of diagrams as a multiplicative factor. Thus the division by the denominator is achieved by simply deleting all such disconnected “vacuum” diagrams from the expansion of the numerator.
Chapter 16

Fields, Charges, and Masses of the Standard Model
## 16.1 Particles and “Particles” of the Standard Model

<table>
<thead>
<tr>
<th>Leptons</th>
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<tbody>
<tr>
<td>Particle</td>
<td>Charge</td>
<td>Spin</td>
<td>Mass</td>
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<tr>
<td>$e^\pm$</td>
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<td>$1/2$</td>
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<td>$\mu^\pm$</td>
<td>$\pm e$</td>
<td>$1/2$</td>
<td>106 MeV</td>
</tr>
<tr>
<td>$\tau^\pm$</td>
<td>$\pm e$</td>
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<tr>
<td>$\nu_1$</td>
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<td>$m_1$</td>
</tr>
<tr>
<td>$\nu_2$</td>
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<td>$1/2$</td>
<td>$[m_1^2 + 8 \times 10^{-5}]^{1/2}$ eV</td>
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<tr>
<td>$\nu_3$</td>
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<td>$1/2$</td>
<td>$[m_1^2 \pm 2 \times 10^{-3}]^{1/2}$ eV</td>
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<tr>
<th>Electro-weak Bosons</th>
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<tr>
<td>$\gamma$</td>
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<td>0</td>
</tr>
<tr>
<td>$W^\pm$</td>
<td>$\pm e$</td>
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<td>80 GeV</td>
</tr>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>91 GeV</td>
</tr>
<tr>
<td>$h$</td>
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<td>0</td>
<td>125 GeV</td>
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<table>
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<tr>
<th>Quarks</th>
<th>Charge</th>
<th>Spin</th>
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<tr>
<td>$d$ $(\times 3)$</td>
<td>$-e/3$</td>
<td>$1/2$</td>
<td>“4.7” MeV</td>
</tr>
<tr>
<td>$s$ $(\times 3)$</td>
<td>$-e/3$</td>
<td>$1/2$</td>
<td>“94” MeV</td>
</tr>
<tr>
<td>$b$ $(\times 3)$</td>
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<td>$1/2$</td>
<td>“4.2” GeV</td>
</tr>
<tr>
<td>$u$ $(\times 3)$</td>
<td>$2e/3$</td>
<td>$1/2$</td>
<td>“2.2” MeV</td>
</tr>
<tr>
<td>$c$ $(\times 3)$</td>
<td>$2e/3$</td>
<td>$1/2$</td>
<td>“1.3” GeV</td>
</tr>
<tr>
<td>$t$ $(\times 3)$</td>
<td>$2e/3$</td>
<td>$1/2$</td>
<td>“173” GeV</td>
</tr>
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<th>Strong Bosons</th>
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<th></th>
<th></th>
</tr>
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<tbody>
<tr>
<td>$G$ $(\times 8)$</td>
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<td>1</td>
<td>“0”</td>
</tr>
</tbody>
</table>

The preceding table lists all of the particles in the standard model for which fundamental fields are introduced. We have listed quarks and gluons as “particles” because none of them can exist in isolation: they are trapped (confined) inside of hadrons. Quarks always come in groups of three or quark antiquark pairs. The gluons come in groups of two or more. Because of confinement the notion of mass is not precise and hence we put the mass values in quotes. They are theoretical parameters that roughly correspond to our heuristic notions of mass.
Chapter 17

Feynman Rules for QED

17.1 Lagrangian (density) for QED

Spinor QED

\[ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad (17.1) \]

\[ \mathcal{L}^{1/2}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi} (m - i\gamma^\mu (\partial_\mu - iQ_0 A_\mu)) \psi \quad (17.2) \]

Scalar QED

\[ \mathcal{L}^0_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (\partial + iQ_0 A) \phi^\dagger \cdot (\partial - iQ_0 A) \phi - \mu^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2 \quad (17.3) \]

17.2 Rules in Coordinate Space

17.2.1 Propagators

\[ S_F(x - y)_{ab} = -i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - i\epsilon} e^{i(x - y) \cdot k} \]

\[ D_{F\mu\nu}(x - y) = -i \int \frac{d^4k}{(2\pi)^4} \frac{\eta_{\mu\nu} - (1 - \alpha)(k_\mu k_\nu/k^2)}{k^2 - i\epsilon} e^{i(x - y) \cdot k} \]

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\[ \Delta_F(x - y) = -i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - i\epsilon} e^{i(x-y) \cdot k} \]

### 17.2.2 Vertices

The vertices are determined from the nonquadratic terms of \( i \) times the Lagrange density.

\[ iQ_0 A_\mu \bar{\psi} \gamma^\mu \psi, \quad \text{Spinor} \quad (17.5) \]
\[ Q_0 A_\mu (\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi) - iQ_0^2 A \cdot A \phi^\dagger \phi \quad (17.6) \]

![Diagram showing vertices and propagators](image)

### 17.2.3 Rules for Calculating \( \langle \psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_n) \cdots \bar{\psi}(y_1) A(z_1) \cdots A(z_m) \rangle \)

1. Draw all possible graphs connecting together the points \((x_i, y_i, z_i)\). Drop all disconnected vacuum bubbles. Associate with each graph the product of propagators and vertices according to the above table. Integrate over all internal points.

2. Each distinct graph has a weight \( \pm 1 \) as follows:
   a) For each closed fermion loop include a factor \((-1)\).
   b) Two graphs of identical structure except for a permutation of the \(x_i\)'s or of the \(y_i\)'s have a relative minus (plus) sign if the total permutation is odd (even).

### 17.2.4 Rules for Scattering Amplitudes

1. Drop all corrections to all external lines.

2. Replace propagators associated with external lines by the following factors:

   **Outgoing Electron:**
   \[
   \frac{\sqrt{Z_2}}{(2\pi)^{3/2} \sqrt{2\omega(p)}} \bar{u}_\lambda(p) e^{-i \cdot p} 
   \]

   **Incoming Electron:**
3. $Z_2$ and $Z_3$ are obtained from examining the poles in the F.T. of the two point functions:

\[
\int d^4x e^{-iq\cdot x} \langle 0 | T[\psi(x)\bar{\psi}(0)] | 0 \rangle \sim \frac{Z_2}{(2\pi)^{3/2} \sqrt{2\omega(p)}} u_\lambda(p) e^{iy\cdot p}
\]

\[
\int d^4x e^{-iq\cdot x} \langle 0 | T[A_\mu(x)A_\nu(0)] | 0 \rangle \sim \frac{Z_3}{(2\pi)^{3/2} \sqrt{2\omega(p)}} \epsilon^\ast_{\lambda}(q) e^{iy\cdot q}
\]

\[
\int d^4x e^{-iq\cdot x} \langle 0 | T[A_\mu(x)(1-\alpha' q\cdot q/q^2)] | 0 \rangle \sim \frac{Z_3}{(2\pi)^{3/2} \sqrt{2\omega(p)}} \epsilon_{\lambda}(q) e^{iy\cdot q}
\]

Note that $\alpha' \neq \alpha$ in general. Also $Z_3$ is gauge invariant whereas $Z_2$ is not.
17.2.5 Cross Sections and Decay Rates

Let $\mathcal{M}$ be the amplitude obtained from the above rules by dropping the factors $1/[(2\pi)^{3/2}\sqrt{2E}]$ associated with the external lines and also dropping the overall momentum conserving delta function factor $(2\pi)^4\delta(\sum_f p'_f - \sum_i p_i)$. Then the differential cross section for $2 \rightarrow N$ particle scattering is:

$$d\sigma = \frac{d^3p'_1 \cdots d^3p'_N}{(2\pi)^32E'_1 \cdots (2\pi)^32E'_N}(2\pi)^4\delta(\sum_f p'_f - \sum_i p_i) \frac{1}{4E_1 E_2|v_2 - v_1|}|\mathcal{M}|^2.$$  \hfill (17.9)

Here $v_i$ are the velocities of the particles in the initial state. The initial state momenta are $p_1, p_2$ and the final state momenta all carry primes.

For a decay process $1 \rightarrow N$ the differential decay rate in the rest frame of the decaying particle is

$$d\Gamma = \frac{d^3p'_1 \cdots d^3p'_N}{(2\pi)^32E'_1 \cdots (2\pi)^32E'_N}(2\pi)^4\delta(\sum_f p'_f - p) \frac{1}{2M}|\mathcal{M}|^2,$$  \hfill (17.10)

where $p = (M, 0)$.

To obtain total cross sections and total rates one must integrate these expressions over all final state momenta and summing over all final spin states. In addition, if some of the final state particles are identical one must include the statistical factor $1/r_k!$ for each subset $k$ of $r_k$ identical particles to compensate for multiple counting of identical states when these final state sums are carried out independently.

17.3 Rules in Momentum Space

17.3.1 Propagators

$$S_F(p)_{ab} = -i \frac{m\delta_{ab} - \gamma_{ab} \cdot p}{m^2 + p^2 - i\epsilon}.$$  

$$D_F^{\mu\nu}(k) = -i \frac{\eta^{\mu\nu} - (1 - \alpha)(k^\mu k^\nu/k^2)}{k^2 - i\epsilon}.$$  

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1. Each line carries a momentum obeying the constraint that energy-momentum is conserved at each vertex.

2. The rules for drawing graphs are identical to those in coordinate space.

3. Each unconstrained internal momentum is integrated with weight $d^4p/(2\pi)^4$.

4. The external line factors for $\mathcal{M}$ are simply

Outgoing Electron:

\[ \sqrt{Z_2 \bar{u}_\lambda(p)} \]

Incoming Electron:

\[ \sqrt{Z_2 u_\lambda(p)} \]

Outgoing Positron

\[ \sqrt{Z_2 v_\lambda(p)} \]

Incoming Positron:

\[ \sqrt{Z_2 \bar{v}_\lambda(p)} \]

Outgoing Photon:

\[ iQ_0 \gamma_{ab}^\mu. \]
17.4 Divergence structure of Feynman diagrams

Since ultraviolet divergences are unavoidable in quantum field theory, it is useful to have a criterion for deciding the degree of divergence of a given (connected) diagram. This is the (naive) degree of divergence $D$, let $L$ be the number of loops; $V_n$ be the number of vertices with $n$ powers of momentum, $I_B$ ($E_B$) be the number of internal (external) boson lines $I_F$ ($E_F$) the number of internal (external) fermion lines. Then

$$D = DL - I_F - 2I_B + \sum_n nV_n$$

where $D$ is the spacetime dimension. The number of loops is $I_B + I_F$ less the number of vertices $\sum_n V_n$ plus 1, because there is a momentum conserving delta function at each vertex, but one of them simply conserves overall momentum and doesn’t reduce the number of momentum integrals. With this information we have

$$D = D(I_B + I_F - \sum_n V_n + 1) - I_F - 2I_B + \sum_n nV_n$$

$$= D + I_B(D - 2) + I_F(D - 1) + \sum_n (n - D)V_n$$

To go further we need more detailed information about the vertices. So first consider spinor QED. Then the only vertex has $n = 0$ and two fermion lines and one boson line. Each vertex has 1 end of a boson line. The number of such ends available is $E_B + 2I_B$ so $V = E_B + 2I_B$. On the other hand each vertex has two ends of a fermion line so $V = E_F/2 + I_F$. Thus

$$I_B = \frac{V - E_B}{2}, \quad I_F = V - \frac{E_F}{2}$$

$$D = D + \frac{D - 4}{2}V - E_B \frac{D - 2}{2} - E_F \frac{D - 1}{2}$$
For $D < 4$ adding vertices and/or external lines makes the degree of divergence less. For $D = 4$ (our dimension) adding vertices doesn’t improve $\mathcal{D}$. However adding external lines does. This is the criterion for a renormalizable theory. (The first case is superrenormalizable).

Coming back to spinor QED, we see that $\mathcal{D} \geq 0$ for the limited number of processes $E_F = 0, E_B = 2, 3, 4$, and $E_F = 2, E_B = 0, 1$. We shall find that the situation is even better than that: because of gauge invariance, the effective $\mathcal{D}$ is negative for $E_F = 0, E_B = 4$, and the process $E_B = 3, E_F = 0$ is zero by Furry’s theorem (Charge conjugation invariance. Furthermore the effective $\mathcal{D}$ for vacuum polarization is only 0, rather than 2.

1 Loop Examples: 2 photons, 3 photons, 4 photons, 2 electrons, 4 electrons, 2 photons and 2 electrons, 2 electrons and 1 photon

Exercise: Find the formula for $\mathcal{D}$ for scalar electrodynamics, scalar $\phi^3$ theory, and scalar $\phi^4$ theory in arbitrary spacetime dimension $D$. In each case draw the one loop diagrams that are divergent in 4 spacetime dimensions, and also, for the $\phi^3$ case in 6 spacetime dimensions.
Chapter 18

Scattering Amplitudes in Quantum Field Theory

The observables of quantum field theory are local quantum fields\(^1\). For example the energy momentum can be expressed as an integral of the local energy momentum tensor.

\[ P^\mu = (P^0, \mathbf{P}) = \int d^3x \theta^{\mu 0}(x,t) \quad (18.1) \]

where \( \theta^{\mu \nu}(x) \), the energy momentum tensor, is a local operator:

\[ [\theta^{\mu \nu}(x), \theta^{\rho \sigma}(y)] = 0 \quad (x - y)^2 > 0. \quad (18.2) \]

The treatment of scattering in quantum field theory which I describe in this chapter follows the method of Francis Low, developed in the late 1950’s. His key innovation was the recognition that wave packets must smear in time as well as space.

**Single Particle States**

\( |\mathbf{p},a\rangle \), with \( \langle \mathbf{p},a|\mathbf{p}',b\rangle = \delta_{ab} \delta^3(\mathbf{p} - \mathbf{p}') \) are characterized by possessing a unique energy for each momentum:

\[ P^\mu |\mathbf{p},a\rangle = \left( \sqrt{m_a^2 + \mathbf{p}^2}, \mathbf{p} \right) |\mathbf{p},a\rangle. \quad (18.3) \]

Single particles can be elementary like leptons, and gauge bosons, or they can be composites like nucleons, mesons, nuclei, and even atoms or molecules.

The space-time picture of scattering processes requires the use of wave packet single particle states:

\[ |f, a\rangle \equiv \int d^3p f(\mathbf{p})|\mathbf{p},a\rangle. \quad (18.4) \]

Their physical properties follow by considering for any local operator \( \Omega(x) \),

\[ \langle f, a|\Omega(x)|f, a\rangle = \int d^3p'd^3p f^*(\mathbf{p}') f(\mathbf{p}) e^{i(p-p')x} \langle \mathbf{p}', a|\Omega(0)|\mathbf{p}, a\rangle, \quad (18.5) \]

\(^1\)Our discussion of scattering follows the treatment by Francis Low, late 1950’s.
where we used translation invariance in space and time \( \Omega(x) = e^{-iP \cdot x} \Omega(0) e^{iP \cdot x} \). This means we are restricting consideration to vanishing external fields. Assuming \( f \) is peaked at \( p_0 \), over an interval small compared to important variations of \( \langle p', a | \Omega(0) | p, a \rangle \), we have

\[
\langle f, a | \Omega(x) | f, a \rangle \approx \langle p_0, a | \Omega(0) | p_0, a \rangle \left[ \int d^3p' f^*(p') e^{-i(p' \cdot x - \omega_a(p') t)} \right] \\
\approx \left| f(x, t) \right|^2 \langle p_0, a | \Omega(0) | p_0, a \rangle .
\]

We can select \( f(p) \) so that at some particular time, \( t \),

\[
f(x, t) = \int d^3p f(p)e^{i(p \cdot x - \omega_a(p) t)}
\]
is confined to some spatial volume \( V(t) \). By construction \( f(x, t) \) satisfies a free particle Schrödinger equation:

\[
i \partial_t f = \omega_a (-i \nabla) f.
\]

As time evolves this volume will move and spread. The center of the packet will follow a straight line trajectory at the group velocity

\[
v_g = \frac{d \omega_a(p)}{dp} \bigg|_{p=p_0} = \frac{p_0}{\omega_a(p_0)} = v_0
\]

and spreading will be negligible over a time interval \( \ll \omega_a(p_0) / (\Delta p)^2 < \Delta x / \Delta v \) where \( \Delta p \) is the width of the peak in \( f_a \) at \( p_0 \).

**Creation Operators for Single Particle States**

We assume that for each single particle state, \( a \), there is a local operator \( \Omega_a(x) \) such that

\[
\langle 0 | \Omega_a(0) | p, a \rangle \neq 0
\]

\( \Omega_a \) must carry the quantum numbers of \( a \), but is otherwise unspecified. A given \( \Omega \) need not be a fundamental field, and in the case of composite particles will not be. For example, the \( \Omega \) for a proton at least must be a product of 3 quark fields, such as \( q_u q_d \).

Introduce wave packets \( f_a(p) \) peaked about some momentum with a narrow width \( \Delta p \) assumed much smaller than any variation in matrix elements or \( \omega_a(p) \). These packets are selected so that

\[
f_a(x, t) = \int \frac{d^3p}{(2\pi)^3} f_a(p) e^{i(p \cdot x - \omega_a(p) t)}
\]
is confined to a volume \( V_a(t) = 0 (1/\Delta p^3) \) with negligible spreading in the interval \(-T/2 < t < +T/2\). Introduce also a switching function \( F(t) \) with shape
and normalized by

$$\int_{-\infty}^{\infty} dt F(t) = \tilde{F}(0) = 1, \quad \tilde{F}(\omega) = \int dt F(t)e^{i\omega t}. \quad (18.12)$$

We stipulate that $1/T_0$ is much smaller than any important momentum variation in matrix elements or the function $\omega_a(p)$. In particular, $1/T_0$ is much smaller than any mass differences. This means that $\tilde{F}(\omega)$ the Fourier transform of $F$ can be chosen sharply peaked in $\omega$ about $\omega = 0$. The width of this peak is only limited by $\Delta \omega > 1/T_0$. Then, we define (for $-T/2 < t_0 < T/2$)

$$A^\dagger_a(t_0) \equiv \int d^4x f^*_a(x, t) F(t - t_0)\Omega^\dagger_a(x) \quad (18.13)$$

This idea of smearing the packets over time was Low’s innovation. The essential point here is that $\int d^4x$ forces $A^\dagger_a(t_0)$ to create energy-momentum in a narrow range about $(p_0, \omega_a(p_0))$. It then follows:

1. $A^\dagger_a(t_0)$ creates a particle, $a$, in a wave packet $f_a$ at $t_0$.
2. $A_a(t_0)$ destroys a particle, $a$, in a wave packet $f_a$ at $t_0$.

Proof:

$$\langle 0 | A^\dagger_a(t_0) | p, b \rangle = \int d^4x f^*_a(x, t) F(t - t_0) \langle 0 | \Omega_a(0) | p, b \rangle e^{ip_0 \cdot x} \quad (18.14)$$

$$= \int_{-\infty}^{\infty} dt f^*_a(p) e^{i(\omega_a(p) - \omega_b(p))t} F(t - t_0) \langle 0 | \Omega_a(0) | p, b \rangle \quad (18.15)$$

$$= f^*_a(p) \langle 0 | \Omega_a(0) | p, b \rangle e^{i(\omega_a(p) - \omega_b(p))t} \tilde{F}(\omega_a(p) - \omega_b(p)) \neq 0 \text{ only when } m_a = m_b$$

$$\approx \delta_{ab} f^*_a(p) \langle 0 | \Omega_a(0) | p_0, b \rangle \quad (18.16)$$

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The last approximate equality assumes each mass value is non degenerate, and is based on the narrow peaking of $\tilde{F}$. The equality is exact as long as the support of $\tilde{F}$ lies entirely within the gap between $\omega_a$ and any other energy available to the system. The fundamental assumption is that each single particle for which such a creation operator can be defined has a mass separated by a finite gap from any other mass. If the state $b$ is a multiparticle state, its energy is not discrete but must in any case be greater than the sum of the masses of the particles it describes. This minimum multiparticle energy is also assumed to be separated by a gap from $m_a$. One can handle degeneracies in mass by carefully choosing the operators $\Omega_a$ so that they decouple from the single particle states degenerate with particle $a$. Clearly the construction does not completely succeed if there are massless particles in the theory because then a multiparticle state containing particle $a$ and several low energy massless particles can have an energy arbitrarily close to $m_a$. In fact, in this case it is experimentally impossible to give a definite meaning to “single particle state” so the limitations of the theory are appropriate.

Normalization:

\[
\langle 0 | A_a(t_0) A_a^\dagger(t_0) | 0 \rangle \approx \int d^3p \langle 0 | A_a(t_0) | p, a \rangle \langle p, a | A_a^\dagger(t_0) | 0 \rangle
= \int d^3p \left| f_a(p) \right|^2 \langle 0 | \Omega_a(0) | p, a \rangle^2
\]

(18.17)

\[
\approx \left| \langle 0 | \Omega_a(0) | p_0, b \rangle \right|^2 \quad \text{since } f_a \text{ is sharply peaked.}
\]

The first approximate equality follows because after insertion of a complete set of states between $A$ and $A^\dagger$, the smearing functions in the definition of the latter focus the energy and momentum of the intermediate states to be (within the allowed windows $\sim (1/T_0, \Delta p)$) those of the single particle state $a$. Thus only those give a significant contribution. The last approximate equality is based on the narrow peaking of the wave packet $f_a$.

### 18.1 Multiparticle States

We can use our creation operators to construct multiparticle states. For example

\[
A_b^\dagger(t_0) A_a^\dagger(t_0) | 0 \rangle
\]

(18.18)

creates a two particle state. Of course, there may be interactions between the two particles, so this interpretation is only meaningful when packet, $a$, is spacelike separated from packet, $b$. In this case, we have

\[
\left[ A_a^\dagger(t_0), A_b^\dagger(t_0) \right] = 0 \quad \text{(Since } [\Omega_a(x), \Omega_b(y)] = 0 \text{ for } (x - y)^2 > 0) \quad (18.19)
\]

from which it follows that the norm of the two particle state factorizes into the product of single particle norms:

\[
\langle 0 | A_a(t_0) A_b(t_0) A_a^\dagger(t_0) A_b^\dagger(t_0) | 0 \rangle \approx \langle 0 | A_a(t_0) A_b^\dagger(t_0) | 0 \rangle \langle 0 | A_b(t_0) A_a^\dagger(t_0) | 0 \rangle
\]
Furthermore, for any operator of the form \( \Omega = \int d^3x f(x) \Omega(x, t_0) \), we have

\[
\langle 0| A_a(t_0) A_b(t_0) \Omega A^\dagger_a(t_0) A^\dagger_b(t_0)|0\rangle \approx \langle 0| A_a(t_0) \Omega A^\dagger_a(t_0)|0\rangle \langle 0| A_b(t_0) A^\dagger_b(t_0)|0\rangle + \langle 0| A_a(t_0) A^\dagger_a(t_0)|0\rangle \langle 0| A_b(t_0) \Omega A^\dagger_b(t_0)|0\rangle
\]

This follows since we can write effectively

\[
\Omega \sim \int_{V_a(t_0)} d^3x f(x) \Omega(x, t_0) + \int_{V_b(t_0)} d^3x f(x) \Omega(x, t_0)
\]

and the two pieces act independently on particles \( a \) and \( b \). Since all the observables of a quantum field theory are local fields such as \( \Omega(x) \), this justifies the multiparticle interpretation. It should be clear that one can extend this construction to states with any number of spatially separated particles.

### 18.2 Reduction Formulae

Assume \( -\frac{T}{2} < t_1, t_2, \ldots, t_n < \frac{T}{2} \)

\[
T \left\{ A_a \left( \frac{T}{2} \right) \Omega_1(x_1) \ldots \Omega_n(x_n) \right\} - T \left\{ \Omega_1(x_1) \ldots \Omega_n(x_n) A_a \left( -\frac{T}{2} \right) \right\} = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \frac{d}{dt_0} T \{ A_a(t_0) \Omega_1(x_1) \ldots \Omega_n(x_n) \} \quad (18.21)
\]

\[
= \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \int d^4x F(t - t_0) \frac{\partial}{\partial t} f_a^*(x, t) T \{ \Omega_a(x) \Omega_1(x_1) \ldots \Omega_n(x_n) \}
\]

\[
= \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \int d^4x F(t - t_0) f_a^*(x, t) \left( i\omega_a (-i\nabla) + \frac{\partial}{\partial t} \right) T \{ \Omega_a(x) \Omega_1(x_1) \ldots \Omega_n(x_n) \}
\]

\[
T \left\{ A^\dagger_a \left( \frac{T}{2} \right) \Omega_1(x_1) \ldots \Omega_n(x_n) \right\} - T \left\{ \Omega_1(x_1) \ldots \Omega_n(x_n) A^\dagger_a \left( -\frac{T}{2} \right) \right\} = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \int d^4x F(t - t_0) \frac{\partial}{\partial t} f_a(x, t) T \{ \Omega^\dagger_a(x) \Omega_1(x_1) \ldots \Omega_n(x_n) \} \quad (18.22)
\]

\[
= \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \int d^4x F(t - t_0) f_a(x, t) \left( -i\omega_a (i\nabla) + \frac{\partial}{\partial t} \right) T \{ \Omega^\dagger_a(x) \Omega_1(x_1) \ldots \Omega_n(x_n) \}.
\]
In these reduction formulae we have used the fact that by construction $f_a$ satisfies a Schrödinger equation with hamiltonian $\omega_a(-i\nabla) = \sqrt{m_a^2 - \nabla^2}$. Spatial integration by parts then allows the spatial derivatives to be transferred to $\Omega(x)$. In most textbooks, the form of the reduction procedure is that of LSZ (Lehman, Symanzik, and Zimmermann), which does not take advantage of Francis Low’s smearing over time. Instead they must imagine the actual scattering process to extend from $t = -\infty$ to $+\infty$, with the assumption that asymptotic states will always be spatially separated single particles. Low’s approach is more physical: the scattering process only need endure for a finite time interval; no assumption about asymptotic states need be made; and it treats the case of massless particles appropriately.

### 18.3 Single Particle States are Handled Consistently

A single particle state prepared at early times should remain a single particle state for all times. This follows from the reduction formulae:

\[
\langle 0|A_a \left(\frac{T}{2}\right) A_b^\dagger \left(\frac{T}{2}\right) |0\rangle = \langle 0|A_a \left(-\frac{T}{2}\right) A_b^\dagger \left(-\frac{T}{2}\right) |0\rangle
\]

\[
+ \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \int d^4x F(t-t_0) f_a^*(x, t) \left(\frac{\partial}{\partial t} + i\omega_a(-i\nabla)\right) \langle 0|\Omega_a(x) A_b^\dagger \left(-\frac{T}{2}\right) |0\rangle
\]

But $\langle 0|\Omega_a(x) A_b^\dagger \left(-\frac{T}{2}\right) |0\rangle \approx \int d^3p \langle 0|\Omega_a(0) |p, b\rangle \langle p, b| A_b^\dagger \left(-\frac{T}{2}\right) |0\rangle e^{i(p\cdot x - \omega_b(p)t)}$.

So the second term becomes

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 \int dt \int d^3p f_a^*(p) F(t-t_0) i (\omega_a(p) - \omega_b(p)) e^{i(\omega_a(p) - \omega_b(p))t} \langle 0|\Omega_a(0) |p, b\rangle \langle p, b| A_b^\dagger \left(-\frac{T}{2}\right) |0\rangle
\]

\[
= \int d^3p \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0 e^{i(\omega_a(p) - \omega_b(p))t_0} f_a^*(p) \tilde{F}(\omega_a(p) - \omega_b(p)) i (\omega_a(p) - \omega_b(p)) \langle 0|\Omega_a(0) |p, b\rangle \langle p, b| A_b^\dagger |0\rangle \approx 0
\]

Since $\tilde{F} \approx 0$ if $b \neq a$ and the r.h.s. is identically zero if $a = b$ because of the factor $\omega_a(p) - \omega_b(p)$. The approximate equalities become exact in the limit

\[
T, T_0 \to \infty \quad \text{with} \quad \frac{1}{\delta m} \ll T_0 \ll T \ll T_0^2 m
\]

(The latter inequality insures negligible wave packet spread.)

We therefore obtain the required result:

\[
\langle 0|A_a \left(\frac{T}{2}\right) A_b^\dagger \left(\frac{T}{2}\right) |0\rangle \approx \lim \langle 0|A_a \left(-\frac{T}{2}\right) A_b^\dagger \left(-\frac{T}{2}\right) |0\rangle
\]

\[
\approx \left|\langle 0|\Omega_a(0) |p_0, a\rangle\right|^2 \delta_{ab}
\]

where the approximate equality becomes exact as $T_0, T \to \infty$.  

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18.4 Two Particle Scattering Amplitudes

The scattering process will consist of preparation at a very early time $-T'/2$ of two separated single particle wave packets with well defined momentum, aimed to collide at roughly time $t \approx 0$, and the subsequent observation at a much later time $+T/2$ of two well separated wave packets.

Let $f_a, f_b$ describe the incoming packets with momentum $p_a$ and $p_b$ respectively; $f_c, f_d$ the outgoing packets with momentum $p_c, p_d$. The information we want is contained in the matrix element

$$\langle 0 | A_c \left( \frac{T}{2} \right) A_d \left( \frac{T}{2} \right) A_a^\dagger \left( -\frac{T'}{2} \right) A_b^\dagger \left( -\frac{T'}{2} \right) | 0 \rangle \approx \langle 0 | \Omega_c | p_c \rangle \langle 0 | \Omega_d(0) | p_d \rangle \langle 0 | \Omega_a^\dagger(0) | 0 \rangle \langle 0 | \Omega_b^\dagger(0) | 0 \rangle$$

(18.26)

$$\int d^3 p' \int d^3 q' \int d^3 p \int d^3 q f_c^*(p') f_d^*(q') f_a(p) f_b(q) \langle p', c; q', d | S | p, a; q, b \rangle,$$

which can be taken as the definition of the $S$-matrix. Now, we apply the Reduction Procedure.

$$\langle 0 | A_c \left( \frac{T}{2} \right) A_d \left( \frac{T}{2} \right) A_a^\dagger \left( -\frac{T'}{2} \right) A_b^\dagger \left( -\frac{T'}{2} \right) | 0 \rangle = \langle 0 | A_c \left( \frac{T}{2} \right) A_d \left( -\frac{T'}{2} \right) A_a^\dagger \left( -\frac{T'}{2} \right) A_b^\dagger \left( -\frac{T'}{2} \right) | 0 \rangle$$

(18.27)

$$+ \int_{-T/2}^{T/2} dt_0 \int d^4 x_1 F(t_1 - t_0) f_d^*(x_1) \left( \frac{\partial}{\partial t_1} + i \omega_d(i \nabla_1) \right)$$

(18.28)

$$\langle 0 | A_c \left( \frac{T}{2} \right) \Omega_d(x_1) A_a^\dagger \left( -\frac{T'}{2} \right) A_b^\dagger \left( -\frac{T'}{2} \right) | 0 \rangle$$

(18.29)

The first term on the RHS contains the amplitude that nothing happens. The final packet $d$ has been extrapolated back as a free particle to time $-T/2$. If no scattering has occurred this extrapolated packet can overlap with initial packet $a$ or packet $b$ but not both since $a$ and $b$ are spatially separated. Thus $A_d(-T/2)$ will commute with $A_a(-T'/2)$ or $A_b(-T'/2)$.
or both. Both possibilities are included by writing

\[ \langle 0 | A_c \left( \frac{T}{2} \right) A_d \left( -\frac{T}{2} \right) A^\dagger_a \left( -\frac{T'}{2} \right) A^\dagger_b \left( -\frac{T'}{2} \right) | 0 \rangle \]

\[ \approx \delta_{m_a,m_d} \langle 0 | A_c \left( \frac{T}{2} \right) A^\dagger_b \left( -\frac{T}{2} \right) | 0 \rangle \langle 0 | \Omega_d(0) | p_d \rangle \langle \Omega^\dagger_a(0) | 0 \rangle \int d^3p f^*_a(p) f_a(p) \]

\[ + \delta_{m_b,m_d} \langle 0 | A_c \left( \frac{T}{2} \right) A^\dagger_a \left( -\frac{T'}{2} \right) | 0 \rangle \langle 0 | \Omega_d(0) | p_d \rangle \langle \Omega^\dagger_b(0) | 0 \rangle \int d^3p f^*_a(p) f_b(p) \]

\[ \approx \left[ \delta_{m_a,m_d} \delta_{m_b,m_c} \int d^3p f^*_a(p) f_a(p) \int d^3p f^*_c(p) f_b(p) \right] \]

\[ + \delta_{m_a,m_c} \delta_{m_b,m_d} \int d^3p f^*_a(p) f_b(p) \int d^3p f^*_c(p) f_a(p) \]

\[ \times \langle 0 | \Omega_d(0) | p_d \rangle \langle 0 | \Omega_c(0) | p_c \rangle \langle \Omega^\dagger_b(0) | 0 \rangle \langle p_b | \Omega^\dagger_b(0) | 0 \rangle \]

where \( \delta_{mm'} = 0 \) if \( m \neq m' \), 1 if \( m = m' \). Of course if the extrapolated packet overlaps with neither \( a \) nor \( b \) this term contributes nothing, as the above formula states because then \( f_d \) will be orthogonal to both \( f_a \) and \( f_b \).

In the 2nd term we now reduce particle \( c \):

\[ \langle 0 | A_c \left( \frac{T}{2} \right) \Omega_d(x_1) A^\dagger_a \left( -\frac{T'}{2} \right) A^\dagger_b \left( -\frac{T'}{2} \right) | 0 \rangle \]

\[ \approx \langle 0 | \Omega_d(x_1) A_c \left( -\frac{T}{2} \right) A^\dagger_a \left( -\frac{T'}{2} \right) A^\dagger_b \left( -\frac{T'}{2} \right) | 0 \rangle \]

\[ + \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_0^{(2)} \int d^4x_2 F(t_2 - t_0^2) f^*_c(x_2) \left( \frac{\partial}{\partial t_2} + i\omega_c(i\nabla_2) \right) \]

\[ \times \langle 0 | T[\Omega_c(x_2) \Omega_d(x_1)] A^\dagger_a \left( -\frac{T'}{2} \right) A^\dagger_b \left( -\frac{T'}{2} \right) | 0 \rangle \]

The first term will give negligible contribution because the factor \( \left( \frac{\partial}{\partial t} + i\omega_c(-i\nabla) \right) \) will yield a factor \( i(\omega_d + \omega_c - \omega_a - \omega_b) \) which will vanish within the support of the factor \( F(\omega_d + \omega_c - \omega_a - \omega_b) \). (A nonvanishing contribution would require either \( c = a \) or \( c = b \), because \( A_c \) is a destruction operator and then the support of \( F \) requires \( d = b \) or respectively \( d = a \).) In this second reduction we have tacitly written \( A_c \left( \frac{T}{2} \right) \Omega_d(x_1) = T[A_c \left( \frac{T}{2} \right) \Omega_d(x_1)] \) which is not strictly true since \( A_c \left( \frac{T}{2} \right) \) can involve \( \Omega_c(y) \) at some times earlier than \( t_1 \). However the integrands include wave packets which are spacelike separated for times near \( \frac{T}{2} \). Since \( \Omega_c \) and \( \Omega_d \) commute at spacelike separations no mistake is made by this procedure. Similarly the replacement

\[ T \left[ \Omega_d(x) A_c \left( -\frac{T}{2} \right) \right] = \Omega_d(x) A_c \left( -\frac{T}{2} \right) \]

is validated because the wave packets extrapolated back to times near \( -\frac{T}{2} \) will again be spacelike separated. (We assume packets are aimed to overlap at times near 0.)
To continue reducing, we need to replace
\[ T[\Omega_c(x_2)\Omega_d(x_1)]A^\dagger_a(-T'/2) \]
by
\[ T[\Omega_c(x_2)\Omega_d(x_1)]A^\dagger_b(-T'/2) \]
. The above argument could be applied for nonforward but not quite for forward scattering
because then one of the final packets extrapolated to time \(-T/2\) could well overlap one of
the initial packets. However, by choosing \(T'/2 > T/2 + T_0\), we force \(-T'/2\) to be earlier than
either \(t_1\) or \(t_2\) and likewise \(T'/2\) to be later than \(t_1\) or \(t_2\). Then the required replacements
are valid. As long as \(-T/2\) is well before and \(T/2\) well after the collision, such a choice is
completely satisfactory.

With this in mind we can reduce particles \(a\) and \(b\). We drop immediately terms where
\(A^\dagger\) stands next to \(\langle 0\rangle\), because they could only produce a negative energy state which we
assume does not exist. The final result is:

\[
\langle 0|\Omega_c(0)|p_c\rangle\langle 0|\Omega_d(0)|p_d\rangle\langle p_a|\Omega^\dagger_a(0)|0\rangle\langle p_b|\Omega^\dagger_b(0)|0\rangle \int d^3p'd^3q' \int d^3pd^3q \\
\times \text{f}^*_c(p')f_d^*(q')f_a(p)f_b(q)|p', c; q', d|(S - I)|p, a; q, b\rangle \\
\approx \int_{-T/2}^{T/2} dt_0^{(1)}dt_0^{(2)} \int_{-T/2}^{T/2} dt_0^{(3)}dt_0^{(4)} \int d^4x_1d^4x_2d^4x_3d^4x_4 \\
F(t_1 - t_0^1)F(t_2 - t_0^2)F(t_3 - t_0^3)F(t_4 - t_0^4)\text{f}_a^*(x_1)f_b^*(x_2)a(x_3)f_a(x_4) \\
\left(\frac{\partial}{\partial t_1} + i\omega_d\right)\left(\frac{\partial}{\partial t_2} + i\omega_c\right)\left(\frac{\partial}{\partial t_3} - i\omega_a\right)\left(\frac{\partial}{\partial t_4} - i\omega_b\right) \\
\langle 0|T[\Omega_c(x_2)\Omega_d(x_1)\Omega^\dagger_a(x_3)\Omega^\dagger_b(x_4)]|0\rangle \tag{18.32}
\]
Now define
\[ T[q_2q_1; q_3q_4] \equiv \int d^4x_1d^4x_2d^4x_3d^4x_4 e^{-i(q_2x_2 + q_1x_1 - q_3x_3 - q_4x_4)} \\
\langle 0|T[\Omega_c(x_2)\Omega_d(x_1)\Omega^\dagger_a(x_3)\Omega^\dagger_b(x_4)]|0\rangle \tag{18.33}
\]
and the r.h.s. becomes
\[
\int_{-T/2}^{T/2} dt_0^{(1)}dt_0^{(2)} \int_{-T/2}^{T/2} dt_0^{(3)}dt_0^{(4)} \prod_{m} \frac{d^4q_m}{(2\pi)^4} F(\omega_d - q_0^1)F(\omega - q_0^2)F(\omega_a - q_0^3)F(\omega_b - q_0^4) \\
\text{e}^{i(\omega_d - q_0^1)\delta_0^1\epsilon^1(\omega - q_0^2)\delta_0^2\epsilon^2(\omega_a - q_0^3)\delta_0^3\epsilon^3(\omega_b - q_0^4)\delta_0^4} T[q_2, q_1; q_3, q_4] \\
\sim \int \frac{d^3q_1d^3q_2d^3q_3d^3q_4}{(2\pi)^3(2\pi)^3(2\pi)^3(2\pi)^3} \text{f}^*_a(q_1)f^*_c(q_2)f_a(q_3)f_b(q_4), \quad T, T' \rightarrow \infty \\
\lim (\omega_d - q_0^1)(\omega_c - q_0^2)(\omega_a - q_0^3)(\omega_b - q_0^4)T[q_2, q_1; q_3, q_4], \tag{18.34}
\]
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where the limit taken is \( q_1^0 \to \omega_d, q_2^0 \to \omega_c, q_3^0 \to \omega_a, q_4^0 \to \omega_b \), and is forced by delta functions arising from the large \( T, T' \) limits of the integrals over the \( t_0^a \). We see that there is nonzero scattering only if \( T \) has poles in all of the \( q_i^0 \) at the energies of the respective incoming and outgoing particles.

Taking the limit of infinitely narrow packets we may express our result as follows

\[
T[p', q'; p, q] \to \omega_c \to \omega_a \to \omega_d \to \omega_b \equiv \omega_c(2\omega_d)(2\omega_a)(2\omega_b) |\Omega_c(0)\rangle \langle 0| (2\omega_c) |\Omega_a(0)\rangle \langle 0| |\Omega_b(0)\rangle \langle q'|
\]

\[
\times \langle b| |c\rangle \times \left[ \sum_{\lambda, p} \langle b|\Omega_b(0)\rangle \langle \lambda, p| |\Omega_a(0)\rangle \times \frac{-i}{p^0 - q^0 - i\epsilon} (2\pi)^3 \delta(p - q) - \delta_{ac}\delta_{bd}3(p' - p)\delta^3(q' - q) - \delta_{ad}\delta_{bc}3(p - q')\delta^3(p' - q) \right] \]

(18.35)

Note that translation invariance implies that \( T \) has an overall factor of \( (2\pi)^4\delta^4(p' + q' - p - q) \) expressing energy momentum conservation. It should now be clear how the generalization to an arbitrary number of incoming and outgoing particles should be expressed. The F.T. of the T.O.P. of interpolating fields will have a pole factor \(-i/(q_a^2 + m_a^2)\) associated with each external line. The coefficient of all these pole factors is proportional to the desired scattering amplitude. Furthermore, the factors of proportionality are clear: a factor \( 2\omega_a(2\pi)^3 \langle p_a|\Omega_a(0)\rangle |0\rangle \) for each incoming line and a factor \( 2\omega_a(2\pi)^3 \langle 0|\Omega_a(0)\rangle |p_a\rangle \) for each outgoing line.

We are left with the task of computing the matrix elements of the interpolating fields between the vacuum and one particle states. This information is contained in the F.T. of the two point functions

\[
\int d^4x e^{-i\vec{q}\cdot \vec{x}} \langle 0|T[\Omega_a(x)\Omega_a^\dagger(0)]|0\rangle
\]

\[
\quad = \sum_{\lambda, p} \langle b|\Omega_b(0)\rangle \langle \lambda, p| |\Omega_a(0)\rangle \langle 0| |\Omega_c(0)\rangle \langle 0| (2\pi)^3 \delta(p - q) - \delta_{ac}\delta_{bd}3(p' - p)\delta^3(q' - q) - \delta_{ad}\delta_{bc}3(p - q')\delta^3(p' - q) \]

(18.36)

\[
\pm \sum_{\lambda, p} \langle b|\Omega_b(0)\rangle \langle \lambda, p| |\Omega_a(0)\rangle \langle 0| |\Omega_c(0)\rangle \langle 0| (2\pi)^3 \delta(p + q) - \delta_{ac}\delta_{bd}3(p' - p)\delta^3(q' - q) - \delta_{ad}\delta_{bc}3(p - q')\delta^3(p' - q) \]

The sums over states include all states with any number of particles. But the single particle states are singled out by their unique values of energy for fixed three momentum. Thus only these states will produce poles in the variable \( q^0 \). (Multiparticle states produce cuts). Single particle contributions in the first term produce poles at positive values \( q^0 = +\omega_c(q) \) whereas in the second term they produce poles at negative values \( q^0 = -\omega_c(q) \). The states contributing to the second term are the charge conjugates (antiparticles) of those contributing to the second term. Focusing on the positive energy poles, we see that the single particle \( c \) gives a pole with structure

\[
\frac{-i}{q^2 + m_c^2 - i\epsilon} (2\pi)^3 2\omega(q) \langle 0|\Omega_a(0)\rangle c|c, q\rangle \langle c, q|\Omega_b(0)\rangle |0\rangle.
\]

(18.37)
Choosing \( a = b = c \) this reads, in the case that \( \Omega_c \) is a scalar operator:

\[
\frac{-i}{q^2 + m_c^2 - i\epsilon} (2\pi)^3 2\omega(q) \langle 0 | \Omega_c(0) | c, q \rangle \langle c, q | \Omega_c^\dagger(0) | 0 \rangle \equiv \frac{-iZ_c}{q^2 + m_c^2 - i\epsilon}, \tag{18.38}
\]

from which we learn that

\[
| \langle 0 | \Omega_c(0) | c, q \rangle | = \frac{\sqrt{Z_c}}{(2\pi)^{3/2} \sqrt{2\omega_c}} \tag{18.39}
\]

If \( \Omega_c \) is a fermion field, Lorentz invariance dictates that the matrix element has the structure \( \langle 0 | \Omega_c | q \rangle \propto u(q) \), so the corresponding \( Z_c \) is defined by setting the pole term equal to \(-iZ_c/(m_c + \gamma \cdot q)\). Correspondingly,

\[
\langle 0 | \Omega_c | q \rangle = \frac{\sqrt{Z_c}}{(2\pi)^{3/2} \sqrt{2\omega_c}} u(q).
\]

In either case, the treatment of external lines in perturbation theory of the scattering amplitude, is to drop all external line corrections, replace the external line propagator with \( \sqrt{Z} \) times the appropriate wave function (1 for scalars; \( u, \bar{u}, v, \) or \( \bar{v} \) for fermions; \( \epsilon \) or \( \epsilon^* \) for photons).
Chapter 19

One Loop Corrections in QED

19.1 Photon self energy

Recall that in our discussion of vacuum polarization we calculated the matrix element \( \langle 0 | T j^\mu(x) j^\nu(0) | 0 \rangle \) which also gives the one loop correction to the photon propagator. We need its Fourier transform

\[
i \Pi^{\mu\nu} = i^2 \int d^4 x e^{-i q \cdot x} \langle 0 | T j^\mu(x) j^\nu(0) | 0 \rangle = (q^{\mu} q^{\nu} - q^2 \eta^{\mu\nu}) i \Pi(q^2)
\]

\[
\Pi(q^2) = \frac{Q_0^2}{2 \pi^2} \int_0^1 dx x (1 - x) \ln \frac{\Lambda^2}{m^2 + x(1 - x) q^2 - i \epsilon}
\]

Feeding this into the calculation of the photon propagator

\[
D_F' = \alpha \frac{-i q^{\mu} q^{\nu}}{q^4} + \frac{-i (\eta^{\mu\nu} - q^{\mu} q^{\nu} / q^2)}{q^2} \left[ 1 - \Pi(q^2) + \cdots \right]
\]

\[
\rightarrow \frac{-i q^{\mu} q^{\nu}}{q^4} + \frac{-i (\eta^{\mu\nu} - q^{\mu} q^{\nu} / q^2)}{q^2 (1 + \Pi(q^2))}
\]

If \( i \Pi^{\mu\nu} \) is taken as the sum of all 1 particle irreducible two photon diagrams, this expression is exact! This is because in terms of this \( \Pi \) the rest of the two photon propagator is a geometric sum. We shall see that the absence of corrections to the longitudinal part of the photon propagator is a consequence of gauge invariance.

The pole location in the exact photon propagator has not shifted due to the transverse structure of \( \Pi^{\mu\nu} \). But the strength of the residue has changed: the photon wave function renormalization is \( Z_3 = 1/(1 + \Pi(0)) < 1 \).
19.2 Proper Vertex Function

Give the photon a small mass $\lambda$ as an infrared cutoff. Then in Feynman gauge the proper vertex is:

$$\Lambda^\mu \equiv \Gamma^\mu(p',p) - \gamma^\mu$$

$$= -iQ^2_0 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\rho(m - \gamma \cdot (p' - k))\gamma^\mu(m - \gamma \cdot (p - k))\gamma^\rho}{(k^2 - \lambda^2 - i\epsilon)((p' - k)^2 + m^2 - i\epsilon)((p - k)^2 + m^2 - i\epsilon)}$$

The first step is to combine denominators using

$$\frac{1}{D_1 D_2 \cdots D_n} = \int_0^\infty dT_1 \cdots dT_n e^{-\sum_k T_k D_k}$$

$$= \int T^{n-1} dT \int_0^1 dx_1 \cdots dx_n \delta(1 - \sum_k x_k) e^{-\sum_k x_k D_k}$$

$$= \Gamma(n) \int_0^1 dx_1 \cdots dx_n \delta(1 - \sum_k x_k) \frac{1}{[\sum_k x_k D_k]^n}$$

The denominators of the $\Lambda$ integrand are combined with $x_2 = x$, $x_3 = y$ and $x_1 = 1 - x - y$:

$$\frac{1}{D_1 D_2 D_3} = 2 \int_{x+y\leq 1} dx dy \frac{1}{[k^2 - i\epsilon + \lambda^2(1 - x - y) - 2k \cdot (xp' + yp)]^3}$$

$$= 2 \int_{x+y\leq 1} dx dy \frac{1}{[\hat{k}^2 - i\epsilon + \lambda^2(1 - x - y) - (xp' + yp)^2]^3}$$

where in the last line we completed the square with $\hat{k} = k - xp' - yp$. By assuming the external lines are on shell $p^2 = p'^2 = -m^2$, we can simplify

$$(xp' + yp)^2 = -m^2(x^2 + y^2) + 2xy p' \cdot p = -m^2(x + y)^2 - xyq^2$$

where $q = p' - p$ is the momentum transfer. Change integration variables to $\hat{k} = k - xp' - yp$ in terms of which the numerator of the integrand becomes

$$\gamma_\rho(m - \gamma \cdot ((1 - x)p' - yp - \hat{k}))\gamma^\mu(m - \gamma \cdot ((1 - y)p - xp' - \hat{k}))\gamma^\rho$$

$$= \gamma_\rho(m - \gamma \cdot ((1 - x)p' - yp))\gamma^\mu(m - \gamma \cdot ((1 - y)p - xp' - \hat{k}))\gamma^\rho + \gamma_\rho \hat{k} \cdot \gamma^\mu \hat{k} \cdot \gamma^\rho$$

$$+ \text{Linear terms in } \hat{k}$$

Inside the $\hat{k}$ integral the terms linear in $\hat{k}$ give zero and $\hat{k}_\mu \hat{k}_\nu$ can be replaced by $\eta_{\mu\nu} \hat{k}^2/4$. Hence the numerator can be replaced with

$$\gamma_\rho(m - \gamma \cdot ((1 - x)p' - yp))\gamma^\mu(m - \gamma \cdot ((1 - y)p - xp' - \hat{k}))\gamma^\rho + \hat{k}^2\gamma^\rho$$

Only the last term of this expression contributes to the ultraviolet divergence of $\Gamma^\mu$. 

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19.2.1 UV divergences

Inserting the last term of the numerator into the integral for $\Gamma^\mu - \gamma^\mu$, we find

$$\Lambda_{uv}^\mu = -2iQ_0^2 \int dxdy \int \frac{d^4k}{(2\pi)^4} \frac{k^2 \gamma^\mu}{[\hat{k}^2 - i\epsilon + \lambda^2(1 - x - y) + m^2(x + y)^2 + xyq^2]^3}$$

Now use

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^2}{[\hat{k}^2 + A - i\epsilon]^3} = \frac{i}{32\pi^2 A} \Lambda^2$$

(19.12)

where the factor of $i$ comes from the Wick rotation: $k^0 \rightarrow ik^4$, to arrive at

$$\Lambda_{uv}^\mu = \gamma^\mu \frac{Q_0^2}{8\pi^2} \int dxdy \left( \ln \frac{A^2}{\lambda^2(1 - x - y) + m^2(x + y)^2 + xyq^2 - \frac{3}{2}} + O(m^2/A^2) \right)$$

(19.13)

19.2.2 UV finite remainder

Use

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[\hat{k}^2 + A - i\epsilon]^3} = \frac{i}{32\pi^2 A}$$

(19.14)

to evaluate the $\hat{k}$ integral for the remainder of the vertex part. For this we rearrange

$$\gamma^\mu m - \gamma \cdot ((1 - x)p' - yp)\gamma^\mu (m - \gamma \cdot ((1 - y)p - xp'))\gamma^\mu$$

$$= 2m^2\gamma^\mu - 4m[(1 - 2x)p' + (1 - 2y)p]\gamma^\mu + 2\gamma \cdot [(1 - y)p - xp']\gamma^\mu[(1 - x)p' - yp] \cdot \gamma$$

(19.15)

If we sandwich $\Gamma^\mu$ between on shell spinors $\bar{u}'\Gamma^\mu u$ satisfying $\bar{u}'(m + \gamma \cdot p') = 0$ and $(m + \gamma \cdot p)u = 0$, use $p = p' - q$ or $p' = p + q$ as appropriate, and use the symmetry of the rest of the integrand under $x \leftrightarrow y$, we can replace the numerator with

$$2m^2\gamma^\mu - 4m(1 - x - y)[p' + p]^\mu$$

$$+ 2[-(1 - x - y)m - (1 - y)q \cdot \gamma] \gamma^\mu[-(1 - x - y)m + (1 - x)q \cdot \gamma]$$

$$= 2m^2(1 + (1 - x - y)^2)\gamma^\mu - 4m(1 - x - y)[p' + p]^\mu - 2(1 - x)(1 - y)q^2\gamma^\mu$$

$$- 2m(1 - x - y)[(1 - x)\gamma^\mu q \cdot \gamma - (1 - y)q \cdot \gamma^\mu]$$

(19.16)

under these same circumstances we can eliminate the $p + p'$ term using

$$[q \cdot \gamma, \gamma^\mu] \rightarrow (-m - p \cdot \gamma)\gamma^\mu - \gamma^\mu(m + p' \cdot \gamma)$$

$$\rightarrow -4m\gamma^\mu + 2(p + p')^\mu$$

(19.17)
with which the numerator can be replaced with

\[ 2m^2 \left( (x + y)^2 - 2(1 - x - y) \frac{q^2}{m^2} (1 - x)(1 - y) \right) \gamma^\mu \]

\[ + m(1 - x - y)(x + y)[\gamma^\mu, q \cdot \gamma] \]

Putting everything together we get

\[ \bar{u}' T^\mu(p', p) u \equiv \bar{u}' \left[ \gamma^\mu F_1(q^2) + \frac{[\gamma^\mu, q \cdot \gamma]}{4m} F_2(q^2) \right] u \] (19.18)

\[ F_1(q^2) = 1 + \frac{Q_0^2}{8\pi^2} \int dxdy \left( \ln \frac{\Lambda^2}{\lambda^2(1 - x - y) + m^2(x + y)^2 + xyq^2} - \frac{3}{2} \right. \]

\[ + \left. \frac{m^2((x + y)^2 - 2(1 - x - y) - q^2(1 - x)(1 - y))}{\lambda^2(1 - x - y) + m^2(x + y)^2 + xyq^2} \right) \] (19.19)

\[ F_2(q^2) = \frac{Q_0^2}{4\pi^2} \int dxdy \frac{m^2(x + y)(1 - x - y)}{\lambda^2(1 - x - y) + m^2(x + y)^2 + xyq^2} \] (19.20)

The UV divergences is entirely in \( F_1(q^2) \). We define \( F_1(0) \equiv Z_1^{-1} \):

\[ \frac{1}{Z_1} = 1 + \frac{Q_0^2}{8\pi^2} \int dxdy \left( \ln \frac{\Lambda^2}{\lambda^2(1 - x - y) + m^2(x + y)^2} - \frac{3}{2} \right. \]

\[ + \left. \frac{m^2((x + y)^2 - 2(1 - x - y))}{\lambda^2(1 - x - y) + m^2(x + y)^2} \right) \] (19.21)

\[ = 1 + \frac{Q_0^2}{8\pi^2} \int_0^1 du \left( \ln \frac{\Lambda^2}{\lambda^2(1 - u) + m^2u^2} - \frac{3}{2} + \frac{m^2(u^2 - 2(1 - u))}{\lambda^2(1 - u) + m^2u^2} \right) \]

then we can write

\[ F_1(q^2) = \frac{1}{Z_1} + F_1(q^2) - F_1(0) \] (19.22)

with the UV divergence residing in the first term. In contrast \( F_2(q^2) \) is both UV and IR finite. In particular

\[ F_2(0) = \frac{Q_0^2}{4\pi^2} \int_0^1 du \frac{m^2u^2(1 - u)}{\lambda^2(1 - u) + m^2u^2} \] (19.23)

\[ \to \frac{Q_0^2}{4\pi^2} \int_0^1 du (1 - u) = \frac{Q_0^2}{8\pi^2} \approx \frac{\alpha}{2\pi}, \quad \lambda = 0 \] (19.24)

### 19.2.3 Anomalous magnetic moment of the electron

We already know that before radiative corrections the \( g \) factor for the electron is 2. To see what radiative corrections do, we use (19.17) to write

\[ \bar{u}' \gamma^\mu u = \bar{u}' \left[ \frac{(p + p')^\mu}{2m} + \frac{1}{4m} [\gamma^\mu, q \cdot \gamma] \right] u \] (19.25)
The second term is responsible for the spin magnetic moment which we know corresponds to \( g = 2 \). By comparing coefficients it follows that radiative corrections give \( g - 2 = 2F_2(0) \approx \alpha/\pi \) a result first obtained by Schwinger. Experiment gives an extremely precise measurement (accurate to 3 parts in \( 10^{13} \)) for \( g - 2 \) which agrees with QED calculated to order \( \alpha^4 \), which requires the evaluation of over 12K Feynman diagrams!

19.3 Self energy of the electron

The fermion self-energy part \( \Sigma \) is defined as \( i \) times the sum of all one particle irreducible two point function. If it were known exactly, the exact fermion propagator would be

\[
S'_F = \frac{-i}{m + \gamma \cdot p} \left[ 1 - \frac{1}{m + \gamma \cdot p + \Sigma} + \cdots \right]
\]

and hence the physical mass of the fermion is determined by finding \( p \) such that \( m + \gamma \cdot p + \Sigma(p) \) has a zero eigenvalue. To 1 loop order we have

\[
-i\Sigma = Q_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu(m - \gamma \cdot (p - k))\gamma^\mu}{(k^2 + \lambda^2 - i\epsilon)(m^2 + (p - k)^2 - i\epsilon)} + O(Q_0^4)
\]

\[
= -iQ_0^2 \int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4} \frac{4m + 2\gamma \cdot ((1 - x)p - \hat{k})}{|\hat{k}^2 + \lambda^2(1 - x) + xm^2 + x(1 - x)p^2|^2}
\]

\[
= \frac{Q_0^2}{8\pi^2} \int_0^1 dx (2m + (1 - x)\gamma \cdot p) \ln \frac{\Lambda^2 e^{-1}}{\lambda^2(1 - x) + xm^2 + x(1 - x)p^2}
\]

To interpret this result, first imagine that we have \( \Sigma \) to all orders. Although \( \Sigma \) is a matrix, by Lorentz covariance it must be a function of the single matrix \( \gamma \cdot p \). Moreover, since \( p^2 = -(\gamma \cdot p)^2 \), all the \( p \) dependence is through \( \gamma \cdot p \). If \( M \) is the physical mass, we may expand \( m + \gamma \cdot p + \Sigma \) in a power series in \( M + \gamma \cdot p \), starting at linear order:

\[
m + \gamma \cdot p + \Sigma = (M + \gamma \cdot p)(1 + \Sigma') + O((M + \gamma \cdot p)^2)
\]

(19.29)

where \( \Sigma' \) is the first derivative of \( \Sigma \) regarded as a function of \( \gamma \cdot p \) evaluated at \( \gamma \cdot p = -M \). With the one loop approximation to \( \Sigma \), we have

\[
m - M + \frac{Q_0^2}{8\pi^2} \int_0^1 dx (2m - M(1 - x)) \ln \frac{\Lambda^2 e^{-1}}{\lambda^2(1 - x) + xm^2 - x(1 - x)M^2} = 0
\]

which determines the mass shift

\[
M - m = \frac{Q_0^2}{8\pi^2} \int_0^1 dx M(1 + x) \ln \frac{\Lambda^2 e^{-1}}{\lambda^2(1 - x) + x^2M^2} + O(Q_0^4)
\]

(19.30)
where we used \( M - m = O(Q_0^2) \) to set \( m = M \) on the right side.

\[
\Sigma'_{\text{1-loop}} = \frac{Q_0^2}{8\pi^2} \int_0^1 dx \left[ (1 - x) \ln \frac{\Lambda^2 e^{-1}}{\lambda^2(1 - x) + x^2 M^2} - 2 \frac{x(1 - x)(1 + x)M^2}{\lambda^2(1 - x) + x^2 M^2} \right]
\]

This result is intimately related to \( F_1(0) = 1/Z_1 \). To see this write the coefficient of the ln term

\[
(1 - x) = x + 1 - 2x = x + \frac{d}{dx}[x(1 - x)]
\] (19.31)

and integrate the contribution of the second term by parts:

\[
\Sigma'_{\text{1-loop}} = \frac{Q_0^2}{8\pi^2} \int_0^1 dx \left[ x \ln \frac{\Lambda^2 e^{-1}}{\lambda^2(1 - x) + x^2 M^2} + x(1 - x) \frac{-\lambda^2 - 2M^2}{\lambda^2(1 - x) + x^2 M^2} \right]
\]

\[
= \frac{Q_0^2}{8\pi^2} \int_0^1 dxx \ln \frac{\Lambda^2}{\lambda^2(1 - x) + x^2 M^2} - 2 + \frac{M^2 x^2 - 2M^2(1 - x)}{\lambda^2(1 - x) + x^2 M^2}
\]

\[
= \frac{1}{Z_1} - 1 - \frac{Q_0^2}{32\pi^2}
\] (19.32)

This is not a fluke. The identity

\[
\frac{\partial}{\partial p^\mu} m + \gamma \cdot p = -m + \gamma \cdot p m + \gamma \cdot p \]

enabled Ward to prove that, to all orders in perturbation theory,

\[
\frac{\partial}{\partial p^\mu} = \Gamma^\mu(p, p) - \gamma^\mu, \quad \text{Ward Identity} \]

(19.34)

To one loop order this result is almost obvious, provided the loop momentum is assigned such that the external momentum \( p \) is routed to always follow the electron line. Our 1 loop result disagrees with the identity only by an additive constant, which can be attributed to the linear divergence of the self-energy loop integral. We shall see that the Ward identity can be derived from gauge invariance or current conservation. Recall that the photon self energy calculation also showed a polynomial discrepancy with gauge invariance.

### 19.4 Ward Identities in QED

#### 19.4.1 Coordinate space derivation

The Ward identities are generally a consequence of gauge invariance. In QED gauge invariance implies em current conservation \( \partial_{\mu} j^\mu = 0 \) as an operator statement in Heisenberg picture. To obtain the implication of this for amplitudes, we insert the current into the time
ordered product of a string of charged fields and take its divergence outside the time ordering
symbol, e.g.:
\[
\partial_\mu \left< 0 | T[j^\mu (x) \psi(y) \bar{\psi}(z)] | 0 \right> = \left< 0 | T[\partial_\mu j^\mu (x) \psi(y) \bar{\psi}(z)] | 0 \right> + \delta(x^0 - y^0) \left< 0 | T[j^\mu (x), \psi(y) \bar{\psi}(z)] | 0 \right> + \delta(x^0 - z^0) \left< 0 | T[\psi(y) j^0 (x), \bar{\psi}(z)] | 0 \right>
\]
where the terms on the second line come from the time derivatives acting on the time ordering
symbol. Those terms involve equal time commutators of the charge density with local fields:
\[
\delta(x^0 - y^0)[j^0 (x), \psi(y)] = \delta^4(x - y) Q_\psi \psi(x)
\]
For the case of the Dirac electron, \( Q_\psi = e \) and \( Q_\bar{\psi} = -e \). So the identity becomes
\[
\partial_\mu \left< 0 | T[j^\mu (x) \psi(y) \bar{\psi}(z)] | 0 \right> = e \delta(x - y) \left< 0 | T[\psi(x) \bar{\psi}(z)] | 0 \right> - e \delta(x - z) \left< 0 | T[\psi(y) \bar{\psi}(z)] | 0 \right>
\]
\[
= e \delta(x - y) S_F'(x - z) - e \delta(x - z) S_F'(y - x)
\]
where \( S_F' \) is the exact (all orders) propagator for the Dirac field. Fourier transforming both
sides w.r.t. \( x, y, z \) gives
\[
-i q_\mu \int d^4 x e^{i x q - i y p'} + i z p \left< 0 | T[j^\mu (x) \psi(y) \bar{\psi}(z)] | 0 \right> = e [S_F'(p) - S_F'(p')] (2\pi)^4 \delta(q - p' + p)
\]
We can recognize the the coefficient of \( i q_\mu \) on the left side as
\[
S_F'(p') e^{\Gamma^n(p', p)} S_F(p) \left( \eta_\mu^n + D_F'(p' - p)_{\nu}^\mu \Pi_\nu^n \right) (2\pi)^4 \delta(q - p' + p)
\]
so we can write the Ward identity as
\[
i q_\mu \Gamma^n(p', p) = i(m + \gamma \cdot p' + \Sigma(p')) - i(m + \gamma \cdot p + \Sigma(p))
\]
where we used \( q_\mu \Pi_\nu^n = 0 \).

### 19.4.2 Ward Identities from Diagrams in momentum space

An external photon in a diagram is always attached to a charged line. Replacing the polariza-
tion vector with \( k^\mu \), the photon momentum should give zero for any scattering amplitude.
The attachment vertex \( \gamma^\mu \) is sandwiched between two propagators so we observe
\[
\frac{-i}{m + \gamma \cdot (p + k)} k \cdot \gamma \frac{-i}{m + \gamma \cdot p} = \frac{-i}{m + \gamma \cdot (p + k)} (m + (p + k) \cdot \gamma
\]
\[
- \frac{-i}{m + \gamma \cdot p} \frac{-i}{m + \gamma \cdot p} = \frac{-i}{m + \gamma \cdot (p + k)} \frac{-i}{m + \gamma \cdot (p + k)}
\]
\[
S_F(p + k) k \cdot \gamma S_F(p) = -i (S_F(p) - S_F(p + k))
\]
(19.39)
The external photon line in a diagram contributing to the proper vertex function attaches to a charged line which could start and end on the external fermion lines or could close on itself as a fermion loop. In the first case the factors associated with the charged line in \( k_{\mu} \Gamma\mu(p + k, p) \) take the form

\[
S_F(p + k) \gamma^\mu S_F(p + k + \sum_{j=1}^{n-1} k_j) \gamma^{\mu_{n-1}} \cdots \gamma^\mu S_F(p + k + \sum_{j=1}^{l-1} k_j) k \cdot \gamma
\]

\[
S_F(p + \sum_{j=1}^{n-1} k_j) \gamma^{\mu_{n-1}} S_F(p + \sum_{j=1}^{l-2} k_j) \cdots \gamma^\mu S_F(p)
\]

\[
-i S_F(p + k) \gamma^\mu S_F(p + k + \sum_{j=1}^{n-1} k_j) \gamma^{\mu_{n-1}} \cdots \gamma^\mu [S_F(p + \sum_{j=1}^{l-1} k_j)
\]

\[
-S_F(p + k + \sum_{j=1}^{n-1} k_j)] \gamma^{\mu_{n-1}} S_F(p + \sum_{j=1}^{l-2} k_j) \cdots \gamma^\mu S_F(p)
\]

where the \( k_j \) are the momenta of various internal photon lines that attach to the charged line, satisfying \( \sum_{j=1}^{n} k_j = 0 \). In this example the external vertex is between \( l - 1 \) and \( l \) but one must of course sum over all attachment locations. The right side of the above equation is the difference of two expressions which could be a charged line contributing to a propagator except that in the first expression the momentum \( k \) is added to the momenta of all the propagators to the left of \( \gamma^\mu \) and the second expression has \( k \) added to the momenta of all the propagators to the left of \( \gamma^{\mu_{n-1}} \). When this difference is summed over all attachment locations (i.e. the choice of \( l \)) these terms will cancel in pairs, leaving uncanceled the expressions

\[
-i [S_F(p) \gamma^\mu S_F(p + \sum_{j=1}^{n-1} k_j) \gamma^{\mu_{n-1}} \cdots \gamma^{\mu_2} S_F(p + k_1) \gamma^{\mu_1} S_F(p)]
\]

\[
S_F(p + k) \gamma^\mu S_F(p + k + \sum_{j=1}^{n-1} k_j) \gamma^{\mu_{n-1}} \cdots \gamma^{\mu_2} S_F(p + k + k_1) \gamma^{\mu_1} S_F(p + k)]
\]

The first expression in square brackets can be identified as the corresponding charged line in a diagram contributing to the full propagator \( S'_F(p) \) whereas the second expression would contribute similarly to \( S'(p + k) \). If the vertex attaches to a closed fermion loop the expressions would be traced and the momentum \( p \) would be integrated. If the change of variables \( p \rightarrow p + k \) is valid the two terms on the right side would cancel, leaving no contribution and the Ward identity would be established. Anomalies to the Ward identity emerge if that \( p \) integral is linearly divergent or worse.
19.5 Charge and mass renormalization to 1 Loop

We have seen that the physical mass of a charged fermion is in general different from the Lagrangian mass $m$, called the bare mass. In the following we call the physical mass $m$ and the bare mass $m_0$. A good systematic way to take this into account is to write $m_0 = m - \delta m$ and break the mass term in the Lagrangian into two pieces:

$$L_m = -m \bar{\psi} \psi + \delta m \bar{\psi} \psi \quad (19.42)$$

and include the second term in the interaction, which adds a two point vertex to the Feynman rules. Then as one executes perturbation theory one chooses $\delta m$ so that to each order the physical mass stays equal to $m$.

Suppose we have managed to calculate $\Sigma$ to all orders. Then the exact propagator can be written

$$S_F' = \frac{-i}{m_0 + \gamma \cdot p + \Sigma(p)} = \frac{-i}{(m + \gamma \cdot p)(1 + \Sigma') + O((m + \gamma \cdot p)^2)} \equiv \frac{-iZ_2}{m + \gamma \cdot p} + O(1) \quad (19.43)$$

$Z_2$ is called the electron wave function renormalization, just as we have called $Z_3 = 1/(1 + \Pi(0))$ the photon wave function renormalization. If we normalize each of the external propagators entering the vertex to unity, this means we absorb the $Z$ factors in the definition of the coupling constant:

$$\bar{\Gamma}^\mu(p',p) \equiv Z_2 \sqrt{Z_3} \Gamma^\mu(p',p) \quad (19.44)$$

$$\to \frac{Z_2 \sqrt{Z_3}}{Z_1} \gamma^\mu, \quad p' \to p \quad (19.45)$$

Thus the physical charge, as measured, for example, in scattering processes is given by

$$e = e_0 \frac{Z_2 \sqrt{Z_3}}{Z_1} = e_0 \sqrt{Z_3} \quad (19.46)$$

because the Ward identity implies $Z_1 = Z_2$. Separately, $Z_1$ and $Z_2$ depend on the nature of the charged particle. The importance of the Ward identity is that the charge renormalization is universal: every charged particle no matter what its properties is renormalized by the same factor $\sqrt{Z_3}$. The cancellation of renormalization effects due to $Z_1$ and $Z_2$ is a consequence of gauge invariance. For example replacing the photon with a spin zero field $\phi$ gives Feynman diagrams very similar to QED. But in this case $Z_1 \neq Z_2$ and all three wave function renormalization constants contribute to coupling renormalization.

After renormalization the physical effect of the corrections resides entirely in the new momentum dependence on $q^2$. At low energy and momenta we can expand in powers of $q^2$. 

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For instance

\[
F_1(q^2) - F_1(0) \\
\approx \frac{Q_0^2}{8\pi^2 q^2} \int dx dy \left[ -\frac{2xy + 1 - x - y}{\lambda^2(1-x-y) + m^2(x+y)^2} \right] \\
\approx \frac{Q_0^2}{8\pi^2 q^2} \int_0^1 du \int_0^u dy \left[ -\frac{2(u-y)y + 1 - u}{\lambda^2(1-u) + m^2u^2} \right] \\
\approx \frac{Q_0^2}{8\pi^2 q^2} \int_0^1 du \left[ \frac{u^3/3 + u(1-u)}{\lambda^2(1-u) + m^2u^2} - \frac{m^2(6u^2 - 2(1-u))}{(\lambda^2(1-u) + m^2u^2)^2} \right] \\
\approx \frac{Q_0^2}{8\pi^2 q^2} \int_0^1 du \left[ -\frac{1}{m^2} \left( u - \frac{2}{3} \right) \right. \\
\left. - \frac{u}{\lambda^2(1-u) + m^2u^2} + \frac{m^2u^3/3}{[\lambda^2(1-u) + m^2u^2]^2} \right] \\
\text{(19.47)}
\]

where we have set \( \lambda = 0 \) in the terms with no IR divergence. The remaining \( \lambda \) dependence may be evaluated as follows:

\[
\int_0^1 du \frac{u}{\lambda^2(1-u) + m^2u^2} \sim \int_0^1 du \frac{1}{m^2u} + \int_0^\epsilon du \frac{u}{\lambda^2 + m^2u^2} \\
\approx \frac{1}{m^2} \ln \frac{\epsilon}{\lambda} + \frac{1}{2m^2} \ln \frac{m^2\epsilon^2}{\lambda^2} = \ln \frac{m}{\lambda} \\
\text{(19.48)}
\]

\[
\int_0^1 du \frac{m^2u^3/3}{(\lambda^2(1-u) + m^2u^2)^2} \sim \int_0^1 du \frac{1}{3m^2u} + \int_0^\epsilon du \frac{m^2u^3/3}{(\lambda^2 + m^2u^2)^2} \\
\approx \frac{1}{3m^2} \ln \frac{\epsilon}{\lambda} + \frac{1}{6m^2} \ln \frac{m^2\epsilon^2}{\lambda^2} - \int_0^\epsilon du \frac{\lambda^2/6}{(\lambda^2 + m^2u)^2} \\
\approx \frac{1}{3m^2} \ln \frac{m}{\lambda} - \frac{1}{6m^2} \left( \frac{1}{(\lambda^2)} - \frac{1}{\lambda^2 + m^2\epsilon^2} \right) \\
\text{(19.49)}
\]

Then

\[
F_1(q^2) - F_1(0) \approx \frac{Q_0^2}{8\pi^2 m^2} \left[ -\frac{2}{3} \ln \frac{m}{\lambda} + \frac{1}{4} \right]
\text{(19.50)}
\]

This infrared divergence is occurring in the correction to elastic electron scattering. It is noteworthy that the sign is negative. We shall find that if an unobserved low energy photon is also emitted, that inelastic event also has an infrared divergence, necessarily positive, and indeed exactly cancels this contribution to the elastic amplitude! The vertex correction modifies the Coulomb potential which binds the hydrogen atom together: it adds a \( V_0\delta(r) \) much as the Vacuum polarization does. Here \( V_0 = O(\alpha^2/m^2)\ln(m/\lambda) \). But the relevant infrared cutoff \( \lambda \) should be of order the typical electron momentum in the atom \( m\alpha \) rather than the photon mass. This reasoning leads to an order of magnitude estimate shift of the \( s \) levels \(+m\alpha^2\ln(1/\alpha)\) (larger and in the opposite direction from the effect of vacuum polarization.)
19.6 Atomic bound states

QED includes much more than the interactions of electrons and photons. Electrons can bind to nuclei to form atoms which can in turn bind into molecules. In NR quantum mechanics we use the Schrödinger equation to describe this physics. So we must see how this works in the context of QED. We already know that the Dirac equation in the presence of a Coulomb potential gives a very good description of hydrogen. In this calculation we treat the Dirac wave function as a (first quantized) Schrödinger wave function. In QED we have instead the Dirac field operator $\psi$. Using Feynman diagrams we can study the interaction of an electron with a very heavy positively charged nucleus. If we imagine the limit of infinite mass, it seems intuitive that the effect of the nucleus could be replaced by the static Coulomb potential centered at the nucleus location. One can confirm this explicitly by studying Feynman diagrams with one electron line one heavy nucleus line and the exchange of any number of photons. Here one can in first instance neglect vacuum polarization and radiative corrections. But those can be put back in perturbatively and a consistent description is attained.

In this context it is very convenient to set up a new interaction picture ("bound state interaction picture") in which $H_0$ is the Hamiltonian of free photons and the Dirac field equation including the Coulomb potential of the heavy nucleus as an external field. In this case the creation operators create an electron (or positron) in a Coulomb eigenstate, either a discrete bound state or an unbound state in the continuum. Then the spectrum of $H_0$ includes both bound and unbound electrons as well as free photons. Time dependent perturbation theory then puts back in radiative effects, order by order, including the physics of the Lamb shift.

Unlike the NR Schrödinger equation the Dirac equation doesn’t have a useful multi-body generalization. This makes an elegant discussion of positronium, charmonium, etc a challenge, because both particles have the same mass–far from one being very heavy compared to the other. One approach is to identify the bound states as poles in a partial summation of Feynman diagrams—the Bethe-Salpeter equation.
Chapter 20

Soft Bremsstrahlung and Infrared divergences

20.1 One Soft Photon

Let us consider the emission of an extra photon in the scattering of an electron in an external potential. To regulate infrared divergences we temporarily introduce a small photon mass \( \lambda \). To lowest order in the external potential the diagrams are:

Writing the Feynman amplitude for this process \( \mathcal{M} = Q_0 \mathcal{M}^\nu \tilde{A}_\nu (p' + k - p) \), we have

\[
\mathcal{M}^\nu = iQ_0 \bar{u} \left[ \gamma^\nu \frac{m - \gamma \cdot (p' + k)}{m^2 + (p' + k)^2} \gamma^\nu + \gamma^\nu \frac{m - \gamma \cdot (p - k)}{m^2 + (p - k)^2} \gamma^\nu \right] u
\]

\[
= iQ_0 \bar{u} \left[ \frac{2 \epsilon^* \cdot p' - \epsilon^* \cdot \gamma \gamma \cdot k}{2p' \cdot k - \lambda^2} \gamma^\nu + \gamma^\nu \frac{2 \epsilon^* \cdot p + \gamma \cdot k \epsilon^* \cdot \gamma}{-2p \cdot k - \lambda^2} \right] u.
\]

In this chapter we are mainly interested in the case of soft photon emission, since that is intimately involved with the cancellation of infrared divergences in physical processes. In the limit of very small \( k, \lambda \), the terms involving three gamma matrices are negligible and \( \mathcal{M}^\nu \) becomes a numerical momentum dependent factor

\[
Q_0 \left[ \frac{2 \epsilon^* \cdot p'}{2p' \cdot k - \lambda^2} - \frac{2 \epsilon^* \cdot p}{2p \cdot k + \lambda^2} \right] \equiv Q_0 \epsilon^* \cdot J
\]

times the lowest order elastic scattering amplitude. If the soft photon is unobserved and its energy is smaller than the energy resolution \( \Delta \), one must include the contribution of
the emission when calculating the cross section for electron scattering. In this case we must obviously sum over polarization and integrate over photon angles. The soft photon contribution to the differential cross section is then approximately given by

$$\frac{d\sigma}{d\Omega_{\text{soft photon}}} \approx \frac{d\sigma}{d\Omega_{\text{Elastic}}} Q_0^2 \int_{|k| \leq \Delta} \frac{d^3k}{(2\pi)^3 2\omega} \left( J \cdot J + \frac{(k_\mu \cdot J)^2}{\lambda^2} \right),$$

where we used the identity

$$\sum_{\text{Pol}} \epsilon_\mu \epsilon_\nu^* = \epsilon_{\mu\nu} + \frac{k_\mu k_\nu}{\lambda^2}$$

(20.5)
to evaluate the sum over polarizations. Here we use $k^2 = -\lambda^2$ as well as $k \cdot \epsilon = 0$, which is true even for a massive photon.

Clearly $J \cdot J = O(1/k^2)$ for small $k$. On the other hand, $k \cdot J = O(\lambda^2/k)$, so the second term in the integrand is a factor of $\lambda^2$ smaller than the first, and can be dropped. Moreover, the smallest in magnitude $k \cdot p$ or $k \cdot p'$ can be is $m\lambda$ so the $\lambda^2$ in the denominators of the expression for $J$ can be dropped, leading to

$$J \cdot J \approx -\frac{2p' \cdot p}{p' \cdot kp \cdot k} - \frac{m^2}{(p' \cdot k)^2} - \frac{m^2}{(p \cdot k)^2}.$$ 

(20.6)

Since the last two terms are special cases of the first, we only need to evaluate one integral:

$$K \equiv \int_{|k| \leq \Delta} \frac{d^3k}{(2\pi)^3 2\omega} \frac{1}{p' \cdot kp \cdot k}.$$ 

(20.7)

To extract the divergent contribution of this integral, first scale $|k| \to \lambda|q|$. In the new variables $|q| \leq \Delta/\lambda$. The divergence as $\lambda \to 0$ then comes from the integration region $q_0 \leq |q| \leq \Delta/\lambda$, where $q_0 >> 1$ is fixed. For this region the $q$ dependence is simply $1/q^3$ so the $q$ integral just gives a factor $\ln(\Delta/q_0\lambda)$. Thus we have

$$K = \text{Finite} + \ln(\Delta/\lambda) \frac{1}{16\pi^3} \int d\Omega_k \frac{1}{(p' \cdot k-E')(p \cdot k - E')}.$$ 

(20.8)

Define

$$I(p', p) \equiv \int d\Omega_k \frac{1}{(p' \cdot k-E')(p \cdot k - E')}.$$ 

(20.9)

Then

$$\frac{d\sigma}{d\Omega_{\text{soft photon}}} \approx \text{Finite} + \ln(\Delta/\lambda) \frac{d\sigma}{d\Omega_{\text{Elastic}}} Q_0^2 \frac{1}{16\pi^3} \left[-2p' \cdot pI(p', p) - m^2I(p', p') - m^2I(p, p)\right].$$ 

(20.10)
The integral defining $I$ can be simplified by combining the denominators with the Feynman trick, and then choosing the $z$-axis for the polar angles to be parallel to the vector $xp' + (1 - x)p$. Then $I$ reduces to

$$I(p', p) = 4\pi \int_0^1 dx \frac{1}{(x E' + (1 - x)E)^2 - (xp' + (1 - x)p)^2}. \quad (20.11)$$

The denominator is the Lorentz scalar $-(xp + (1 - x)p)^2$ which evaluates to $(x^2 + (1 - x)^2)m^2 - 2x(1 - x)p' \cdot p$. Using $q^2 \equiv (p' - p)^2 = -2m^2 - 2p' \cdot p$, we then obtain

$$I(p', p) = 4\pi \int_0^1 dx \frac{1}{m^2 + x(1 - x)q^2}, \quad (20.12)$$

from which we see that $I$ depends only on $q^2$. Taking account of this information, we can write

$$\frac{d\sigma}{d\Omega} \approx \text{Finite} + \ln(\Delta/\lambda) \frac{d\sigma}{d\Omega_{\text{Elastic}}} \times \left(1 + \frac{\alpha}{4\pi^2} [q^2 I(q^2) + 2m^2(I(q^2) - I(0))]\right). \quad (20.13)$$

What we have computed is the contribution to the scattering cross section for electron scattering with the emission of an unobservable soft photon. Adding this to the contribution with no photon emission gives to this order the total electron scattering cross section

$$\frac{d\sigma}{d\Omega} \approx \alpha \times \text{Finite} + \frac{d\sigma}{d\Omega_{\text{Elastic}}} \times \left(1 + \ln(\Delta/\lambda) \frac{\alpha}{4\pi^2} [q^2 I(q^2) + 2m^2(I(q^2) - I(0))]\right). \quad (20.14)$$

The unobserved photon emission contribution seems to make this cross section diverge for $\lambda \to 0$. However, in this limit the elastic cross section is not directly measurable, since soft photon emission can not even in principle be experimentally vetoed. It is only a physical quantity (in principle) for a non zero photon mass. We shall find that the one loop radiative correction to the elastic amplitude has a divergence as $\lambda \to 0$ which precisely cancels the one in the above formula.

To see this cancellation, let’s re-examine the one loop vertex function in the presence of a finite photon mass.

$$\Gamma^\mu(p', p) - \gamma^\mu = -iQ_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\rho(m - (p' - k) \cdot \gamma)\gamma^\mu (m - (p - k) \cdot \gamma)\gamma^\rho}{(k^2 + \lambda^2 - i\epsilon)(k^2 - 2k \cdot p - i\epsilon)(k^2 - 2k \cdot p' - i\epsilon)}. \quad (20.15)$$

We are interested in the infrared divergence in this expression when $\lambda \to 0$. Since we want to present the result in a form suitable for comparison to the photon emission amplitude, it is appropriate to first evaluate the $k^0$ integration by contours. Examination of the denominators reveals six simple poles at the following values of $k^0$:

$$\pm(\sqrt{k^2 + \lambda^2 - i\epsilon})$$

$$E \pm \sqrt{E^2 + k^2 - 2k \cdot p - i\epsilon} \quad E' \pm \sqrt{E'^2 + k^2 - 2k \cdot p' - i\epsilon}. \quad (20.16)$$
Three poles are in the upper half plane and three in the lower half plane. We can close the contour up or down, but the best choice is down in the lower half plane. The reason is that only one of the poles in the lower plane is near the origin for small $k$, the region responsible for the infrared divergence. The other two poles are far from the origin at small $k$ and it is simple to see that their residues do not contribute to the divergence. Thus closing in the lower half plane we only need to consider the residue of the pole at $k^0 = +\sqrt{k^2 + \lambda^2} - i\epsilon$.

\[ \Gamma^\mu(p', p) - \gamma^\mu \approx Q_0^2 \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{\gamma_\nu(m-p' \cdot \gamma)\gamma_\rho(m-p \cdot \gamma)\gamma_\mu}{(-\lambda^2 - 2k \cdot p)(-\lambda^2 - 2k \cdot p')} \tag{20.17} \]

where we have dropped terms linear in $k$ in the numerator, since they won’t contribute to the infrared divergence. When the vertex is sandwiched between on-shell spinors, the numerator simplifies to $4p' \cdot p$ by moving $\gamma \cdot p'$ to the left and $\gamma \cdot p$ to the right. To extract the divergent part as $\lambda \to 0$ we compute the integral over the region $|k| \leq \kappa$ for some fixed $\kappa$. This integral is identical to the one encountered in the soft photon calculation, so by comparison, we find

\[ \Gamma^\mu(p', p) \approx \gamma^\mu \left( 1 + \frac{Q_0^2}{2(2\pi)^3} p' \cdot p' \ln \frac{\kappa}{\lambda} (q^2) \right). \tag{20.18} \]

Finally, we have to separate from the vertex correction the part that should be absorbed into $1/Z_1$. This is just the value of the correction at $p' = p$, so we write (Recall $2p \cdot p' = -q^2 - 2m^2$.)

\[ \Gamma^\mu(p', p) = \frac{1}{Z_1} \gamma^\mu \left( 1 - \frac{\alpha}{8\pi^2} \ln \frac{\kappa}{\lambda} [q^2 I(q^2) + 2m^2 (I(q^2) - I(0))] \right) + \alpha \times \text{Finite} + O(\alpha^2). \tag{20.19} \]

Of course, defining $Z_1$ this way makes it depend on the infrared cutoff. In any regulation scheme respecting the Ward Identity, it will turn out that $Z_1 = Z_2$ so that charge renormalization will not be infrared sensitive, even though $Z_1, Z_2$ separately are. The elastic electron cross section involves the square of $\Gamma$, and since the infrared divergence occurs at order $\alpha$, it is correct to this order to extract the divergent $\lambda$ dependence of the elastic differential cross section as an overall factor:

\[ \frac{d\sigma}{d\Omega_{\text{Elastic}}} = \frac{d\sigma}{d\Omega_{\text{IR Finite}}} \left( (1 - \frac{\alpha}{8\pi^2} \ln \frac{\kappa}{\lambda} [q^2 I(q^2) + 2m^2 (I(q^2) - I(0))]\right)^2 + O(\alpha^2) \]

\[ = \frac{d\sigma}{d\Omega_{\text{IR Finite}}} \left( (1 - \frac{\alpha}{4\pi^2} \ln \frac{\kappa}{\lambda} [q^2 I(q^2) + 2m^2 (I(q^2) - I(0))]\right) + O(\alpha^2). \tag{20.20} \]

Inserting this information into the expression for the electron scattering cross section, we see that the sensitivity to the small photon mass disappears:

\[ \frac{d\sigma}{d\Omega} \approx \frac{d\sigma}{d\Omega_{\text{IR Finite}}} \left( 1 + \ln(\Delta/\kappa) \frac{\alpha}{4\pi^2} [q^2 I(q^2) + 2m^2 (I(q^2) - I(0))]\right) + \alpha \times \text{Finite}. \tag{20.21} \]
Notice, however that the physical effect of soft photons remains, signaled by the unavoidable sensitivity of the electron scattering cross section to the experimental energy resolution \( \Delta \). There is no “infrared catastrophe,” \( i.e. \) the theory gives perfectly finite predictions for all physical measurements.
21.1 Ward Identities

One might think that gauge-fixing would exhaust the implications of gauge invariance on correlation functions. This is not really true because physics must not depend on which gauge is chosen, and this gauge independence implies many relationships among different correlation functions known as Ward identities.

Recalling the gauge fixing procedure, it is clear that a necessary condition for gauge independence is that the part of the path integral describing the coupling of charged fields to the gauge fields, $\langle \text{out}\vert \text{in} \rangle_A$ must be invariant under a gauge change of $A$:

$$iD_\mu \langle \text{out}\vert j_\mu \vert \text{in} \rangle = D_\mu \frac{\delta}{\delta A_\mu(z)} \langle \text{out}\vert \text{in} \rangle_A = 0.$$  \hspace{1cm} (21.1)

Here we have written the condition for a general gauge theory; for QED $D = \partial$. If this is inserted for the $\langle \text{out}\vert \text{in} \rangle_A$ factor in the path integral for a correlation function of gauge invariant operators, it becomes the statement that the correlation function of the divergence of the gauge current ($\partial_\mu J^\mu$) with any set of gauge invariant operators is zero (abelian case) or related to another correlator (nonabelian case).

If gauge noninvariant operators also appear in the correlator, more contributions appear. As an important example of this, take the case of additional charged fields in QED. Then the gauge transform induces a change in the phase of each charged field $\psi \rightarrow e^{iQ_0 A} \psi$. For example, if only $\psi, \bar{\psi}$ appear we find

$$\partial_\mu \langle j^\mu(z)\psi(x)\bar{\psi}(y) \rangle = -Q_0[\delta(x-z) - \delta(y-z)] \langle \psi(x)\bar{\psi}(y) \rangle.$$  \hspace{1cm} (21.2)

The VEV (vacuum expectation) on the left hand side of this Ward identity is simply related to the vertex to which a photon propagator couples. In fact, all diagrams in which the current couples to a closed charged field line contribute nothing to the l.h.s.$^1$ so we can apply the

\hspace{1cm} 

$^1$This follows simply from the gauge invariance of $\langle \text{out}\vert \text{in} \rangle_A$. 

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identity to the one photon line irreducible vertex. To get an identity for the completely one particle irreducible vertex $\Gamma$, we write

$$i \langle j^\mu(z) \psi(x) \bar{\psi}(y) \rangle_{1\gamma IR}$$

$$= \int d^4 x' d^4 y' \langle \psi(x) \bar{\psi}(x') \rangle_i \langle j^\mu(z) \psi(x') \bar{\psi}(y') \rangle_{1PIR} \langle \psi(y') \bar{\psi}(y) \rangle$$

$$= \int d^4 x' d^4 y' \langle \psi(x) \bar{\psi}(x') \rangle_i \langle j^\mu(z) \psi(x') \bar{\psi}(y') \rangle \langle \psi(y') \bar{\psi}(y) \rangle.$$  \tag{21.3}

We state the relevant identity in momentum space, for which we define

$$\langle \psi(x) \bar{\psi}(y) \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip \cdot (x-y)}}{m_0 + \gamma \cdot p + \Sigma(p)}, \tag{21.5}$$

where $\Sigma$ is the electron’s proper self energy part. In momentum space, the Ward identity then reads:

$$(p' - p) \mu \Gamma^\mu(p', p) = \gamma \cdot (p' - p) + \Sigma(p') - \Sigma(p). \tag{21.6}$$

This is actually a generalized form of the original Ward Identity which follows from the generalized one plus an assumption on the absence of certain infrared singularities.

To get it, take the derivative of the above equation with respect to $p'$, and set $p' = p$. Then, assuming that $(p' - p) \mu \partial / \partial p'_\mu \Gamma^\mu \to 0$ in this limit, we obtain

$$\Gamma^\mu(p, p) = \gamma^\mu + \frac{\partial \Sigma(p)}{\partial p^\mu}. \tag{21.7}$$

An example of the sort of singularity that would invalidate this derivation would be a contribution to $\Gamma^\mu$ of the form

$$\frac{(p' - p)^\mu \gamma \cdot (p' - p)}{(p' - p)^2}. \tag{21.8}$$

Such a singularity would imply a zero mass bound state coupled to the current, and can be shown to be absent in QED to any finite order in perturbation theory.

As a simple application of the Ward identity, we use it to show that the only contribution to charge renormalization comes from corrections to the photon propagator. We define charge by the photon coupling at zero photon momentum. Define $Z_1$ by

$$\Gamma^\mu(p, p) = \frac{1}{Z_1} \gamma^\mu + O(m + \gamma \cdot p). \tag{21.9}$$

By the definition of $Z_2$,

$$m_0 + \gamma \cdot p + \Sigma(p) = \frac{1}{Z_2} (m + \gamma \cdot p) + O(m + \gamma \cdot p)^2. \tag{21.10}$$
Then the Ward identity implies \( Z_1 = Z_2 \). Clearly the renormalized (physical) charge is related to the bare one by

\[
Q = \frac{Z_1}{Z_2} \sqrt{Z_3} Q_0 = \sqrt{Z_3} Q_0
\]  

(21.11)

by the Ward identity.

Recall also that the Ward identity for diagrams with only photon external lines required that the vacuum polarization tensor was transverse:

\[
\Pi_{\mu\nu}(q) = - (q^2 \eta_{\mu\nu} - q_\mu q_\nu) \Pi(q^2).
\]  

(21.12)

\( \Pi \) is defined so that the sum of all 1PIR 2 photon diagrams is given by \( i Q_0^2 \Pi_{\mu\nu}(q) \). Thus the photon two point function is given by

\[
\langle A_\mu(x) A_\nu(0) \rangle = \int \frac{d^4q}{(2\pi)^4} e^{iq \cdot x} \left[ -i \alpha q_\mu q_\nu \frac{q^4}{q^4} + \frac{-i}{q^2} \left( \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \right] \frac{1}{1 + Q_0^2 \Pi(q^2)}.
\]

We note two consequences. (1) The longitudinal part of the two point function is not corrected. (2) The photon mass remains zero unless \( \Pi(q^2) \) has a pole at \( q^2 = 0 \). We can read off the value of \( Z_3 = 1/(1 + Q_0^2 \Pi(0)) \).

## 21.2 Ultraviolet Divergences and Gauge Invariance

In our calculation of vacuum polarization, we found that ultraviolet divergences spoiled the gauge invariance of the calculation. In this chapter we look into this question more closely. Our discussion generalizes that of K. Johnson, Brandeis Lectures, 1964.

The focus will be on the vacuum persistence amplitude of a charged field in the presence of a fixed external gauge field. This quantity is a basic ingredient of the path history quantization of gauge fields in interaction with charged fields, and as such must be gauge invariant for consistency. As long as the gauge field path integral is amenable to dimensional regularization, difficulties with gauge invariance must, in fact reside in the charged field sector which is the focus of this chapter.

Let us first sketch the important consequences of gauge invariance. Of course, gauge invariance demands that, in addition to the persistence amplitude itself, the outin matrix element of every gauge invariant observable should likewise be invariant under gauge transformations. For QED (the abelian case), the current operator \( j^\mu(x) \) is one such observable. In the nonabelian case the current operator \( j^\mu_a(x) \) is covariant rather than invariant under gauge transformations since gauge transformations act on the index \( a \).

If we return to the variational equation for \( \langle out|in \rangle \)

\[
\delta \langle out|in \rangle = i \int d^4x \delta A_\mu(x) \langle out|j^\mu(x)|in \rangle
\]  

(21.13)

we see that gauge invariance of \( \langle out|in \rangle \) under infinitesimal gauge transformations that vanish sufficiently rapidly at infinity is equivalent to current conservation \( \partial_\mu j^\mu = 0 \) in the abelian
case. (In the nonabelian case the requirement is \( D_\mu j^\mu = 0 \).) More is in fact true: since the conservation laws hold for arbitrary \( A \), all terms with the same power of \( A \) must cancel among themselves.

We first try to understand how ultraviolet divergences can spoil current conservation. Consider the representation of the current matrix element in terms of the Green function

\[
\langle \text{out} | j^\mu (x) | \text{in} \rangle = -q \text{Tr}[\gamma^\mu S_F(x, x; A)].
\]

The two arguments of the Green function are coincident, but it is apparent from the free Dirac propagator that \( S_F(x - y) \sim 1/(x - y)^3 \) at short distances. The fact that \( A \neq 0 \) is not going to alter this fact. To exert a bit more caution, we should consider the Green function \( S_F(x, y; A) \) at slightly separated points and study what happens as \( y \to x \). The first thing to notice is that \( S_F \) is not gauge invariant because the Dirac field \( \psi \to e^{iq\Lambda(x)}\psi \) under gauge transformations

\[
S_F(x, y; A) \to e^{iq[\Lambda(x) - \Lambda(y)]} S_F(x, y; A).
\]

Superficially, one would think that the above expression for the current matrix element would be gauge invariant because the prefactor formally goes to 1 as \( y \to x \). But the approach to unity is only linear so the fact that \( S_F \) has a cubic short distance singularity means that gauge noninvariance can occur with quadratic or lower singularities. Thus point splitting alone will not yield a gauge invariant definition of the current matrix element let alone the vacuum persistence amplitude.

To obtain a gauge invariant definition of the current, we first notice that

\[
\int_y^{\infty} d\xi^\mu \partial_\mu \Lambda(\xi) = \Lambda(y) - \Lambda(x).
\]

Thus

\[
\bar{S}_F(x, y; A, C) \equiv e^{iq \int_{x,C} d\xi^\mu A_\mu(\xi)} S_F(x, y; A)
\]

is gauge invariant, albeit path dependent. We can attempt to define the current matrix element as the coincident point limit of \( S_F \) averaged in some suitable way over \( C \). The result

\[
\int_y^{\infty} d\xi^\mu \partial_\mu \Lambda(\xi) = \Lambda(y) - \Lambda(x).
\]

As \( y \to x \) this approaches a similarity transformation, but only linearly, so divergences spoil this property.

In the nonabelian case, we can construct the matrix \( P \exp\{ig \int_{x,C} d\xi^\mu A_\mu(\xi)\} \), where the \( P \) denotes path ordering: matrices associated with “later” points on the path \( C \) always stand to the left of “earlier” ones. This matrix transforms under gauge transformations as

\[
P \exp\{ig \int_{x,C} d\xi^\mu A_\mu(\xi)\} \to \Omega(y) P \exp\{ig \int_{x,C} d\xi^\mu A_\mu(\xi)\} \Omega^\dagger(x).
\]
will certainly be gauge invariant, and with a bit of luck might also be conserved. If so we can then define \( \langle \text{out} | \text{in} \rangle \) from this construction through the variational equation.

If we consider the weak field expansion for \( S_F(x, y; A) \), the divergence gets one degree less severe with each extra factor of \( A \). Thus the field independent piece has a cubic divergence, the linear term a quadratic divergence, the quadratic term a linear divergence and the cubic term only a logarithmic divergence. Thereafter all the terms are finite. Since the modification to \( S \) in \( \bar{S} \) vanishes linearly, one only needs to keep terms up to order \( A^3 \) in the modification factor. Actually, charge conjugation invariance makes alternate terms vanish. The electromagnetic current \( \langle \text{out} | j^\mu | \text{in} \rangle \) should have only odd powers of \( A \), whereas the axial current \( j_5^\mu \equiv \bar{\psi} \gamma_5 \gamma^\mu \psi \) should have only even powers of \( A \) in its \( \text{outin} \) matrix element. Thus we tentatively define

\[
J^\mu(x) \equiv \frac{\langle \text{out} | j^\mu(x) | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle} \quad (21.23)
\]

\[
= -q \lim_{\epsilon \to 0} \frac{1}{2} \text{Tr}\{\gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; A) - \gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; -A)\}
\]

\[
J_5^\mu(x) \equiv \frac{\langle \text{out} | j_5^\mu(x) | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle} \quad (21.24)
\]

\[
= - \lim_{\epsilon \to 0} \frac{1}{2} \text{Tr}\{\gamma_5 \gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; A) + \gamma_5 \gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; -A)\}
\]

To check current conservation we need the Green function equations

\[
\left( \frac{1}{i} \gamma \cdot \partial_x + m - q \gamma \cdot A(x) \right) S_F(x, y; A) = -i \delta(x - y) \quad (21.25)
\]

\[
S_F(x, y; A)(-\frac{1}{i} \gamma \cdot \partial_y + m - q \gamma \cdot A(y)) = -i \delta(x - y), \quad (21.26)
\]

Then

\[
\bar{S}_F(x, y; A, C) \equiv P e^{ig \int_{-\epsilon}^{\epsilon} d\xi^\nu A_\nu(x)} S_F(x, y; A) \quad (21.20)
\]

transforms by the similarity transformation \( \bar{S}_F(x, y; A, C) \to \Omega(y) \bar{S}_F(x, y; A, C) \Omega^\dagger(y) \) under gauge transformations.

4 The currents in a nonabelian gauge theory carry a gauge symmetry index, \( j^\mu_a = \bar{\psi} \lambda_a \gamma^\mu \psi \), and \( j_5^\mu_a = \bar{\psi} \lambda_5 \gamma_5 \gamma^\mu \psi \), where \( \lambda_a \) is the generator of the gauge group in the representation carried by \( \psi \). Their \( \text{outin} \) matrix elements can be defined by

\[
J^\mu_a(x) = - \lim_{\epsilon \to 0} \text{Tr}\lambda_a \gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; A) \quad (21.21)
\]

\[
J_5^\mu_a(x) = - \lim_{\epsilon \to 0} \text{Tr}\lambda_a \gamma_5 \gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; A) \quad (22.22)
\]
from which we get
\[ \partial_\mu \text{Tr}[\gamma^\mu S(x - \epsilon/2, x + \epsilon/2; A)] = \] (21.29)
\[ i g \text{Tr}[(A_\mu(x - \epsilon/2) - A_\mu(x + \epsilon/2)) \gamma^\mu S(x - \epsilon/2, x + \epsilon/2; A)]. \] (21.27)

To calculate the gradient of the modification factor, we must first specify what happens to the integration contour under differentiation. Under the change \( x \to x + \delta x \), we globally translate \( C \) parallel to itself by the same amount. We use Stokes theorem in the form
\[ \oint d\xi^\mu A_\mu(\xi) = \int d\sigma^{\mu\nu} F_{\mu\nu}, \] (21.30)
where the integral on the r.h.s. is over a surface spanning the closed curve implicit in the line integral on the l.h.s. To be completely explicit about the conventions in this identity parameterize the surface by \( \xi^\mu(\sigma, \tau) \) with \( \sigma \) and \( \tau \) each ranging from 0 to 1, with \( \sigma \) labeling the abscissa and \( \tau \) the ordinate. Then
\[ d\sigma^{\mu\nu} = \frac{1}{2} d\sigma d\tau \left( \frac{\partial \xi^\mu}{\partial \sigma} \frac{\partial \xi^\nu}{\partial \tau} - \frac{\partial \xi^\nu}{\partial \sigma} \frac{\partial \xi^\mu}{\partial \tau} \right), \] (21.33)
and with these parameters, the line integral runs around the boundary of the unit square of parameter space in a counter-clockwise direction. The factor in parentheses in (21.33) is of course a total divergence:
\[ 2 \left( \frac{\partial \xi^\mu}{\partial \sigma} \frac{\partial \xi^\nu}{\partial \tau} - \frac{\partial \xi^\nu}{\partial \sigma} \frac{\partial \xi^\mu}{\partial \tau} \right) = \frac{\partial}{\partial \sigma} \left( \xi^\mu \frac{\partial \xi^\nu}{\partial \tau} - \xi^\nu \frac{\partial \xi^\mu}{\partial \tau} \right) \] (21.34)
\[ + \frac{\partial}{\partial \tau} \left( \xi^\nu \frac{\partial \xi^\mu}{\partial \sigma} - \xi^\mu \frac{\partial \xi^\nu}{\partial \sigma} \right), \] (21.35)

\[ ^5 \text{In the nonabelian case the gauge fields are of course matrices and the appropriate equation is} \]
\[ \partial_\mu \text{Tr}[\lambda_\alpha \gamma^\mu S(x - \epsilon/2, x + \epsilon/2; A)] = \] (21.27)
\[ i g \text{Tr}[(A_\mu(x - \epsilon/2) - A_\mu(x + \epsilon/2)) \lambda_\alpha \gamma^\mu S(x - \epsilon/2, x + \epsilon/2; A)]. \] (21.28)
Then the coincident point limit gives formal covariant conservation rather than ordinary conservation.

\[ ^6 \text{A gauge covariant form of Stokes theorem can be given in the nonabelian case for an infinitesimal closed loop} \]
\[ P e^{ig \oint d\xi^\mu A_\mu(\xi)} \approx I + ig \int d\sigma^{\mu\nu} F_{\mu\nu}, \quad \text{infinitesimal loop,} \] (21.31)
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \) is the nonabelian field strength. (The quadratic term in \( F \) comes from \( (ig)^2 \oint d\xi^\mu \oint d\xi^\nu P[A_\mu(\xi)A_\nu(\xi')] \) which for an infinitesimal loop at \( z \) over which \( A \) is constant is just \( (ig)^2 A_\mu(z)A_\nu(z) \oint d\xi^\mu \xi^\nu = -(ig)^2 [A_\mu, A_\nu] \oint d\sigma^{\mu\nu} \). Another version of this is a statement of how the path ordered phase changes under an infinitesimal deformation of the curve \( C \) spanning an infinitesimal surface element \( d\sigma^{\mu\nu} \) at the point \( z \):
\[ \delta_z P e^{ig \oint d\xi^\mu A_\mu(\xi)} = ig P[\delta \sigma^{\mu\nu} F_{\mu\nu}(z)e^{ig \oint d\xi^\mu A_\mu(\xi)}] \] (21.32)
where it is understood that \( F(z) \) is included in the path ordering.
which leads to the useful identity

\[ \int d\sigma_{\mu\nu} = \frac{1}{2} \oint \xi^\mu d\xi^\nu = -\frac{1}{2} \oint \xi^\nu d\xi^\mu, \]  

(21.36)

with the contour for the r.h.s. enclosing the surface on the l.h.s. in a counterclockwise sense.

To define the derivative of the modification factor, take the left vertical boundary of parameter space to map onto the curve \( C \), the right vertical boundary to map onto \( C + \delta x \), the curve rigidly translated by the amount \( \delta x \). The top and bottom boundaries then give the displacements of the end points of the curve \( x + \epsilon/2 \) and \( x - \epsilon/2 \) respectively. Using Stokes theorem on this closed contour then gives\(^7\)

\[ \partial_\mu \int_{C_x} d\xi \cdot A = A_\mu(x + \epsilon/2) - A_\mu(x - \epsilon/2) + \int_C d\xi^\nu F_{\mu\nu}(\xi). \]  

(21.38)

\(^7\)For the nonabelian case we get

\[ \partial_\mu Pe^{ig \int d\xi^\nu A_\nu} = ig(A_\mu(x + \epsilon/2)Pe^{ig \int d\xi^\nu A_\nu} - Pe^{ig \int d\xi^\nu A_\nu} A_\mu(x - \epsilon/2) + P[\int d\xi^\nu F_{\mu\nu}(\xi)e^{ig \int d\xi^\nu A_\nu}]. \]  

(21.37)
Combining these results then gives the conservation laws\(^8\)

\[
\partial_\mu J^\mu(x) = -i q^2 \lim_{\epsilon \to 0} \frac{1}{2} \int_C d\xi^\nu F_{\mu\nu}(\xi) \text{Tr}\{\gamma^\mu(\bar{S}(x - \epsilon/2, x + \epsilon/2; A) + S(x - \epsilon/2, x + \epsilon/2; -A))\}
\]

\[
\partial_\mu J^\mu_5(x) = i m \lim_{\epsilon \to 0} \text{Tr}\gamma_5(\bar{S}(x - \epsilon/2, x + \epsilon/2; A) + S(x - \epsilon/2, x + \epsilon/2; -A))
\]

\[
\partial_\mu J^\mu_5(x) = -i q \lim_{\epsilon \to 0} \frac{1}{2} \int_C d\xi^\nu F_{\mu\nu}(\xi) \text{Tr}\{\gamma_5 \gamma^\mu(\bar{S}(x - \epsilon/2, x + \epsilon/2; A) - S(x - \epsilon/2, x + \epsilon/2; -A))\}
\]

\[(21.43) \quad (21.44)\]

Now we consider whether we can choose the contour to give conserved currents. We need to consider the behavior of the factors multiplying \(\int F\). Since the latter is of order \(\epsilon\) only the singular parts of these factors need be retained. First consider \(\text{Tr}\gamma^\mu \bar{S}_F\). Since it is gauge invariant it must be a vector formed from \(e^\mu\), \(F_{\mu\nu}\) and derivatives of \(F\). The field independent term must be of the form \(e^\mu f(\epsilon^2)\) where \(f\) behaves as \(1/\epsilon^4\). Because of the antisymmetry of \(F\), it will drop out if we specify the integration contour to be a straight line connecting the two endpoints \((e^\mu e^\nu F_{\mu\nu} = 0)\). The term linear in \(A\) was a priori quadratically divergent, but in \(\bar{S}\) this is reduced by one power of \(\epsilon\) because a factor of momentum must be provided to form \(F\) out of \(A\). Technically the quadratic and linear divergences in the unadjusted vacuum polarization \(T^{\mu\nu}(\ell)\) turn out to be proportional to \((1 - i\epsilon \cdot \ell) e^\mu e^\nu / \epsilon^4\) in point splitting regularization and therefore give zero when contracted with \(F_{\mu\nu} e^\nu\). Thus the linear term in the “vector” current is at worst logarithmically divergent.

The quadratic term in \(A\) is a priori linearly divergent but gauge invariance in the abelian case requires two factors of momentum to be used in forming \(F^2\) so it and all higher powers must be finite. In the nonabelian case a quadratic term with no derivatives could conceivably be needed to complete the nonabelian field strength that might appear in the linear term in \(A\). But as we note below, the divergence in the “vector” linear term is only logarithmic and so does not contribute. The divergence in the “axial” linear term is linear and then the quadratic term provides the rest of the nonabelian field strength.

In summary, provided we take a straight line contour, the only term of relevance in the abelian case is the linear one. It doesn’t contribute in the divergence of the electromagnetic

\[\text{We quote here the nonabelian results:}\]

\[
\partial_\mu J^\mu_a(x) = -ig \lim_{\epsilon \to 0} \text{Tr}\{\lambda_a, A_\mu(x + \epsilon/2)\} \gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; A)
\]

\[
\partial_\mu J^\mu_5 a(x) = 2i m \lim_{\epsilon \to 0} \text{Tr}\lambda_a \gamma_5 \bar{S}(x - \epsilon/2, x + \epsilon/2; A)
\]

\[
\partial_\mu J^\mu_5 a(x) = -ig \lim_{\epsilon \to 0} \text{Tr}\{\lambda_a, A_\mu(x + \epsilon/2)\} \gamma_5 \gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; A)
\]

\[
\partial_\mu J^\mu_5 a(x) = -ig \lim_{\epsilon \to 0} \text{Tr}\{\lambda_a, A_\mu(x + \epsilon/2)\} \gamma_5 \gamma_5 \gamma^\mu \bar{S}(x - \epsilon/2, x + \epsilon/2; A)
\]

\[(21.39) \quad (21.40) \quad (21.41) \quad (21.42)\]
current because of odd charge conjugation. Even if we don’t explicitly enforce charge conjugation by making $J$ manifestly odd in $A$, we know from our vacuum polarization calculation that the linear term is only logarithmically divergent since the adjustments for gauge invariance are included when we use $\bar{S}$. With point splitting regularization the linear divergence can be present, but its coefficient vanishes for “vector” currents. This is important in the nonabelian case because then the charge conjugation properties are no longer so simple. It is sufficient to know that the term is only log divergent to prove that it won’t contribute to the “vector” conservation law. Thus the e.m. current and, more generally, the “vector” nonabelian currents, defined through $\bar{S}$ with a straight line contour, are conserved after $\epsilon \to 0$, and $\bar{S}$ therefore determines a gauge invariant persistence amplitude.

The same cannot be said about the axial current which is even under charge conjugation. In that case the linear term of $\text{Tr} \gamma_5 \gamma^\mu \bar{S}$ does contribute a linearly divergent factor to cancel the factor of $\epsilon$ implicit in $\int F$, leaving a finite anomalous contribution. This is the celebrated axial anomaly to which we shall return below.

We have shown that choice of a straight line contour suffices to make $\bar{S}_F$ yield a conserved e.m. current. However a direction $\epsilon^\mu$ is singled out breaking Lorentz invariance. Thus, it is convenient to average over all directions. This averaging procedure preserves gauge invariance and current conservation, since those features hold for each fixed direction. Since all terms of order $A^4$ and higher are independent of $\epsilon$ as $\epsilon \to 0$ we only need to apply this averaging procedure for the linear and cubic terms. (The constant and quadratic terms vanish by charge conjugation (Furry’s theorem).) The linear term is the one relevant to vacuum polarization, so let’s look at that one in detail.

### 21.2.1 Vacuum Polarization and Scattering of Light by Light

Vacuum polarization is given by the terms in $J^\mu$ linear in $A$. The linear terms that come from the modification factor

$$iq \int_{x-\epsilon/2}^{x+\epsilon/2} d\xi A_\mu(\xi) \approx iq \epsilon \cdot A + \frac{iq}{24} (\epsilon \cdot \partial)^2 \epsilon \cdot A$$

are multiplied by

$$\text{Tr}[\gamma^\mu S_F(\epsilon)] = -4i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2 - i\epsilon} \frac{p^\mu}{p^2 + m^2 - i\epsilon} \equiv \frac{\epsilon^\mu}{\epsilon^4} f(m^2 \epsilon^2),$$

where $f$ is regular and nonvanishing at zero argument. In the average over directions of $\epsilon$ the following replacements take place

$$\epsilon^\mu \epsilon^\nu \to \frac{1}{4} \eta^{\mu\nu} \epsilon^2$$

$$\epsilon^\mu \epsilon^\nu \epsilon^\rho \epsilon^\sigma \to \frac{\epsilon^4}{24} (\eta^{\mu\nu} \eta^{\rho\sigma} + \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho})$$

(21.45)
Thus
\[
\langle iq \int_{x-\epsilon/2}^{x+\epsilon/2} d\xi A_\mu(\xi) \text{Tr} \gamma^\mu S_F \rangle \approx iq \frac{f(m^2 \epsilon^2)}{\epsilon^4} \left( \frac{\epsilon^2}{4} A^\mu + \frac{\epsilon^4}{24^2} (\partial^2 A^\mu + 2 \partial^\mu \partial \cdot A) \right).
\] (21.51)

These terms represent the adjustment that should be made to the linear term in \(A\) (whose coefficient is related to the vacuum polarization \(T^{\mu \nu}(k)\)). We see that they have the qualitative appearance of the adjustment we actually had to make in our original calculation: a quadratically divergent constant times \(\eta^{\mu \nu}\) and a finite second order polynomial in \(k\). The detailed coefficients cannot be compared because the cutoff procedure was different in that calculation.

The term of order \(A^2\) is absent in \(J^\mu\) because of charge conjugation (Furry’s theorem). Finally there will be a finite adjustment to the term of order \(A^3\), which controls scattering of light by light. These arise from the order \(a^k\) term in the modification factor times the order \(A^{3-k}\) term in \(S_F(A)\) for \(k = 1, 2, 3\). This modification enters the fourth order term in \(\langle \text{out} | \text{in} \rangle\). The unmodified value of this term turns out to be \(UV\) finite but not gauge invariant. In summary, we have seen that a careful gauge invariant definition of the e.m. current has led to modifications in the calculation of only the first few terms in the weak field perturbation series for \(\langle \text{out} | \text{in} \rangle\), specifically the quadratic and order four terms. All higher terms are gauge invariant without modification.

### 21.2.2 Evaluation of the Axial Current Anomaly

Let us return to the axial current conservation law (21.44).

\[
\partial_\mu J_5^\mu(x) = -2mJ_5 - iq \lim_{\epsilon \to 0} \frac{1}{2} \int_C d\xi F_{\mu \nu}(\xi) \text{Tr}\{\gamma_5 \gamma^\mu(\bar{S}(x-\epsilon/2, x+\epsilon/2; A) - \bar{S}(x-\epsilon/2, x+\epsilon/2; -A))\},
\]

where we have defined \(J_5\) as the suitably regularized version of

\[
\langle \text{out} | [\bar{\psi}, i\gamma_5 \psi] | \text{in} \rangle / \langle \text{out} | \text{in} \rangle
\] (21.52)
given by the first term on the r.h.s. of (21.44). We would like to extract the explicit contribution of the anomaly which arises from the linearly divergent term in \(\text{Tr}\gamma_5 \gamma^\mu \bar{S}(x-\epsilon/2, x+\epsilon/2; A)\), which resides (in the abelian case) in the term with only one power of \(A\). When we expand \(S(x-\epsilon/2, x+\epsilon/2; A)\) in \(A\), the order zero term vanishes because one can’t form an axial vector from the only available four vector \(\epsilon^\mu\). The linear term in \(A\) would appear to be quadratically divergent, but a momentum factor must be used along with \(\epsilon^\mu\) and \(A_\nu\) to form an axial vector \(\epsilon^\mu \epsilon^\nu \epsilon^\rho \partial_\rho A_\sigma\). Hence the divergence is only linear. The quadratic term in \(A\) is also linearly divergent, but in the case of an abelian gauge field Bose symmetry (or charge conjugation) kills this leading linear divergence. (We have already
noted that since the linear divergence in this term would not involve derivatives of $A$ it would be inconsistent with abelian gauge invariance.) It does give a contribution to the anomaly in the nonabelian case, where it is needed to complete the nonabelian field strength whose derivative terms come from the linear term.

Since we only need keep the linearly divergent term, the modification factor can be dropped. Thus, for the abelian case we only need to extract the linearly divergent piece of

$$\text{Tr}\gamma_5 \gamma^\mu S(x - \epsilon/2, x + \epsilon/2; A)$$

(21.53)

$$\approx iq \int d^4y \text{Tr}\gamma_5 \gamma^\mu S_F(x - y - \epsilon/2) \gamma \cdot A(y) S_F(y - x - \epsilon/2)$$

(21.54)

$$\approx -iq \int \frac{d^4k}{(2\pi)^4} \tilde{A}(k) e^{i(x+\epsilon/2)k}$$

(21.55)

The trace of $\gamma_5$ times fewer than 4 gamma matrices vanishes and

$$\text{Tr}\gamma_5 \gamma^\rho \gamma^\nu \gamma^\rho \gamma^\nu = -4i\epsilon^{\mu\rho\nu\sigma}.$$  

(21.56)

Thus the trace in the numerator gives simply $+4i\epsilon^{\mu\rho\nu\sigma} p_\rho k_\sigma$. Furthermore, the leading divergence as $\epsilon \to 0$ coming from the integral over $p$ is independent of $k$ and $m$ so the latter can be set to zero, and we only need to evaluate

$$\int \frac{d^4p}{(2\pi)^4} \frac{p_\rho}{(p^2 - i\epsilon)^2} e^{-i\epsilon p} = \frac{-i\epsilon p}{2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - i\epsilon} e^{-i\epsilon p}$$

(21.57)

$$= \frac{\epsilon p}{2} \int \frac{d^4pE}{(2\pi)^4} \frac{1}{p^2} e^{-i\epsilon p}$$

(21.58)

$$= \frac{\epsilon p}{2} \int \frac{d^4pE}{(2\pi)^4} \int_0^\infty dT e^{-T^2 p^2 - i\epsilon p}$$

(21.59)

$$= \frac{\epsilon p}{32\pi^4} \int_0^\infty dT \frac{\pi^2}{T^2} e^{-\epsilon^2/4T}$$

(21.60)

$$= \frac{\epsilon p}{8\pi^2\epsilon^2}$$

(21.61)

where we wrote $p_\rho/(p^2 = i\epsilon)^2 = (-1/2)\partial_\rho(p^2 - i\epsilon)^{-1}$ and an integration by parts in the first line, did a Wick rotation ($d^4p \to id^4p_E$) in the second line, and used a simple representation for $1/p^2$ in the third line.

Collecting these results we obtain

$$\text{Tr}\gamma_5 \gamma^\mu S(x - \epsilon/2, x + \epsilon/2; A)$$

(21.62)

$$\sim 4q \int \frac{d^4k}{(2\pi)^4} \tilde{A}(k) e^{ixk} e^{i\rho\nu\sigma k_\sigma} \frac{\epsilon p}{8\pi^2\epsilon^2}$$

(21.63)

$$= -iq\epsilon^{\mu\rho\sigma} \partial_\sigma A_\nu(x) \frac{\epsilon p}{2\pi^2\epsilon^2}$$

(21.64)

$$= iq\epsilon^{\mu\rho\sigma} F_{\nu\sigma}(x) \frac{\epsilon p}{4\pi^2\epsilon^2}$$

(21.65)
This can now be substituted into our expression for $\partial_\mu J^\mu(x)$, which gives after averaging over directions of $\epsilon$ (giving another $1/4!$)

$$
\partial_\mu J^\mu_5(x) = -2mJ_5 + \frac{q^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\nu\rho}(x)F_{\mu\sigma}(x)
$$

(21.66)

$$
= -2mJ_5 + \frac{\alpha_0}{4\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\rho}(x)F_{\nu\sigma}(x)
$$

(21.67)

More generally it is clear from our general remarks that these results generalize in the nonabelian case to

$$
D_\mu J^\mu_{5a}(x) = -2mJ_{5a} + \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}[\lambda_a F_{\mu\rho}(x)F_{\nu\sigma}(x)].
$$

(21.68)

### 21.3 Gauge Invariant Regulation Procedures

For practical calculations in quantum gauge field theories, it is sufficient to establish a systematic procedure for regulating divergences compatible with gauge invariance. Such a procedure will automatically supply the polynomial modifications needed for gauge invariance. We mention briefly the more popular regulators.

**Pauli-Villars Method.** This method is particularly suited to Abelian gauge theories since it relies on the fact that the violations in gauge invariance are independent of the mass of the charged fields. Thus if we introduce extra charged fields of large mass $M_i$ which contribute to divergent diagrams with negative signs we can adjust the coefficients of their contributions to render the loop integrals finite. Then the gauge violating pieces of each contribution will cancel, and the regulated calculation will be gauge invariant. One then lets $M_i \to \infty$ at the end of the calculation. This may still leave UV divergences, but only those compatible with gauge invariance as we found in the vacuum polarization calculation.

**Dimensional Regularization.** The idea here is that the severity of UV divergences depends on the space-time dimension. In particular, the nature of UV gauge invariance violations is different in each dimension. Thus if we can carry out the calculation in a way that applies to general dimension, the violations of gauge invariance must disappear. This method will obviously not work in theories that can be defined only in particular dimensions. For example the alternating symbol $\epsilon^{\mu_1\cdots\mu_D}$ has a different tensor structure in each dimension. This loophole allows gauge invariance anomalies to creep in. A theory in which it appears in a fundamental way can not be formulated in a general dimension.

To illustrate how dimensional regularization is used, we reconsider the vacuum polarization calculation. Let us first make some general remarks. In $D$ dimensions the charge has units $[mass]^{2-D/2}$. Thus it is convenient to introduce a mass parameter $\mu$ to define a dimensionless coupling $q_I$ in general $D$ via

$$
q^2 \equiv q_I^2 \mu^{4-D}.
$$

(21.69)

The identities for calculating traces of products of gamma matrices carry over to $D$ dimensions except that the constant $4 = \text{Tr} I$ is replaced by $2D/2$. The Feynman trick for handling
denominators of propagators is unchanged in general dimension, and the shift of integration variables to make the denominators depend only on the squares of the loop momenta is still applicable. But then averaging over directions of the loop momenta gives a $D$ dependent factor: For example $\langle p^\mu p^\nu \rangle_{\text{angles}} = \eta^\mu\nu p^2 / D$. After taking all this into account, the vacuum polarization calculation for general $D$, becomes

$$T_D^{\mu\nu}(k) = -2^{D/2} q^2 \mu^{4-D} (2\pi)^D \int_0^1 dx \int d^D p \frac{\eta^{\mu\nu} [p^2 ((2/D) - 1) - m^2 + x(1-x)k^2] - 2x(1-x)k^\mu k^\nu}{[m^2 + p^2 + x(1-x)k^2]^2}$$

Clearly we need to be able to do the integral

$$\int \frac{d^D p p^m}{[p^2 + A^2]^2} = A^{D+m-4} \frac{2\pi D/2}{\Gamma(D/2)} \int_0^\infty \frac{p^{D+m-1} dp}{[p^2 + 1]^2}$$

where we used the identity

$$\int_0^\infty \frac{p^{m+D-1} dp}{[p^2 + 1]^2} = \frac{1}{2} \int_0^1 dx x^{1-(D+m)/2} (1-x)^{(D+m)/2-1}$$

Note that the quantity $\Omega_D \equiv 2\pi^{D/2} / \Gamma(D/2)$ is just the value of the integral over all angles in $D$ dimensions. The following table lists $\Omega_D$ for the $2 \leq D \leq 8$.

<table>
<thead>
<tr>
<th>$D$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_D$</td>
<td>$2\pi$</td>
<td>$4\pi$</td>
<td>$2\pi^2$</td>
<td>$8\pi^2/3$</td>
<td>$\pi^3$</td>
<td>$16\pi^3/15$</td>
<td>$\pi^4/3$</td>
</tr>
</tbody>
</table>

Using these results to do the integrals, we obtain

$$T_D^{\mu\nu}(k) = (k^\mu k^\nu - k^2 \eta^{\mu\nu}) T_D(k^2)$$

$$T_D(k^2) = \frac{q^2}{2\pi^2} \Gamma(2 - D/2) \int_0^1 dx x (1-x) \left[ \frac{m^2 + x(1-x)k^2}{2\pi \mu^2} \right]^{(D-4)/2}.$$
The pole at $D = 4$ represents an infinity which has after renormalization the same fate as the cutoff dependence did in our earlier calculation, namely it disappears after expressing measurable results in terms of measured parameters.

Since the pole at $D = 4$ corresponds to the logarithmic divergence of a direct cutoff procedure, it is useful to establish the relation between the residue of the pole and the coefficient of $\ln(\Lambda^2)$. This follows from the simple integral

$$\int_{\mu}^{\Lambda} dpp^{D-5} = \frac{\Lambda^{D-4} - \mu^{D-4}}{D - 4}$$

$$\sim \begin{cases} 
-\frac{1}{D-4} - \ln \mu & \Lambda \to \infty, D < 4 \\
\ln \frac{\Lambda}{\mu} & D \to 4, \Lambda \text{ fixed},
\end{cases}$$

from which we see that the coefficient of $\ln(\Lambda^2)$ is $-(\text{residue of pole})/2$. This is of course in agreement with our two calculations of vacuum polarization.

### 21.3.1 Minimal subtraction

For practical calculations, dimensional regularization provides a systematic way to specify the separation of the divergent parts which are absorbed into renormalization of parameters and the finite parts which characterize the physics.

The divergences in the integral of any $N$ loop Feynman diagram take the form of poles at $D = 4$, of order up to $N$. Generally the value of an $N$ loop Feynman diagram can take the form

$$\sum_{n=1}^{N} A_n(g_I, m, \mu) \left(4 - D\right)^n + \text{Finite}(g_I, m, \mu)$$

where $D$ has been set to 4 in the finite part. We can assume that the coefficients $A_n(g_I, m)$ are independent of $D$. Then the separation is unique. (Any coefficient $A_n$ that depends on $D$ can be Taylor expanded about $D = 4$ modifying the coefficients for smaller $n$ and the finite part, dropping new terms of order $(D - 4)^{n+1}$.) The minimal subtraction prescription (MS) is to choose the counterterms to precisely cancel the multi-order poles and nothing more. In our example of vacuum polarization the pole at $D = 4$ resides in

$$\Gamma(2 - D/2) = \frac{2}{4-D} \Gamma(3 - D/2)$$

$$\sim \frac{2}{4-D} \left(\Gamma(1) + (2 - D/2)\Gamma'(1) + O((D - 4)^2)\right)$$

$$\sim \frac{2}{4-D} - \gamma + O((D - 4))$$

Here $\gamma \equiv -\Gamma'(1) \approx 0.577$ is the Euler-Mascheroni constant. For example the vacuum
polarization result is then

\[ T_D(k^2) \rightarrow \frac{q_I^2}{2\pi^2} \int_0^1 dx x(1-x) \left[ \frac{2}{4-D} - \gamma - \ln \frac{m^2 + x(1-x)k^2}{2\pi^2} \right]. \]

\[ \rightarrow \frac{q_I^2}{2\pi^2} \left[ \frac{1}{64-D} - \int_0^1 dx x(1-x) \ln \frac{m^2 + x(1-x)k^2}{2\pi^2} \right]. \]

A modified version of MS called \( \overline{\text{MS}} \) defines the finite part by scaling the unphysical mass parameter \( \mu^2 \rightarrow \bar{\mu}^2 e^{\gamma/(4\pi)} \) so that

\[ T_D^{\mu\nu}(k) \rightarrow \frac{q_I^2}{2\pi^2} \left[ \frac{1}{64-D} - \int_0^1 dx x(1-x) \ln \frac{m^2 + x(1-x)k^2}{\bar{\mu}^2/2} \right]. \]

In other words the relation between \( g \) and \( g_I \) is

\[ g^2 = g_I^2 \left[ \frac{\bar{\mu}^2 e^{\gamma}}{4\pi} \right]^{2-D/2}, \quad \overline{\text{MS}} \] (21.81)

Minimal subtraction in dimensional regularization is especially advantageous in theories with no mass scale. For example, the vacuum polarization with zero electron mass becomes

\[ T_D(k^2) \rightarrow \frac{q_I^2}{2\pi^2} \left[ \frac{1}{64-D} - \int_0^1 dx x(1-x) \ln \frac{m^2 + x(1-x)k^2}{\mu^2/2} \right]. \] (21.82)

After the counter terms have cancelled all the pole terms, what is left is the \( \overline{\text{MS}} \) definition of the renormalized vacuum polarization. The couplings and masses that appear become the \( \overline{\text{MS}} \) definition of the renormalized parameters.

Here we see a characteristic feature of apparently scale invariant field theories: the scale invariance is broken by the UV divergences, which allows a nontrivial dependence on the momenta of the process. The argument of the logarithm is made dimensionless with the unphysical scale \( \mu \), which cannot enter any physical quantity. The way this works is that the renormalized couplings and masses \( g_I, m \) are regarded as dependent on \( \mu \) in just such a way that every \textbf{physical} quantity is independent of \( \mu \). For a theory with only one coupling and vanishing masses, before renormalization the theory had only one free parameter, the bare coupling. After renormalization the coupling is not really a free parameter, a scale is determined by say the momentum at which the coupling is unity. That scale cancels out of all mass ratios, so in effect the theory has no free parameters: the one free parameter \( \mu \) does nothing more than set the unit of mass.
Nonabelian Gauge Theory

In the mid 1950’s Yang and Mills invented an analogue of QED which replaced QED’s local phase symmetry, \( \psi(x) \rightarrow e^{iQ \Lambda(x)} \psi(x) \), with a local \( SU(2) \) symmetry. Later it was realized that \( SU(2) \) could be any continuous group. To motivate the construction, first recall how the local phase symmetry works. Terms in \( \mathcal{L} \) with no derivatives, like the mass term \( \bar{\psi} \psi \) are obviously invariance under an \( x \) dependent phase change. However terms with derivatives are not invariant:

\[
\bar{\psi} \gamma \cdot \partial \psi \rightarrow \bar{\psi} \gamma \cdot (\partial + iQ \partial \mu \Lambda(x)) \psi
\]  

(22.1)

This motivates the introduction of a vector potential \( A_{\mu}(x) \) via \( \partial_{\mu} \rightarrow (\partial_{\mu} - iQA_{\mu}) \) which undergoes the gauge transformation \( A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \Lambda \) so that

\[
\bar{\psi} \gamma \cdot (\partial - iQA) \psi \rightarrow \bar{\psi} \gamma \cdot (\partial - iA) \psi
\]  

(22.2)

is invariant. To make the new gauge field dynamical we need gauge invariant terms containing derivatives of \( A \). A simple derivative \( \partial_{\mu} A_{\nu} \rightarrow \partial_{\mu} A_{\nu} + \partial_{\nu} A_{\mu} \) is not invariant but the antisymmetrized derivative \( F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is invariant. Thus the QED Lagrange density

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \bar{\psi} (m - i\gamma \cdot (\partial - iQ A)) \psi
\]  

(22.3)

is properly gauge invariant. Notice that the commutator of the covariant derivatives \( D_{\mu} = \partial_{\mu} - iQA_{\mu} \) is proportional to \( F_{\mu \nu} \): \( [D_{\mu}, D_{\nu}] = -iQ F_{\mu \nu} \).

To achieve the Yang-Mills generalization, we consider an internal multiplet of fields put into a column vector transforming under some unitary matrix representation of a continuous symmetry group \( \psi \rightarrow \Omega(x) \psi \). \( \Omega(x) \in G \). Continuous groups are mostly determined by infinitesimal transformations \( I - i\theta_{a} T_{a} \) where \( T_{a} \) satisfy a Lie algebra

\[
[T_{a}, T_{b}] = i f_{abc} T_{c}, \quad f_{abc} = -f_{bac}, \quad (22.4)
\]

where \( f_{abc} \) are the (real) structure constants of the group. For our discussion the \( T_{a} \) are taken to be Hermitian matrices representing the Lie algebra, and the Lie “product”
\[ [A, B] = AB - BA \] is just the commutator. The Hermiticity condition guarantees that finite group transformations \( e^{-iT_a \theta^a} \) are unitary. The inner product of two elements of the Lie algebra \( \langle A|B \rangle = \text{Tr}(AB) \) is invariant under \( A, B \rightarrow \Omega^\dagger(A, B)\Omega \) and it is convenient to take the basis of generators orthonormal under this metric \( \text{Tr}T_a T_b = \delta_{ab}/2 \). For such a basis \( i f_{abc} = 2 \text{Tr}[T_a T_b T_c] \) which is antisymmetric under the interchange of any pair of \( a, b, c \). (In a nonorthonormal basis the antisymmetry is only under \( a \leftrightarrow b \).)

Returning to the local gauge transformation of \( \psi \), its derivative has the transform

\[
\partial_\mu \psi \rightarrow \Omega(x)(\partial_\mu + \Omega^{-1} \partial_\mu \Omega)\psi
\]

Now \( \Omega^{-1} \partial_\mu \Omega \) is in the Lie algebra of \( G \), generated by \( T_a, a = 1, \ldots, n \). Next we introduce a gauge field \( A_\mu(x) = \sum_a T_a A_\mu^a(x) \) in the Lie algebra. Then the transform

\[
A_\mu \rightarrow \Omega(x)A_\mu\Omega^{-1}(x) + \frac{1}{ig}(\partial_\mu \Omega)\Omega^{-1}
\]

makes \( D_\mu \psi \equiv (\partial_\mu - igA_\mu)\psi \) transform like \( \psi \):

\[
D_\mu \psi \rightarrow \Omega(x)D_\mu \psi
\]

so that, for example, \( \bar{\psi} D_\mu \gamma^\mu \psi \) is invariant. Notice that regarding \( D_\mu \) as a matrix differential operator, we can write its transform as \( D_\mu \rightarrow \Omega D_\mu \Omega \).

To construct gauge covariant derivatives of \( A_\mu \) itself, we examine the commutator

\[
[D_\mu, D_\nu] = -ig\partial_\mu A_\nu + ig\partial_\nu A_\mu + (-ig)^2[A_\mu, A_\nu] \equiv -igF_{\mu\nu}
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]
\]

noting that it produces a field strength \( F_{\mu\nu} \) which transforms as

\[
F_{\mu\nu} \rightarrow \Omega(x)F_{\mu\nu}\Omega^{-1}(x)
\]

so that the trace of any matrix product of the components \( F_{\mu\nu}(x) \), all at the same point \( x \), is gauge invariant. In terms of the gauge field components \( A_\mu^a \) we have

\[
F_{\mu\nu} = \sum_a T_a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) - ig\sum_{b,c} [T_b T_c]A_\mu^b A_\nu^c
\]

\[
= \sum_a T_a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) - ig\sum_{b,c} i f_{abc} T_a A_\mu^b A_\nu^c
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + gf_{abc} A_\mu^a A_\nu^b A_\nu^c
\]

The simplest gauge invariant Lagrangian only has the \( A_\mu^a \) as dynamical variables and is conventionally written, in analogy with QED, as

\[
\mathcal{L} = -\frac{1}{4} \sum_a F_{\mu\nu}^a F^{a\mu\nu} = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu}
\]
It is interesting that this simplest Lagrangian necessarily includes cubic and quartic terms in the vector potentials: the gauge particles interact with each other, not like photons!

To this simplest system one can add additional “matter” fields transforming under various representations $R$ of the gauge group. For them the covariant derivative is $D^R = \partial - igA^R$, where $A^R_\mu \equiv \sum_a A^a_\mu T^R_a$. Possible gauge invariant terms include

$$\bar{\psi}\psi, \quad \bar{\psi}iD^R_\mu \gamma^\mu \psi, \quad \phi^\dagger \phi, \quad (D^R_\mu \phi)^\dagger D^R_\mu \phi, \quad (\phi^\dagger \phi)^2$$

(22.13)

and so on. Here $T^R_a$ are the generators of the Lie algebra in the chosen representation. Note that once we normalize the defining generators $T^a$ according to $\text{Tr}T^a T^b = \delta_{ab}/2$, the normalization for other representations is determined $\text{Tr}T^R_a T^R_b \equiv \delta_{ab}(R)$. For the defining representation $T(R) = 1/2$.

### 22.1 Some gauge groups and their representations

#### 22.1.1 The Adjoint Representation

For any Lie group one can form a matrix representation from the structure constants. This follows from the Jacobi identity for commutators:

$$0 = [T_a, [T_b, T_c]] + [T_c, [T_a, T_b]] + [T_b, [T_c, T_a]]$$

$$= -f_{bcd} f_{ade} T_e - f_{abd} f_{cde} T_e - f_{cad} f_{bde} T_e$$

$$0 = f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde}$$

(22.14)

Now consider the matrices $M^b_{kl} = if_{kld}$ and write out their commutators

$$[M^a, M^b]_{kl} = M^c_{kn} M^b_{nl} - M^b_{kn} M^c_{nl}$$

$$= -(f_{kan} f_{nbd} - f_{kln} f_{nba}) = f_{kln} f_{nba} = i f_{abc} M^a_{kl}$$

(22.15)

showing that $T^a = M^a$ represents the Lie algebra of the given group. This universal representation is called the adjoint representation. The covariant derivative in the adjoint representation is

$$D_\mu V^a = \partial_\mu V^a - ig M^b_{ac} A^b_\mu V^c = \partial_\mu V^a + g f_{abc} A^b_\mu V^c$$

(22.16)

For instance the covariant derivative on the field strength tensor is $D_\mu F^a_{\rho\sigma} = \partial_\mu F^a_{\rho\sigma} + g f_{abc} A^b_\mu F^c_{\rho\sigma}$. If we make a matrix $V \equiv \sum_a T^a V^a$, then

$$D_\mu V = \partial_\mu V + g \sum_a T^a f_{abc} A^b_\mu V^c = \partial_\mu V + g \sum_a (-i[T^b, T^c] A^b_\mu V^c$$

$$= \partial_\mu V - ig [A_\mu, V]$$

(22.17)


22.1.2 Examples

A gauge field theory can be constructed based on any continuous group. The original Yang-Mills work chose $SU(2)$ which every physicist knows well, because it gives the action of the rotation group on spin 1/2 particles. When $SU(2)$ is regarded as an internal symmetry group unrelated to rotations we call the Pauli matrix generators $\tau/2$ rather than $\sigma/2$:

$$
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

so the structure constants of $SU(2)$ are just $\epsilon_{abc}$.

In general the group $SU(N)$ is defined as the set of unitary $N \times N$ matrices with unit determinant. The corresponding Lie algebra is the set of $N \times N$ hermitian traceless matrices. The complete classification of continuous Lie groups includes the infinite sequences $SU(N)$, $SO(2N)$, $SO(2N+1)$, and $Sp(2N)$, together with five exceptional groups, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$. The dimension of a group is the number of infinitesimal generators. The rank of a group is the size of the maximal subset of mutually commuting generators, called the Cartan subalgebra. The subscripts on the exceptional group names is the rank of that group. So $SU(N)$ has dimension $N^2 - 1$ and rank $N - 1$. $SO(N)$ has dimension $N(N - 1)/2$, but $SO(2N)$ and $SO(2N+1)$ both have rank $N$.

The standard model gauge group is $SU(3) \times SU(2) \times U(1)$ Here $SU(3)$, the gauge group of the strong interactions, is the only one that may be new to you. The three factor groups, which are all simple groups, act independently and each have an independent coupling. The whole gauge group is not simple. Larger (simple) groups play a role in unification ideas, but will not figure in the physics of the standard model. For example, $SU(5)$ and $SO(10)$ both contain the standard model gauge group. The dimension of $SU(5)$ is $5^2 - 1 = 24$ whereas that of $SO(10)$ is $10 \times 9/2 = 45$. The dimension of the standard model gauge group is $8 + 3 + 1 = 12$. So in the first case one would introduce 12 new gauge bosons, and in the second case 33 new gauge bosons. Since no trace of them is seen at current energies the new bosons must be extremely heavy.

The defining generators of $SU(3)$ can be taken to be $T_a = \lambda_a/2$, where $\lambda_a$ are the 8 Gell-Mann matrices

$$
\lambda_k = \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda_4 + i\lambda_5 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\lambda_6 + i\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
$$

In addition to gauge fields a gauge theory can possess several fields transforming in various representations of the gauge group. For example the defining representation of $SU(N)$, also
called the fundamental representation, is $N$ dimensional, requiring $N$ field components. Starting with any representation of a group $T_a(R)$ satisfying $[T_a(R), T_b(R)] = i f_{abc} T_c(R)$, $-T_a^*(R)$ is also a representation, not necessarily a new one. For example if $T_a(R)^* = -T_a(R)$ or if $U \dagger T_a(R) U$ satisfies this condition the representation is called real. It is called pseudo real if $-T_a^*(R) = V \dagger T_a(R) V$. otherwise it is called complex. For $SU(2)$ the fundamental representation is pseudoreal, but the fundamental representation of $SU(N)$ is complex for $N > 2$. We call the fundamental irrep of $SU(N)$ $N$ and the complex conjugate representation $\bar{N}$.

### 22.2 Path integrals for gauge theories and gauge fixing

Recall the path integral formula for time ordered products:

$$\langle \Phi \rangle = \frac{\int \mathcal{D} A^a_{\mu} \mathcal{D}( Fields) \, \Phi e^{i \int d^4 x (-F^a_{\mu\nu} F^{a\mu\nu}/4 + \mathcal{L}_{\text{fields}})} \int \mathcal{D} A^a_{\mu} \mathcal{D}( Fields) e^{i \int d^4 x (-F^a_{\mu\nu} F^{a\mu\nu}/4 + \mathcal{L}_{\text{fields}})}}{\int \mathcal{D} A^a_{\mu} \mathcal{D}( Fields) e^{i \int d^4 x (-F^a_{\mu\nu} F^{a\mu\nu}/4 + \mathcal{L}_{\text{fields}})}}.$$

Here $\Phi$ just represents some string of gauge invariant operators. If none of these operators involve time derivatives the functional average just gives the ground state expectation value of the time ordered product of those operators. If some involve time derivatives it gives the $T^*$ product in which all derivatives are taken outside the time ordering symbol.

As we have already discussed for QED the gauge invariance of the functional integrand means it is independent of one of the gauge field components, so both the numerator and denominator contain an infinite factor of the gauge group volume for each space time point. The factor formally cancels between numerator and denominator, but to make the cancellation more rigorous, it is best to fix the gauge by imposing a constraint on the gauge fields. One has a lot of flexibility in choosing this constraint, but in our work we will use a generalized Lorentz gauge $\partial \cdot A^a(x) - f^a(x) = 0$.

Fadeev and Popov introduce such a constraint through the following device. Call the gauge transformed gauge field $A_\Omega$ and define $\Delta_{FP}[A]$ to satisfy

$$\Delta_{FP}(A) \int d\Omega(x) \delta(\partial \cdot A^a_\Omega(x) - f^a(x)) = 1 \quad (22.20)$$

The integration over the gauge group is of course a functional integral, not an ordinary one, and it is assumed to be invariant under the change of variables $\Omega(x) \rightarrow U(x) \Omega(x)$. This implies that $\Delta_{FP}$ is gauge invariant. Next insert the right side of this expression in the path integrals in both the numerator and denominator. Change variables in the gauge functional integral $A \rightarrow A_{\Omega^{-1}}$ and use the gauge invariant measure and observables to get

$$\langle \Phi \rangle = \frac{\int d\Omega(x) \int \mathcal{D} A^a_{\mu} \mathcal{D}( Fields) \, \Phi \Delta_{FP}[A] \delta(\partial \cdot A^a(x) - f^a(x)) e^{i \int d^4 x (-F^a_{\mu\nu} F^{a\mu\nu}/4 + \mathcal{L}_{\text{fields}})}}{\int d\Omega(x) \int \mathcal{D} A^a_{\mu} \mathcal{D}( Fields) \Delta_{FP}[A] \delta(\partial \cdot A^a(x) - f^a(x)) e^{i \int d^4 x (-F^a_{\mu\nu} F^{a\mu\nu}/4 + \mathcal{L}_{\text{fields}})}}.$$

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The volume of the gauge group can now be cancelled, leaving a formally convergent integral over the gauge fields. The presence of the gauge fixing delta function is awkward in practical calculations. It can be finessed by noting that both numerator and denominator by gauge invariance should be independent of $f^a(x)$. Therefore we are free to multiply numerator and denominator by $e^{-i\int d^4x f^a(x)/\left(2\xi\right)}$ and integrate both over $f^a$. When this is done the gauge fixing delta function sets $f^a = \partial \cdot A^a$ inside the functional integral, leading to

$$
\langle \Phi \rangle = \frac{\int D_A^a D(Fields) \Phi \Delta_{FP}[A] e^{i\int d^4x(-F_{\mu\nu}^a F^{a\mu\nu}/4-(\partial \cdot A^a)^2/2\xi+L_{fields})}}{\int D_A^a D(Fields) \Delta_{FP}[A] e^{i\int d^4x(-F_{\mu\nu}^a F^{a\mu\nu}/4-(\partial \cdot A^a)^2/2\xi+L_{fields})}}.
$$

This formula is derived assuming $\Phi$ is gauge invariant. If there are gauge variant operators in $\Phi$, this expression is taken as the definition of those operators in this gauge. It would of course then be different in different gauges.

### 22.2.1 The Fadeev-Popov Determinant

We would like to put $\Delta_{FP}[A]$ in a form suitable for practical calculations. We first note that since it multiplies the gauge fixing delta function we only need it for $A$ satisfying the gauge condition. For $A$ satisfying the gauge condition the integral over $\Omega(x)$ need only include infinitesimal gauge transformations $\Omega = I - iG A_G = A - i[G, A] - g^{-1}\partial G$ or $A^a \rightarrow A^a + f_{abc} C^b A^c - g^{-1}\partial C^a$

$$
\int d\Omega \delta(\partial \cdot A^a - f^a) = \int dG \delta(\partial \cdot (g^{-1}(\delta_{ab}\partial - gf_{abc} A^c)) G^b) = \det^{-1}[\partial \cdot (\delta_{ab}\partial - gf_{abc} A^c)]
$$

Then we infer

$$
\Delta_{FP}[A] = \det[-\partial \cdot (\delta_{ab}\partial - gf_{abc} A^c)]
$$

Now we can represent the determinant as a Grassmann functional integral over two different Grassmann variables;

$$
\det[-\partial \cdot (\delta_{ab}\partial - gf_{abc} A^c)] = \int dB^a dC^a e^{-i\int d^4x \partial B^a(x) \cdot (\partial C^a + gf_{abc} A^b C^c)}
$$

We note that $D^\mu C^a = \partial^\mu C^a + gf_{abc} A^b C^c$ is just the covariant derivative of a gauge field in the adjoint representation.

### 22.3 Feynman Rules for a Nonabelian Gauge Theory

As always the propagators are determined by the quadratic terms in the Lagrangian:

$$
\mathcal{L}_0 = -\frac{1}{2} \partial^\mu A^a_\nu (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - \frac{1}{2\xi}(\partial \cdot A^a)^2 - \partial B^a(x) \cdot \partial C^a
$$

$$
= -\frac{1}{2} \partial^\mu A^a_\nu (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - \frac{1-\xi}{2\xi}(\partial \cdot A^a)^2 - \partial B^a(x) \cdot \partial C^a
$$

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The gauge field propagator is \( \delta_{ab}D_{\mu\nu} \) in momentum space where
\[
(p^2 \eta_{\mu\nu} + \frac{1 - \xi}{\xi} p_\mu p_\nu)D^{\nu\lambda} = -i\eta^{\lambda}_\mu. \tag{22.25}
\]

To solve put \( D_{\nu\lambda} = A(p^2)\eta_{\nu\lambda} + B(p^2)p_\nu p_\lambda \) and plug in
\[
A\eta_{\mu\lambda}p^2 + Bp^2 p_\mu p_\lambda + \frac{1 - \xi}{\xi} p_\mu(A + Bp^2)p_\lambda = -i\eta_{\mu\lambda}
\]
\[
A = -\frac{i}{p^2 - i\epsilon} \quad (A + Bp^2)(1 - \xi) + \xi Bp^2 = 0
\]
\[
\langle A^a_\mu A^b_\nu \rangle = -i\delta_{ab} \frac{\eta_{\mu\nu} + (\xi - 1)p_\mu p_\nu/p^2}{p^2 - i\epsilon} \tag{22.26}
\]

The ghost propagator is easily read off:
\[
\langle C^a B^b \rangle = -i\delta_{ab} \frac{p^2 - i\epsilon}{p^2 - i\epsilon} \tag{22.27}
\]

The Feynman gauge is \( \xi = 1 \) and the Lorentz or Landau gauge is \( \xi = 0 \).

The vertices are obtained from the cubic and quartic terms in \( \mathcal{L} \)
\[
\mathcal{L}_{3A} = -\frac{g}{2} f_{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^b_\mu A^c_\nu = -gf_{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} \tag{22.28}
\]
From which we infer the cubic gauge vertex, with all \( k_i \) outgoing momenta,
\[
\langle A^d_\kappa(k_1)A^e_\rho(k_2)A^f_\sigma(k_3) \rangle = -gf_{def}(k_1\rho\eta_{\kappa\sigma} - k_1\sigma\eta_{\kappa\rho} + k_2\sigma\eta_{\rho\kappa} - k_2\kappa\eta_{\rho\sigma}
+ k_3\kappa\eta_{\rho\sigma} - k_3\rho\eta_{\kappa\sigma}) = -gf_{def}(k_1 - k_2)\rho\eta_{\kappa\sigma} + (k_2 - k_1)\sigma\eta_{\rho\kappa}
+ (k_3 - k_2)\kappa\eta_{\rho\sigma}) \tag{22.29}
\]
Note that the labels of the gluons enjoy a cyclic symmetry.

The cubic ghost term is
\[
\mathcal{L}_{3G} = -gf_{abc}\partial_\mu B^{b\mu}A^c \tag{22.30}
\]
which implies the vertex
\[
\langle C^d(k_1)A^e_\rho(k_2)B^f(k_3) \rangle = -gf_{def}k_1\rho \tag{22.31}
\]
We must remember to treat the ghosts as fermions, meaning a factor of \(-1\) for each ghost loop. Finally the quartic gauge term is
\[
\mathcal{L}_4 = -\frac{g^2}{4} f_{abc} f_{ade} A^b_\mu A^e_\nu A^{d\mu}A^{e\nu} \tag{22.32}
\]
and the corresponding vertex is
\[
\langle A^a_\alpha A^a_{\lambda} A^a_{\rho} A^a_{\sigma} \rangle = -ig^2 [f_{abc} f_{aef} (\eta_{\kappa \rho} \eta_{\lambda \sigma} - \eta_{\kappa \sigma} \eta_{\lambda \rho}) \\
+ f_{adf} f_{ae\gamma} (\eta_{\kappa \lambda} \eta_{\rho \sigma} - \eta_{\kappa \sigma} \eta_{\lambda \rho})
+ f_{adg} f_{ae\delta} (\eta_{\kappa \lambda} \eta_{\rho \sigma} - \eta_{\kappa \sigma} \eta_{\lambda \rho})] \quad (22.33)
\]

Additional Dirac fermions in some representation of the group would add the vertex \( igT_R a \gamma^\mu \). And of course scalar fields would introduce their own cubic couplings as well as a two scalar two gauge boson quartic coupling.

### 22.4 UV divergences in nonabelian gauge theory

In this section we examine the structure of UV divergences in nonabelian gauge theories. We first remark that the superficial degree of divergence of a given diagram is essentially that of scalar electrodynamics: the cubic gauge particle vertex as well as the cubic ghost vertex has one derivative and the quartic vertex has none. Therefore

\[
D = D - \frac{D - 2}{2} E_B + \frac{D - 4}{2} (2V_0 + V_1)
\rightarrow 4 - E_B, \quad D \rightarrow 4 \quad (22.34)
\]

The most striking difference between nonabelian gauge theory and QED is that the coupling renormalization has the opposite sign to vacuum polarization in QED. The latter sign corresponds to a screening of the bare charge: the charge grows weaker at longer distances. If the effect has the opposite sign it corresponds to antiscreening: the coupling grows larger at longer distances. Conversely, in QED the charge grows larger at shorter distances whereas in nonabelian gauge theory it weakens at shorter distances. These opposite effects are evident in one loop corrections. However the calculations which reveal it are quite lengthy.

It is somewhat simpler to renormalize coupling of the gauge field to a Dirac fermion in the fundamental representation of the gauge group than that of the self interaction of the gauge field. The relevant diagrams are shown at the end of this section.

#### 22.4.1 QED-like Diagrams

The diagrams in the first row involve the same loop integrals as the corresponding ones in QED. To adapt them to nonabelian theory one replaces the QED charge with the generators \( T_a \) of the gauge group. For example the fermion self energy bubble replaces \( e^2 \) in QED by

\[
g^2 T_a \delta_{ab} T_b = g^2 \sum_a T_a^2 = g^2 C_F I \quad \text{the Casimir operator for the fundamental representation.}
\]

Remembering our normalization \( \text{Tr} T_a T_b = \delta_{ab} / 2 \) for the fundamental of \( SU(N) \) shows that \( C_F = (N^2 - 1)/(2N) \). For the QED-like vertex diagrams the gauge group factors are

\[
\sum_a T_a T_b T_a = C_F T_b + i f_{abc} T_c T_a = C_F T_b + \frac{1}{2} i f_{abc} i f_{cde} T_d
\]

\[
= \left( C_F - \frac{C_A}{2} \right) T_b. \quad (22.35)
\]
For $SU(N)$ one can regard the adjoint plus a singlet as the tensor product of an $N$ and an $\bar{N}$. Then

$$(T_a^N + T_a^\bar{N})^2 = 2C_F + 2 \sum_a T_a^N T_a^\bar{N} = C_A P_A + C_S P_S = C_A P_A$$

(22.36)

Taking the trace of both sides leads to

$$2N^2C_F = C_A(N^2 - 1), \quad C_A = N.$$  

(22.37)

Apart from these color factors the loop integrals are the same as QED. If we calculate in Landau gauge $\xi = 0$ these two QED like diagrams do not contribute to the UV renormalization of the coupling. (Exercise). The fermion contribution to the gauge particle self energy is obtained from the QED calculation by replacing $e^2$ with $g^2 \text{Tr} T_a T_b = g^2 T(R) \delta_{ab}$.

In Landau gauge the coupling renormalization due to gauge sector 1 loop diagrams, i.e. not including the vacuum polarization diagrams due to matter fields, is determined by the three diagrams on the second line. For now we focus on isolating the coefficient of $(4 - D)^{-1}$ or alternatively the coefficient of $\ln(\Lambda/\mu)$ in these diagrams. We do them in reverse order.

### 22.4.2 Fermion Gluon Vertex

In Landau gauge the value of the last diagram is, with the gluon momentum $q$ outgoing and the fermion momenta $p, p'$ following the charge flow,

$$(-i)^3 (ig)^2 g f_{abc} T_b T_c \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\rho(-p + k) \cdot \gamma^\sigma}{k^2(q + k)^2(p + k)^2} \left( \eta_{\rho\sigma'} - \frac{(k + q)^\rho(k + q)^{\rho'}}{(k + q)^2} \right) \left( \eta_{\sigma \sigma'} - \frac{k^\sigma k^{\sigma'}}{k^2} \right) \left( \eta^\rho_{\mu} (k - q)^{\rho'} + \eta^{\rho'}_{\mu} (2q + k)^{\sigma'} - \eta^{\rho' \sigma'}(q + 2k)_\mu \right)$$

(22.38)

To capture the log divergence, we only need the leading term as $k \rightarrow \infty$:

$$(-i)^3 (ig)^2 g f_{abc} T_b T_c \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\rho(-k \cdot \gamma^\sigma)}{k^6} \left( \eta_{\rho\sigma'} - \frac{k^\rho k^{\rho'}}{k^2} \right) \left( \eta^{\rho}_{\mu} k^{\rho'} + \eta^{\rho'}_{\mu} k^{\sigma'} - \eta^{\rho' \sigma'} 2k_\mu \right)$$

$$= (-i)^3 (ig)^2 g f_{abc} T_b T_c \int \frac{d^4k}{(2\pi)^4} \frac{2k_\mu \gamma^\rho k \cdot \gamma^\sigma}{k^6} \left( \eta_{\rho\sigma'} - \frac{k^\rho k^{\sigma'}}{k^2} \right)$$

$$= -ig^3 f_{abc} T_b T_c \int \frac{d^4k}{(2\pi)^4} \frac{6k_\mu k \cdot \gamma}{k^6} = \frac{3g^3}{16\pi^2} f_{abc} T_b T_c \gamma_\mu \ln \frac{\Lambda}{\mu}$$

(22.39)

Now note

$$f_{abc} T_b T_c = \frac{1}{2} f_{abc} [T_b , T_c] = -i \frac{1}{2} f_{abc} f_{cda} T^d = \frac{1}{2} i (T_b^A T_b^A)_{cd} T_d = \frac{iC_A}{2} T_a$$

(22.40)
so the last diagram on the second line has a log divergence

\[ \frac{3g^2 C_A}{32\pi^2} igT_a \gamma_\mu \ln \frac{\Lambda}{\mu} \]  

(22.41)

In general \( \xi \) gauge, this diagram plus the QED-like vertex diagram, which now has a log divergence proportional to \( \xi \), changes the 2 quark 1 gluon diagram by the substitution \( 3 \rightarrow 3 + 5\xi \).

22.4.3 FP ghost loop

Next we consider the second diagram on the first line, the contribution of the FP ghosts:

\[ -(\gamma^2 g^2 \epsilon_{bed} \epsilon_{dea}) \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{k_\mu k_\nu - x(1-x)q_\mu q_\nu}{k^2 + x(1-x)q^2} \]

\[ \sim g^2 \epsilon_{bed} \epsilon_{dea} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{k^2 \eta_{\mu\nu} / 4 - x(1-x)q_\mu q_\nu}{k^4} \left[ 1 - 2x(1-x)\frac{q^2}{k^2} \right] \]

\[ \sim g^2 C_A \delta_{ab} \int \frac{d^4k}{(2\pi)^4} \left[ -\frac{\eta_{\mu\nu}}{4k^2} + \frac{1}{12} \left( \eta_{\mu\nu} q^2 + 2q_\mu q_\nu \right) \right] \]  

(22.42)

The log divergence is in the second term in square brackets:

\[ i \frac{C_A}{12} \frac{g^2}{8\pi^2} \ln \frac{\Lambda}{\mu} (q^2 \eta_{\mu\nu} + 2q_\mu q_\nu), \quad \text{FP ghosts} \]  

(22.43)

22.4.4 Gluon loop

By far the most tedious diagram to evaluate is the first one on the second line, the gluon propagator correction (i.e. the gluon self energy) with gluons circulating in the loop. The upshot is that it adds to the factor \( q^2 \eta_{\mu\nu} + 2q_\mu q_\nu \) in the ghost contribution the quantity

\[ 24(q^2 \eta_{\mu\nu} - q_\mu q_\nu) + q^2 \eta_{\mu\nu} - 4q_\mu q_\nu \]  

(22.44)

giving the net result

\[ i \frac{26C_A}{12} \frac{g^2}{8\pi^2} \ln \frac{\Lambda}{\mu} (q^2 \eta_{\mu\nu} - q_\mu q_\nu), \quad \text{gluon self – energy} \]  

(22.45)

For general \( \xi \) this net result changes according to \( 26 \rightarrow 26 - 6\xi \). When this corrected propagator is attached to the tree vertex one includes a gluon propagator and a factor of \( 1/2 \) (corresponding to \( \sqrt{Z_3} \)), giving the net result, remembering that \( q_\mu \bar{u} \gamma^\mu T_a u = 0 \)

\[ ig\gamma^\mu T_a \frac{13C_A}{12} \frac{g^2}{8\pi^2} \ln \frac{\Lambda}{\mu} \]  

(22.46)
Adding the result for the vertex correction, adds $3/4$ to $13/12$:

$$ig\gamma^\mu T_a \frac{22C_A}{12} \frac{g^2}{8\pi^2} \ln \frac{\Lambda}{\mu} \quad (22.47)$$

or written in terms of the renormalized coupling

$$g_R = g \left( 1 + \frac{11C_A}{12} \frac{g^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) \quad (22.48)$$

For general $\xi$ all of the $\xi$ dependence from the vertex and quark and gluon propagators cancel in the final result: in other words the net coupling renomalization is gauge independent, at least to one loop.

A fermion loop correction in representation $R$ of the gauge group has the effect of replacing $\frac{11C_A}{12} \to \frac{11C_A}{12} - \frac{4}{\pi^2} T(R)$. For example in QCD the $n_F$ quarks are in the fundamental representation of the gauge group $SU(3)$, for which $C_A = 3$ so $\frac{11C_A}{12} - \frac{4}{\pi^2} T(3) = 33 - 2n_F$ which becomes negative for $n_F > 16$. Fortunately, there is evidence for only 6 flavors of quark.

Increasing the renormalization scale $\mu$ decreases the renormalized coupling. This is what is meant by asymptotic freedom. Another way to write this is in terms of $\alpha = g^2/(4\pi)$:

$$\alpha_R = \alpha \left( 1 + \frac{11C_A}{6} \frac{\alpha}{\pi} \ln \frac{\Lambda}{\mu} \right) \sim \frac{1}{\alpha^{-1} - (11C_A/(6\pi)) \ln(\Lambda/\mu)} \quad (22.49)$$

We see that the $\ln \Lambda$ can be absorbed in $\alpha$ while maintaining $\alpha_R > 0$.

Details of one loop gluon propagator. In Landau gauge we need a generalization of the Feynman trick

$$\frac{1}{A^2B^2} = 2 \int_0^1 dx \frac{x(1-x)}{[Ax + B(1-x)]^4} \quad (22.50)$$

Then the numerator of the loop integrand is

$$N_{\mu\nu} = \left[ \eta_{\mu\sigma}(2q + k)_{\sigma} - \eta_{\rho\sigma}(q + 2k)_{\rho} + \eta_{\mu\sigma}(k - q)_{\rho} \right]$$

$$\left[ \eta_{\nu\rho'}(2q + k)_{\rho'} - \eta_{\nu\rho'}(q + 2k)_{\rho} + \eta_{\nu\rho'}(k - q)_{\rho'} \right]$$

$$\left[ \eta^{\rho\rho'}(k + q)^2 - (k + q)^\rho(k + q)^{\rho'} \right]$$

$$\left[ \eta^{\rho\rho'}(k + q)^2 - (k + q)^\rho(k + q)^{\rho'} \right]$$

$$= \left[ \eta_{\mu\rho}2q_{\sigma} - \eta_{\rho\sigma}(q + 2k)_{\mu} - \eta_{\mu\rho}2q_{\rho} \right]$$

$$\left[ \eta_{\nu\rho'}2q_{\sigma'} - \eta_{\nu\rho'}(q + 2k)_{\nu} - \eta_{\nu\rho'}2q_{\rho'} \right]$$

$$\left[ \eta^{\rho\rho'}(k + q)^2 - (k + q)^\rho(k + q)^{\rho'} \right]$$

$$\left[ \eta^{\rho\rho'}(k + q)^2 - (k + q)^\rho(k + q)^{\rho'} \right] \quad (22.51)$$

In the second form we have used the transverse projectors to set $k_{\sigma} = k_{\sigma'} = 0$ and $k_\mu = k_{\rho'} = -q$. To simplify writing in what follows it is convenient to define

$$R_{\mu\rho \sigma} \equiv \eta_{\mu\rho}2q_{\sigma} - \eta_{\rho\sigma}q_{\mu} - \eta_{\mu\sigma}2q_{\rho} \quad (22.52)$$
so the numerator can be written
\[ N_{\mu\nu} = [R_{\mu\rho\sigma} - 2\eta_{\rho\sigma}k_{\mu}][R_{\nu\rho'\sigma'} - 2\eta_{\rho'\sigma'}k_{\nu}] \]
\[ \frac{[\eta_{\rho\rho'}(k + q)^2 - (k + q)^\rho(k + q)^\rho'][\eta_{\sigma\sigma'}k^2 - k^\sigma k'^\sigma]}{(k + xq)^2 + (x(1 - x)q^2)^4} \]

Meanwhile after combining denominators the denominator is
\[ [(k + xq)^2 + x(1 - x)q^2]^4 \]

Then after the change of variables \( k \to k - xq \), the terms odd in \( k \) can be dropped and averaging over directions of \( k \) allows the replacements
\[ \langle k_{\mu}k_{\nu} \rangle = k^2\frac{\eta_{\mu\nu}}{D}, \quad \langle k_{\mu}k_{\nu}k_{\rho}k_{\sigma} \rangle = k^4\frac{\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}}{D(D + 2)} \]

The term in the numerator with 6 factors of \( k \) only has two free indices so we don’t need the average of six \( k_s \). That term reduces to
\[ 4k_{\mu}k_{\nu}k^2(2\eta_{\rho} - k_{\rho}k_{\rho}) = 12k^4k_{\mu}k_{\nu} \rightarrow 12k^6\frac{\eta_{\mu\nu}}{D} \]

The full numerator will include terms of order \( k^6, k^4q^2, k^2q^4, q^6 \) but the log divergence is contained only in the \( k^4q^2 \) term in the numerator and in the \( k^6 \) term multiplied by \(-4x(1 - x)q^2/k^2 \) coming from expanding the denominator in powers of \( q^2/k^2 \). The zeroth order term of this expansion is a quadratic divergence which dimensional regularization sets to zero. After keeping only the first two terms in this expansion the \( x \) integral is just that of a polynomial which can be immediately done. After completing all these steps the result quoted above is found.

Putting in the factors of \(-i)^2 \) from the two propagators and \( g^2f_{\text{bed}}f_{\text{dca}} = -C_A g^2\delta_{ab} \) from the two vertices, and the symmetry factor 1/2, the gluon propagator correction is so far
\[ \frac{g^2C_A}{2} \int \frac{d^Dk}{(2\pi)^D} \int_0^1 dx \frac{x(1 - x)}{[k^2 + x(1 - x)q^2]^4} \]
\[ [R_{\mu\rho\sigma} - 2\eta_{\rho\sigma}(k - xq)_{\mu}][R_{\nu\rho'\sigma'} - 2\eta_{\rho'\sigma'}(k - xq)_{\nu}] \]
\[ [\eta_{\rho\rho'}(k + q(1 - x))^2 - (k + q(1 - x))^\rho(k + q(1 - x))^\rho'] \]
\[ [\eta_{\sigma\sigma'}(k - xq)^2 - (k - xq)^\sigma(k - xq)^\sigma'] \]

To capture the log divergence we simply collect terms quadratic in \( q \). For instance
\[ (R_{\mu\rho\sigma} + 2x\eta_{\rho\sigma}q_{\mu})(R_{\nu\rho'\sigma'} + 2x\eta_{\rho'\sigma'}q_{\nu})(\eta_{\rho\rho'}k^2 - k_{\rho}k_{\rho'})(\eta_{\sigma\sigma'}k^2 - k_{\sigma}k_{\sigma'}) \]
\[ \rightarrow k^4\left( \frac{\eta_{\rho\rho'}\eta_{\sigma\sigma'}}{1 - \frac{2}{D}} + \frac{1}{D(D + 2)}(\eta_{\rho\sigma}\eta_{\rho'\sigma'} + \eta_{\rho\rho'}\eta_{\sigma\sigma'} + \eta_{\rho\sigma'}\eta_{\sigma\rho'}) \right) \]

After isolating the quadratic term in the numerator, one easily obtains the log divergence by setting \( D = 4 \) and \( q^2 = 0 \) in the denominator. And of course one must include the quadratic term from the expansion of the one of the denominator to first order in \( q^2/k^2 \) and setting \( q = 0 \) in the numerator.

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22.5 BRST Invariance

As we know from QED the gauge invariance implies that correlation functions satisfy the Ward identities. They can be used to prove gauge independence as well as the decoupling of negative norm particle states present in covariant gauges. The corresponding identities known as Ward-Takahashi identities in nonabelian gauge theories are considerably more involved because unlike the photon, the gauge particles themselves are adjoints in the gauge group and therefore interact with each other. Fortunately Becchi, Rouet, and Stora and independently Tyutin discovered a remarkable symmetry (BRST) of the gauge fixed action involving the FP ghost fields, which captures in an elegant way the important decouplings and gauge independence required for consistency of any gauge theory.

Recall that an infinitesimal gauge transformation is given by
\[
\delta A^a_\mu = \partial_\mu \theta^a + g f_{abc} A^b_\mu \theta^c \tag{22.59}
\]
which of course is not a symmetry of the gauge fixed action. The BRST discovery was that if the gauge parameter is replaced by the ghost field \( C^a \), then one can find transformations of the other fields so that the gauge-fixed action is invariant!

Denote the BRST transform of a field \( \Phi \) by \( s \Phi \) Then we have
\[
\begin{align*}
\delta A^a_\mu &= \partial_\mu C^a + g f_{abc} A^b_\mu C^c = D^{ab}_\mu C^b \tag{22.60}
\end{align*}
\]
where the last step uses the Jacobi identity. Now we can say that with
\[
\begin{align*}
\delta A^a_\mu &= \partial_\mu C^a + g f_{abc} A^b_\mu C^c \\
\delta B^a &= 1/\xi (\partial \cdot A^a) \tag{22.65}
\end{align*}
\]
where the second replacement uses the Grassmann nature of \( C \). Now we need the product law for covariant derivatives
\[
\begin{align*}
D^{ac}_\mu (f_{ebc} C^b C^c) &= f_{ebc} (\partial_\mu C^b) C^c + f_{ebc} C^b \partial_\mu C^c + g f_{ace} f_{ebc} A^b C^b C^c \tag{22.63}
\end{align*}
\]
where the last step uses the Jacobi identity. Now we can say that with \( sC^a = -1/2 f_{ebc} C^b C^c / \)

The gauge fixed Lagrangian is BRST invariant. In summary, the BRST transform is
\[
\begin{align*}
sA^a_\mu &= \partial_\mu C^a + g f_{abc} A^b_\mu C^c = D^{ab}_\mu C^b \tag{22.64} \\
$sB^a &= 1/\xi (\partial \cdot A^a) \tag{22.65} \\
$sC^a &= -1/2 f_{ebc} C^b C^c \tag{22.66}
\end{align*}
\]
A remarkable property of the BRST transformation is that it is nilpotent: applied twice it gives zero:

\[ s^2 A^a_\mu = D^{ab}_\mu sC + gf_{abc} (sA^b)C^c = 0 \]  

(22.67)

by the same calculation we did in showing the invariance of the FP ghost Lagrangian.

\[ s^2 B^a = \frac{1}{\xi} \partial^\mu D^a_\mu C^b = 0 \]  

(22.68)

by the field equation for \( C \). And

\[ s^2 C^a = -\frac{1}{2} f_{abc} (sC^b)C^c + \frac{1}{2} f_{abc} C^b sC^c \]

\[ = \frac{1}{4} f_{abc} f_{bde} C^d C^e C^c - \frac{1}{4} f_{abc} C^b f_{cde} C^d C^e \]

\[ = \frac{1}{2} f_{abc} f_{bde} C^d C^e C^c = \frac{1}{6} (f_{abc} f_{bde} + f_{abc} f_{bcd} + f_{abc} f_{bce}) C^d C^e C^c = 0 \]

by the Jacobi identity.

The BRST transformation of the various fields in the system will be generated by an operator \( Q \) in the operator formalism. This operator should commute with the Hamiltonian \([Q, H] = 0\) and all physical observables, and it should also be nilpotent \( Q^2 = 0 \). BRST invariance implies the Ward-Takahashi identities. It is the Hilbert space realization of the redundancy of gauge fields that are pure gauge.

The state space includes the action of all four components of the gauge field \( A^a_\mu \) the ghosts \( C^a \) and antighosts \( B^a \) on the vacuum. The physical states are required to satisfy \( Q|\text{Phys}\rangle = 0 \). But any vector in state space of the form \( Q|\Lambda\rangle \) is also annihilated by \( Q \). But due to \( Q^2 = 0 \) such states will be null, whereas a true physical state is required to have finite norm. Thus any pair of states differing by \( Q|\Lambda\rangle \) will be identified. States annihilated by \( Q \) are said to be in the kernel of \( Q \). So a physical state is in the kernel of \( Q \) mod \( Q|\lambda\rangle \). It is appropriate to use the language of differential geometry: \( Q \) playing a role analogous to the exterior derivative \( d \). both of which are nilpotent. Then we can say that the space of physical states is the cohomology of \( Q \), consisting of the kernel of \( Q \) mod \( Q|X\rangle \).

### 22.6 Gauge theory of the Standard Model

Having learned how to describe a general gauge theory, we now turn to its major application in this course: the standard model of elementary particle physics. In this model the gauge group is the non simple group \( SU(3) \times SU(2) \times U(1) \). There is an independent gauge field for each factor group as follows:

\[ SU(3) : \quad A_\mu = \sum_a \frac{\lambda^a}{2} A^a_\mu, \quad F^a_\mu\nu = \partial^\mu A^a_\nu - \partial^\nu A^a_\mu + g_3 f_{abc} A^b_\mu A^c_\nu \]

\[ SU(2) : \quad W_\mu = \sum_a \frac{\tau^a}{2} W^a_\mu, \quad W^a_\mu_\nu = \partial^\mu W^a_\nu - \partial^\nu W^a_\mu + g_2 \epsilon_{abc} W^b_\mu W^c_\nu \]

\[ U(1) : \quad B_\mu, \quad B_\mu_\nu = \partial^\mu B_\nu - \partial^\nu B_\mu \]  

(22.69)
Each gauge field has its own coupling constant $g_3, g_2, g_1$ respectively. Notice that $g_{2,3}$ are physically meaningful in the pure gauge theory because $SU(2, 3)$ are nonabelian. The coupling $g_1$ of the abelian gauge group $U(1)$ only becomes meaningful with the addition of further charged fields.

The $SU(3)$ gauge field mediates the strong interactions and couples only to quarks and itself. It is referred to as the color group. The electroweak gauge group, mediating the electromagnetic and weak interactions, is $SU(2) \times U(1)$. The $SU(2)$ factor is referred to as weak isospin and the $U(1)$ factor as weak hypercharge. This terminology is borrowed from the familiar isospin and hypercharge of strong interactions.

The electroweak part of the standard model gauge group was strongly motivated by the phenomenology of the electromagnetic and weak interactions of the elementary particles. In the 1950’s and 60’s studies of nuclear beta decay, e.g. $n \to p + e + \bar{\nu}$ and muon decay $\mu \to e + \bar{\nu}_e + \nu_\mu$ established that the weak interactions had a current-current structure strongly suggesting that they were mediated by a charged heavy vector boson, called $W^\pm$. This together with the long understood mediation of electromagnetic interactions by the photon $A_\mu$ suggested a gauge group with at least a three dimensional adjoint representation. For a time in the early 1960’s physicists attempted to do with just $SU(2)$. But Salam and Weinberg soon (1967) built a successful model based on $SU(2) \times U(1)$ which turned out to be the right answer. Their model had an additional neutral vector boson, which would mediate neutral current weak interactions. The experimental discovery of neutral current interactions (several years after the theory was proposed) clinched the Weinberg-Salam model as the gauge theory of electroweak interactions. As we shall see the photon field is not simply the $B_\mu$ field but is a linear combination of $B_\mu$ with $W^3_\mu$.

22.6.1 Gauge group representations of quarks and leptons

The electroweak fields couple to quarks and leptons as well as to each other. The gauge part of the Lagrangian is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu}$$  \hspace{1cm} (22.70)

The fermionic “matter fields” are organized in three families, each consisting of a left handed $SU(2)$ doublet of quarks, a left handed $SU(2)$ doublet of leptons, with the corresponding right-handed fermions in $SU(2)$ singlets. The quarks are fundamental triplets under $SU(3)$ whereas all of the leptons are singlets under $SU(3)$. The pattern of irreducible representation assignments for each family are tabulated below.
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</tbody>
</table>

where the subscript $L$ on a fermion field $\psi$ means $(1 - \gamma_5)\psi/2$ and the subscript $R$ means $(1 + \gamma_5)\psi/2$. The assignment of different gauge group representations to the left and right components of a Fermi field characterizes the standard model as chiral. Dirac mass terms are therefore forbidden in the standard model by gauge invariance. A Majorana mass term is allowed for a right handed neutrino $\nu_R$, should it exist, because that field is a gauge singlet under the whole gauge group. Indeed if that Majorana mass is sufficiently large, it would “explain” why such a particle has not been seen. The doublet left handed pattern of couplings was strongly motivated by the beta decay processes already mentioned. For the baryons, the doublet structure was initially taken to be $(p, n)$ but once the quark model was established the composite nucleons $(p, n)$ were replaced by the more “fundamental” quarks $(u, d)$.

The assignments of $U(1)$ charge, called weak hypercharge, shown in the table is dictated by the electric charge each fermion should carry. It is normalized according to the relation $Q = (\tau_3 + Y)/2$, which will later be seen to be a consequence of the Higgs mechanism. So the left handed neutrino $(T_3 = +1/2)$ and its lepton partner $(T_3 = -1/2)$, which have 0 and -1 units of charge, would have $Y = -1$, whereas the right handed electron would have $Y = -2$. The righthanded neutrino, should it exist would have $Y = 0$. For the quarks, the up type quark has electric charge 2/3 corresponding to $Y = 1/3$ for the left handed one and $Y = 4/3$ for the right handed one. The right handed down quark has $Y = -2/3$. There are two more lepton/quark families $(\nu_{\mu}, \mu)$, $(c, s)$ and $(\nu_{\tau}, \tau)$, $(t, b)$ with identical pattern of electroweak irreps.

### 22.6.2 Masses for Gauge particles: Higgs mechanism.

As presented so far this model seems far from reasonable: it has 12 massless gauge particles, and a number of necessarily massless chiral fermions. The only massless gauge boson in nature is the photon. The only near massless fermions are the neutrinos, but the presence of neutrino oscillations provides evidence that even the neutrinos have a (tiny) nonzero mass. So we need another ingredient to give masses to all but one particle in the standard model. This missing ingredient is the Higgs mechanism, which uses additional scalar fields with
nonzero constant values in the vacuum. If they are judiciously coupled to the fields we have encountered so far, the mass parameters will be proportional to these constant values.

To motivate these couplings consider a potential electron mass term $\bar{e}_L e_R + \bar{e}_R e_L$. Since the left handed electron is a member of an $SU(2)$ doublet and the righthanded one an $SU(2)$ singlet, this is not gauge invariant because it has weak isospin $1/2$ and nonzero weak hypercharge. If we introduce in the model an $SU(2)$ doublet scalar field $\phi = (\phi_1, \phi_2)$, we can form an $SU(2)$ singlet $\lambda \phi^* \phi^\dagger \bar{e}_R (\nu e_L e_L) = \lambda \phi^* \phi^\dagger \bar{e}_R \nu e_L + \lambda \phi^* \phi^\dagger e_R e_L$ so an effective mass $\lambda \langle \phi^* \phi^\dagger \rangle$ for the electron would arise if $\langle \phi^* \phi^\dagger \rangle \neq 0$ and $\langle \phi^* \phi^\dagger \rangle = 0$. So this simple way of giving mass to fermions requires an $SU(2)$ doublet scalar field.

Now examine the scalar kinetic term of such an $SU(2)$ doublet, assumed to be a singlet under $SU(3)$, $-(D^\mu \phi)^\dagger D\mu \phi$, where

$$D\mu \phi = (\partial - ig_2 W - ig_1 Y_h/2) \phi$$

Since this is the first appearance of $g_1$ we may define $Y_h = -1$ as a convention. Then if $\langle \phi \rangle \equiv v$ is a constant spinor, the shift $\phi = v + \hat{\phi}$ leads to

$$D\mu \phi = (\partial - ig_2 W + ig_1 Y_h/2) \hat{\phi} + (-ig_2 W + ig_1 Y_h/2)v$$

Then

$$(D\mu \phi)^\dagger D\mu \phi = \partial^\dagger \partial \hat{\phi} + \partial^\dagger (-ig_2 W + ig_1 Y_h/2)v + v^\dagger (ig_2 W - ig_1 Y_h/2) \cdot \partial \hat{\phi}$$

$$+ v^\dagger \left( \frac{g_2^2}{4} W^{a2} + \frac{g_1^2}{4} B^2 - \frac{ig_1 g_2}{2} \tau^a W^a \cdot B \right) v + \text{cubic terms}$$

and the last term is a quadratic form in the gauge fields:

$$v^\dagger \left( \frac{g_2^2}{4} W^{a2} + \frac{g_1^2}{4} B^2 - \frac{ig_1 g_2}{2} \tau^a W^a \cdot B \right) v$$

It is a basic property of the doublet representation of $SU(2)$ that any spinor is the eigenvector of $\hat{n} \cdot \tau$ with eigenvalue $+1$ for some unit vector $\hat{n}$. We can then fix our 3-axis in the $\hat{n}$ direction\(^1\). Then the above quadratic form becomes $v^\dagger v$ times

$$\frac{g_2^2}{4} (W_1^2 + W_2^2) + \frac{g_2^2}{4} W_3^2 + \frac{g_1^2}{4} B^2 - \frac{ig_1 g_2}{2} B \cdot W_3$$

$$= \frac{g_2^2}{4} (W_1^2 + W_2^2) + \frac{1}{4} (g_2 W_3 - g_1 B)^2$$

\(^1\)Since the dynamics is invariant under $SU(2)$, if $v \neq 0$ minimizes the energy, then so does $Uv$ with $U$ in $SU(2)$.  

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With our conventions the component of the scalar doublet we have taken to be a nonzero constant is the upper one with $t_3 = +1/2$. Three of the gauge bosons gain a mass: $W^1, W^2$ or $W^\pm = (W^1 \pm iW^2)/\sqrt{2}$ gain a mass squared of $M_W^2 = v^\dag v g_2^2/2$ and the linear combination

$$g_2 W^3_\mu - g_1 B_\mu \equiv g_2 (W^3_\mu - B_\mu \tan \theta_W) = \frac{g_2}{\cos \theta_W} (W^3_\mu \cos \theta_W - B_\mu \sin \theta_W)$$

$$\equiv \frac{g_2}{\cos \theta_W} Z_\mu$$

(22.77)

gains a mass squared $M_Z^2 = M_W^2/\cos^2 \theta_W$. The orthogonal linear combination

$$A_\mu = B_\mu \cos \theta_W + W^3_\mu \sin \theta_W, \quad \tan \theta_W = \frac{g_1}{g_2}$$

(22.78)

remains massless and can be identified with the photon. (The orthogonality is dictated by the desire to keep the quadratic derivative terms of the gauge fields diagonal.) We can now confirm the formula for the charge by eliminating $W^3$ and $B$ in favor of $A$, $Z$ in the gauge couplings:

$$g_2 W^3 T^3 + g_1 \frac{Y}{2}$$

$$= g_2 T^3 (Z \cos \theta_W + A \sin \theta_W) + g_1 \frac{Y}{2} (-Z \sin \theta_W + A \cos \theta_W)$$

$$= A \left( g_1 \frac{Y}{2} \cos \theta_W + g_2 T^3 \sin \theta_W \right) + Z \left( g_2 T^3 \cos \theta_W - g_1 \frac{Y}{2} \sin \theta_W \right)$$

$$= eA \left( T^3 + \frac{Y}{2} \right) + Z g_2 \cos \theta_W \left( T^3 - \frac{Y}{2} \tan^2 \theta_W \right)$$

(22.79)

where we have identified the electric charge $e = g_2 \sin \theta_W$.

After the Higgs mechanism we have a massless photon and 8 massless gluons. The electroweak part of the gauge group has been spontaneously broken by the Higgs mechanism, with the QED gauge symmetry unbroken. In addition the color $SU(3)$ part of the gauge group is unbroken. The reason we don’t see massless gluons has to do with the confinement phenomenon, not the Higgs mechanism. According to the confinement hypothesis only color singlet bound states have finite energy. Quarks and gluons carrying nonsinglet color have infinitely long flux tubes attached which makes them infinitely massive.

### 22.6.3 Gauge-fixing with Higgs mechanism

Fixing the $\xi$ gauge for the $SU(3)$ gauge fields is done as usual. But the electroweak quadratic terms after the shift include mixing between the scalar and gauge fields Let’s collect the undesirable quadratic terms in the Lagrangian after the shift $\phi \to v + \hat{\phi}$.

$$\frac{1}{2} \left[ (\partial \cdot B)^2 + (\partial \cdot W^a)^2 \right] + \hat{\phi}^\dag T_a v \left( -ig_2 \partial \cdot W^a \right) + \left( \hat{\phi}^\dag v g_1 \partial \cdot B/2 \right)$$

$$+ ig_2 \partial \cdot W^a v^\dag T_a \hat{\phi} - ig_1 \partial \cdot B/2) v^\dag \hat{\phi}$$

(22.80)
't Hooft had the clever idea to choose gauge fixing terms
\[
-\frac{1}{2\xi} [\partial \cdot B + i\xi g_1 (\hat{\phi}^\dagger v - v^\dagger \hat{\phi})/2]^2
\]
\[
-\frac{1}{2\xi} [\partial \cdot W^a - ig_2\xi (\hat{\phi}^\dagger T_a v - v^\dagger T_a \hat{\phi})]^2
\] (22.81)

The added terms are engineered so that the cross terms cancel the scalar-gauge mixing terms! For calculations $\xi = 1$ has obvious advantages because then the gauge propagators are simply $-i\eta_{\mu\nu}/(k^2 + M^2)$. In that case what remains of the gauge fixing term gives mass to $\hat{\phi}_2$ equal to the $W$ boson mass, and gives mass to the imaginary part of $\hat{\phi}_1$ equal to the $Z$ boson mass. These masses squared are proportional to $\xi$ when $\xi \neq 1$.

Finally, to construct the FP ghost terms we calculate the change of the gauge fixing condition under an infinitesimal gauge transformation:
\[
\Delta B_\mu = \partial_\mu \eta, \quad \Delta W^a_\mu = D^{ab}_\mu \theta^b,
\]
\[
\Delta \hat{\phi} = ig_2 T_a (v + \hat{\phi}) \theta^a - \frac{ig_1}{2} (v + \hat{\phi}) \eta
\] (22.82)

The terms involving $v$ in the FP determinant, expressed in terms of FP ghost fields, lead to masses for the FP ghost fields which match the ones for the scalar fields discussed above. In particular when $\xi = 1$ there is a scalar and FP ghost field for $W^1, W^2, Z$ with matching masses. There is only a massless FP ghost field but no scalar for the em field $A$. For each of the gauge fields the FP ghosts cancel two polarizations, leaving 3 for each massive gauge field and only two for the photon field, as required by Lorentz invariance. The scalar field serves as the 0 helicity polarization needed for a massive vector particle.

### 22.6.4 Massive vector boson couplings

We have determined the couplings of the neutral $Z$ boson in the process of determining the photon couplings. Since charge assignments are easier to remember than hypercharge, it is useful to give it in two forms:
\[
g_2 \cos \theta_W \left( T_3 - \frac{Y}{2} \tan^2 \theta_W \right) = g_2 \cos \theta_W \left( T_3 (1 + \tan^2 \theta_W - Q \tan^2 \theta_W) \right)
\]
\[
= \frac{g_2}{\cos \theta_W} \left( T_3 - Q \sin^2 \theta_W \right)
\] (22.83)

It is noteworthy that the relative couplings depend on $\theta_W$, unlike the electric charge.

The massive $W$ couplings are $g_2(T_1 W^1 + T_2 W^2)$, but it is useful to express the $W$ field in terms of charge eigenstates $W = (W^1 + i W^2)/\sqrt{2}$ or $W^1 = (W + W^\dagger)/\sqrt{2}$ and $W^2 = (W - W^\dagger)/(i \sqrt{2})$. Then
\[
g_2(T_1 W^1 + T_2 W^2) = \frac{g_2}{\sqrt{2}} (WT_- + W^\dagger T_+)
\] (22.84)
where $T_\pm = T_1 \pm iT_2$ are the standard raising and lowering operators for $SU(2)$. For instance on the electron doublet
\[
T_+ \left( \nu_e \right)_L = \frac{1 - \gamma_5}{2} \begin{pmatrix} e \\ 0 \end{pmatrix}, \quad T_- \left( \nu_e \right)_L = \frac{1 - \gamma_5}{2} \begin{pmatrix} 0 \\ \nu_e \end{pmatrix}
\] (22.85)
so we can write out the $W$ couplings to a lepton doublet
\[
\frac{g_2}{2\sqrt{2}} \left[ \bar{\nu}_l W^+ \cdot \gamma(1 - \gamma_5)l + \bar{l} W \cdot \gamma(1 - \gamma_5)\nu_l \right]
\] (22.86)
In the early days (1930’s - 1970’s) of interpreting the weak interactions, the energies were so low e.g. tenths of a GeV, that the momentum dependence of the vector boson propagator was negligible: it could be replaced by $-i\eta_{\mu\nu}/M_W^2$. The physics could be completely understood by ignoring the $W$ boson and replacing the $W$ coupling with an effective four fermion interaction
\[
G_F \sqrt{2} \sum l \left( \bar{\nu}_l \gamma_\mu(1 - \gamma_5)l \right) \sum \nu \left( \bar{\nu}_\nu \gamma_\mu(1 - \gamma_5)\nu_l \right) = \frac{G_F}{\sqrt{2}} J_\mu J^\mu, \quad G_F \sqrt{2} = \frac{g_2^2}{8M_W^2}
\]
This precise form was determined experimentally through painstaking study of muon and nuclear beta decay. In the standard model the coupling to quarks was analogous:
\[
\frac{g_2}{2\sqrt{2}} \left[ \bar{u}_l W^+ \cdot \gamma(1 - \gamma_5)d + \bar{d}_l W \cdot \gamma(1 - \gamma_5)u_l \right]
\] (22.87)
which is valid for massless quarks. As we shall see once fermions are given mass there will be a flavor mixing matrix.

### 22.6.5 Fermion masses

As we have seen, fermion (Dirac) mass terms are forbidden in the standard model because of the chiral coupling patterns. But with the Higgs scalar having a vacuum expectation value, one can form gauge invariant Yukawa coupling terms which act like mass terms for constant VEV’s. To do this for all the necessary mass terms we shall use not only $\phi = (\phi_0, \phi_-)$, which has $Y = -1$, but also its charge conjugate $\phi_c = -i\sigma_2\phi^* = (-\phi^*, \phi^0)$, which has $Y = +1$. (To see that $\phi_c$ transforms the same way under $SU(2)$ as $\phi$, note that $(e^{-i\sigma \xi})^* = \sigma_2 e^{-i\sigma \xi} \sigma_2$)

Let the doublets of the standard model be labeled $E^l$ for the lepton doublets and $Q^l$ for the quark doublets. Then the most general Yukawa interaction terms can be written
\[
G^e_{ij} \bar{E}^i e_R j \phi_c + G^{\nu}_{ij} \bar{E}^i \nu_R j \phi + G^d_{ij} \bar{Q}^i d_R j \phi_c + G^u_{ij} \bar{Q}^i u_R j \phi + h.c.
\] (22.88)
It can be shown (Exercise) that any complex matrix can be brought to diagonal form with nonnegative eigenvalues by a pair of unitary matrices: $G^a = U^a \sum_b G^{bIAG} V^a$. Now suppose $\langle \phi \rangle = (v, 0)$ so $\langle \phi_c \rangle = (0, v)$ with $v$ real. Then define $\phi = \langle \phi \rangle + \delta \phi$. Then the terms involving $v$ become
\[
v G^e_{ij} \bar{e}_{Li} e_R j + v G^\nu_{ij} \bar{\nu}_{Li} \nu_R j + v G^d_{ij} \bar{d}_{Li} d_R j + v G^u_{ij} \bar{u}_{Li} u_R j \phi + h.c.
\] (22.89)
Now diagonalize each of the four coupling matrices \( G^a = U^a_D^a V \), and absorb \( U \) and \( V \) into a redefinition of the corresponding fermion field \( f_R^a \rightarrow V^a f_R^a \), \( f_L^a \rightarrow U^a f_L^a \). This redefinition of fermion fields leaves the kinetic part of the Lagrangian invariant. However the interaction terms are not all invariant under this field redefinition:

1) The \( SU(3) \) gauge couplings are family blind. \( \sum_i \bar{q}_i^{L,R} \gamma \cdot A q_i^{L,R} \) is invariant.

2) The photon couplings are invariant because each pair \( U, V \) is associated with a distinct charge and chirality.

3) The part of the \( Z \) couplings proportional to charge \( Q \) is invariant for the same reason the photon couplings are. The \( T_3 \) couplings are invariant because the \( L \) fermions are all in doublets. Before charm was discovered, there was a doublet partner for only one of the charge \(-1/3\) quarks. Then the other left handed quark would need to be an \( SU(2) \) singlet. Then diagonalizing the \( Q = -1/3 \) mass matrix led to the left handed doublet partner of \( u_L \) being a linear combination of the mass eigenfields \( d_L' = d_L \cos \theta + s_L \sin \theta \) so the \( T_3 \) couplings could change strangeness if \( \theta \neq 0 \). Strangeness changing charge currents were well established by the late 1960’s, and their strength gave an experimental determination of \( \theta \), the Cabibbo angle. The suppression of strangeness changing neutral currents led Glashow, Iliopoulis, and Maiani (GIM) to propose the existence of the charm quark. Then, with all left handed quarks in doublets, the standard model \( T_3 \) couplings don’t mix any flavors because \( U^a U^{a\dagger} = I \).

4) The charged \( W \) couples fermions of different charge, for which the mass diagonalization matrix is different. Thus

\[
\bar{u}_{Li} \gamma \cdot W d_{Lj} \rightarrow \bar{u}_{Li} (U^u U^{dT})_{ij} \gamma \cdot W d_{Lj} \\
\bar{v}_{Li} \gamma \cdot W e_{Lj} \rightarrow \bar{v}_{Li} (U^e U^{e\dagger})_{ij} \gamma \cdot W e_{Lj} 
\]  

(22.90)

[If the neutrinos were massless one could choose \( U^e \) to cancel \( U^{e\dagger} \), with no flavor mixing.] The matrix \( U_{CKM} = U^T_u U^T_d \), for Cabibbo-Kobayashi-Maskawa, allows the charged currents to change quark flavor. The corresponding matrix in the lepton sector requires massive neutrinos, for which there is now indirect evidence.

The possibility of giving a Majorana mass to \( \nu^i_R \) has been mentioned. This is allowed because the right handed neutrinos are gauge singlets under the completes standard model gauge group. If this is combined with the Dirac mass term coming from the Yukawa interactions one has the following neutrino mass terms

\[
\bar{E}^i G^j_{ij} \nu_{Rj} \phi + \frac{1}{2} \nu^T_R M_{ij} (i \gamma_2 \beta) \nu^j_R + h.c. 
\]

(22.91)

where \( M \) is a complex symmetric matrix, because of Fermi statistics and the fact that \( \beta \gamma_2 \) is an antisymmetric matrix. If all eigenvalues of \( M \) are large we can ignore the kinetic terms of \( \nu_R \) amnd integrate out the \( \nu_R \) fields,

\[
\nu^T_R = - \bar{E}^k \delta i \beta \gamma_2 (G^r M^{-1})_{kj} 
\]

(22.92)
generating the terms

$$\frac{1}{2} E_L^{T_k} \phi^T [G^{\nu*} M^* - 1 G^{\nu*} T]_{kj} i \gamma_2 \beta \phi \dagger E_L^j + h.c.$$  \hspace{1cm} (22.93)

Putting $\phi = (v, 0)$ and real leads to the neutrino mass matrix $v^2 G^{\nu*} M^* - 1 G^{\nu*} T$. This mass matrix is is a symmetric but complex matrix. It can be shown (Exercise) that such a matrix can be brought to diagonal form $U^T M U = M_{\text{Diagonal}}$ with a unitary matrix $U$. Note that this is not a similarity transform if $U$ is complex. This “seesaw” structure of neutrino masses being inversely proportional to a very heavy mass, was proposed by Minkowski, and independently by Gell-Mann, Ramond, and Slansky, to account for very tiny neutrino masses, even if the neutrino Yukawa couplings are similar to the charged lepton Yukawas.
Chapter 23

Systematics of Renormalization

23.1 Renormalized perturbation theory

In this chapter we try to explain the general procedures to follow in establishing the success of the renormalization program. Rather than attempting a completely general approach we will restrict attention to the important example of QED. We have in fact carried out renormalization through one loop explicitly so now we want to sketch how the program works at higher orders.

Recall that in QED the superficially divergent 1PIR graphs are the photon self energy (vacuum polarization) $\Pi$ the electron self energy $\Sigma$ and the electron photon vertex. Let's first introduce the notion of a skeleton graph for a process that is not one of these, in which corrections to vertices and propagators are shrunk to a point. Then one gets the complete graph to a given order by inserting the complete propagators and vertices (to the desired order) rather than the lowest order ones.

Suppose we know:

a) $D' = Z_3(\text{uv Finite}) + \text{gauge terms}$

b) $S' = Z_2(\text{uv finite})$

c) $\Gamma = Z_1^{-1}(\text{uv finite})$

d) a), b), c) have no worse large momentum behavior than their zeroth order versions, modulo logarithms. (Weinberg’s Theorem).

Now each vertex in the skeleton has a factor of $e_0$, so the $Z$ factors can be absorbed by renormalizing the charge as $e_R = e_0 Z_2 \sqrt{Z_3} Z_1^{-1}$. If the original graph is primitively convergent then this procedure renders all skeleton subgraphs convergent and then d) allows us to conclude that the overall graph is convergent.

But what about a), b), and c)? We use induction on the number of loops. Define an electron-positron irreducible $e^+e^-$ kernel. Then $\Gamma$ satisfies an inhomogeneous integral
equation with $K$ as the kernel of the structure

$$\Gamma^\mu = \gamma^\mu + \int \Gamma^\mu S'S'K$$  \hspace{1cm} (23.1)$$

Now the ingredients of the second term are all of lower order than the order calculated. Then the rescaling of renormalization applied to these ingredients replaces bare charges with renormalized ones and the renormalized ingredients are all finite, with a net factor of $Z_1^{-1}$ which is accurate to one less order than calculated. With the high momentum assumption a single subtraction of the second term makes the last integral convergent times $Z_1^{-1}$ to lower order. This establishes c):

$$\int \Gamma^\mu S'S'K = \left( \frac{1}{Z_1} - 1 \right) \gamma^\mu + \Delta \int \Gamma^\mu S'S'K$$

$$\Gamma^\mu = \frac{1}{Z_1} \left( \gamma^\mu + \Delta \int \tilde{\Gamma}^\mu \tilde{S}' \tilde{S}' \tilde{K} \right)$$  \hspace{1cm} (23.2)$$

In these equations $\Delta$ denotes the difference of the expression minus its value at the on shell $q = 0$ point, and the tildes indicate the renormalized quantities. The $1/Z_1$ multiplying this term is only needed to one lower order than calculated. That multiplying $\gamma$ is accurate to the order calculated.

For QED one can define $\partial \Sigma/\partial p^\mu = \Gamma$ by the Ward identity: $S', D', \Gamma$ ok to order $\alpha^n$ implies $\Gamma$ ok to order $\alpha^{n-1}$ which by WI implies $S'$ ok to order $\alpha^n$. The WI allows us to bypass the problem of overlapping divergences that we must face in attacking $\Pi$.

For the vacuum polarization, gauge invariance reduces the superficial degree of divergence from $D = 2$ to $D = 0$:

$$D' = -i \left[ \frac{\xi q^\mu q^\nu}{q^4} + Z_3 \frac{\eta^{\mu\nu} - q^\mu q^\nu/q^2}{q^2(1 + e_0^2 Z_3(\Pi(q^2) - \Pi(\mu^2)))} \right]$$  \hspace{1cm} (23.3)$$

So one must prove $[\Pi(q^2) - \Pi(\mu^2)]$ is finite. The problem is overlapping divergences. A symptom of the overlap problem is that the only skeleton diagram is the one loop vacuum polarization.

23.1.1 Two loop example

We illustrate the difficulty by examining the two loop diagrams correcting lowest order vacuum polarization.

$$\int d^4p d^4k \frac{1}{k^2} \text{Tr} \gamma^\mu \frac{1}{m + \gamma \cdot p} \gamma^\lambda \frac{1}{m + \gamma \cdot (p - k)} \gamma^\nu \frac{1}{m + \gamma \cdot (p - k - q)} \gamma^\lambda \frac{1}{m + \gamma \cdot (p - q)}$$

The graph displays only one possible vertex correction. But there are actually two disjoint vertex divergences: one with $k \to \infty$ with $p$ fixed and a second with $k \to \infty$ with $p - k$
fixed. Since they are disjoint regions of integration they both contribute and at this order are additive. Thus even though the graph displays only one vertex correction, their are actually divergences available for both! As one goes to higher order it gets more and more challenging to disentangle the overlapping divergences, but it turns out to work in the end.

Another approach is to fashion an analogue of the Ward identity which handles the problem for the electron self energy. One doesn’t have a Ward identity but one can construct Feynman rules for a new object

\[ q \cdot \frac{\partial}{\partial q} \Pi^{\mu\nu}(q) \]  

which has the character of a forward vertex function like \( \Gamma(p, p) \). Then overlapping divergences are absent. Unlike with the Ward identity, there is not a natural way to choose the routing of the external momentum \( q \) through the diagram. Nonetheless, one can construct a proof that it is rendered finite after renormalization and then recovers \( \Pi^{\mu\nu} \) by integration.

### 23.2 Renormalization group of Gell-Mann and Low

Once one accepts the facts of renormalization it turns out that one can get a surprising amount of information just by consistency of the scheme. It is simplest to apply this notion in a renormalizable theory with no mass scale, e.g. QED with \( m_e = 0 \). But it could equally be QCD with all quark masses set to zero, or indeed pure nonabelian gauge theory. Also the neglect of masses is only a technical simplification, which can be removed with a slightly more complicated procedure: Equivalently one is studying any renormalizable theory at momenta much larger than all masses.

Focussing first on QED (QCD will be the subject of the next chapter), consider the vacuum polarization function \( \Pi(q^2) \). Since it is dimensionless the naive expectation is that it must be independent of \( q \), the only dimensionful scale available. But UV divergences invalidate this conclusion. Since \( m_e = 0 \) it makes no sense to renormalize at \( q = 0 \). Instead, we pick an arbitrary mass \( \mu \) and make the necessary subtraction at \( q^2 = \mu^2 \). We have

\[ D'_{F\mu\nu} = \frac{-i\eta_{\mu\nu}}{q^2(1 + e_0^2\Pi(q^2))} \]  

and define the renormalized charge as \( e^2(\mu) = e_0^2/(1 + e_0^2\Pi(\mu^2)) \). Then we shall examine the momentum dependent quantity

\[ d(q^2, e^2(\mu), \mu^2) = \frac{e^2(\mu)}{1 + e^2(\mu)(\Pi(q^2) - \Pi(\mu^2))} = \frac{e_0^2}{1 + e_0^2\Pi(q^2)} \]  

The far right side makes no reference to the scale \( \mu \) Thus \( d \) is independent of \( \mu \) which is simply a reflection of the success of renormalization: changing \( \mu \) is just a change in the value of \( q \) at which \( e \) is defined. Therefore \( e \) must depend on \( \mu \) according to

\[ \left[ \mu \frac{\partial}{\partial \mu} + \mu \frac{de^2}{d\mu} \frac{\partial}{\partial e^2} \right] d(q^2, e^2(\mu), \mu) = 0 \]
Now define the Gell-Mann-Low function

\[ \psi(e^2, \mu) \equiv \mu \frac{d e^2}{d \mu} = \psi(e^2) \]  

(23.8)

by dimensional analysis\(^1\). But \(d\) is itself dimensionless so it must be a function only of \(q/\mu\) and \(e\). So we write \(d(q^2/\mu^2, e^2)\) and note by definition that \(d(1, e^2) \equiv e^2\). The Gell-Mann-Low renormalization group equation can now be written

\[ \left( -q \frac{\partial}{\partial q} + \psi(e^2) \frac{\partial}{\partial e^2} \right) d(q, e^2) = 0. \]  

(23.9)

or, defining \(t\) by \(dt = de^2/\psi(e^2)\) with \(e^2(0) = e^2\), the equation reads\(^2\)

\[ \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial \ln q} \right) d = 0 \]  

(23.10)

whose general solution is \(d = f(t + \ln(q/\mu))\). \(f\) can be determined by setting \(q = \mu\) where \(d(\mu^2, e^2(t)) = e^2(t)\), so \(d(q^2/\mu^2, e^2(t)) = e^2(t + \ln(q/\mu))\). We can now set \(t = 0\) to get

\[ d(q^2/\mu^2, e^2) = e^2(\ln q/\mu), \quad \ln \frac{q}{\mu} = \int_{e^2}^{\psi(e^2)} \frac{dx}{x^2(1 + cx^2)} \]  

(23.11)

We notice, from the scaling equation, that we can calculate \(\psi(e^2)\) by calculating \(d(q^2)\) in perturbation theory, taking its derivative with respect to \(\ln q\) and setting \(q = \mu\), which sets \(\psi(e^2) \partial d/\partial e^2 = \psi(e^2)\). Recalling our lowest order result for vacuum polarization

\[ e^2(\Pi(q^2) - \Pi(\mu^2)) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1 - x) \ln \frac{m^2 + x(1 - x)\mu^2}{m^2 + x(1 - x)q^2} \to \frac{e^2}{12\pi^2} \ln \frac{\mu^2}{q^2} \]

\[ d(q^2, e^2) = e^2 \left( 1 - \frac{e^2}{12\pi^2} \ln \frac{\mu^2}{q^2} \right) + O(e^6) \]  

(23.12)

from which

\[ \psi(e^2) = \frac{e^4}{6\pi^2} + O(e^6) \]  

(23.13)

It is important to appreciate that solving the RG equations with \(\psi(e^2)\) truncated at some finite order gives some (but of course not all) information about higher order contributions to the quantity calculated. This is evident from our QED example. Look at the exact solution for \(e^2(\ln q/\mu)\) with \(\psi(x) = bx^2 + cx^3\):

\[ b \ln \frac{q}{\mu} = \int_{e^2}^{\psi(e^2)} \frac{dx}{x^2(1 + cx^2)} \approx \frac{1}{e^2} - \frac{1}{e^2(\ln q/\mu)} - c \ln \frac{e^2}{e^2} \]  

(23.14)

\[ e^2(\ln q/\mu) = e^2 \left[ \frac{1}{1 - be^2 \ln q/\mu} + O \left( e^4 \ln \frac{q}{\mu} \right) \right] \]  

(23.15)

\(^1\)Standard dimensional analysis is allowed now because the renormalization process has removed UV divergences.

\(^2\)Note that \(\psi(e^2) = 0\) identically, would imply that \(d\) is independent of \(q\), the consequence of naive scale invariance.
The right side has contributions from all orders in perturbation theory, the expansion parameter being \( e^2 \ln q/\mu \). We sometimes call this a leading log approximation. It can be trusted if \( 1 \ll \ln q/\mu \lesssim 1/e^2 \), because it neglects terms of order \((e^2)^n(\ln q/\mu)^{n-1}\).

Another way to see this is to examine how the equation is solved in perturbation theory. We try a double expansion in \( \alpha = e^2/(4\pi) \) and \( t \equiv \ln q/\mu \):

\[
d = \alpha + \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} c_{mn} \alpha^n t^m
\]

\[
\frac{dd}{dt} = \sum_{m,n} mc_{mn} \alpha^n t^{m-1}
\]

\[
= (b_1 \alpha^2 + b_2 \alpha^3 + \cdots) \left[ 1 + \sum_{1 \leq m < n} n c_{mn} \alpha^{n-1} t^m \right]
\]

We shall return to this in the next chapter.

### 23.2.1 Categories of large momentum behavior

Coming back to the exact equation, we ask what kind of behavior can we expect as \( \ln q/\mu \to \infty \). This requires that the integral \( \int dx/\psi(x) \) diverge. There are several possibilities:

a) If \( e^2 \to \infty \), i.e. the effective charge blows up at short distances, then \( \psi(x) \leq x \) as \( x \to \infty \).

b) Another way is for \( e^2 \) to approach a zero of \( \psi \) as \( \ln q/\mu \to \infty \). Suppose there is a linear zero:

\[
\psi(x) \sim (x - x_0) \psi'(x_0)
\]

Then

\[
\ln q/\mu \approx \int_{e^2}^{x_0-\epsilon} \frac{dx}{\psi(x)} + \frac{1}{\psi'(x_0)} \ln \frac{x_0 - e^2(q)}{\epsilon}
\]

\[
e^2(q) \approx x_0 - Ae^{\psi'(x_0) \ln q/\mu}
\]

The approach to \( x_0 \) as \( q \to \infty \) requires that \( \psi'(x_0) < 0 \). In that case we say that \( x_0 \) is an UV stable fixed point. A theory with this property would “predict” the value of the coupling. If \( \psi'(x_0) > 0 \) it would be called an IR stable fixed point.

c) What about \( e^2 \to 0 \)? The integral certainly diverges there. Suppose \( \psi(x) \to bx^2 \). Then

\[
\ln \frac{q}{\mu} \sim \frac{1}{be^2} - \frac{1}{be^2(q)}
\]

\[
e^2(q) \sim \frac{1}{1/e^2 - b \ln q/\mu}
\]

For the approach to take place in the UV, must have \( b < 0 \)!

This is the famous case of asymptotic freedom, a feature of nonabelian gauge theory! We say that QED has a UV unstable fixed point at the origin. Massless QED has an IR stable fixed point at the origin,
Chapter 24

Renormalization and Short Distance Properties of QCD

In this chapter we focus on QCD neglecting quark masses, so the classical theory is scale invariant. As we have seen this scale invariance is broken by UV divergences and dimensional analysis has to be supplemented by the renormalization group.

24.1 Scaling properties of Green functions in massless gauge theory.

In our discussion of QED we have applied RG ideas to an actual physical quantity: $e_0^2$ times the photon propagator. The presence of $e_0^2$ made the quantity the actual measured charge at a given value of $q$. Had we considered the propagator itself, renormalization would mean that $D'/Z_3$ is what must be finite.

In nonabelian gauge theories it is not so easy to find a gauge invariant physical quantity. The simplest might be correlation functions of singlet operators like $\sum_a F_{a2}^2$. Instead we consider a general (non gauge invariant) correlation function. Then the statement of renormalization is

$$G_{\text{bare}}(q_1, \ldots, q_n; p_1, \ldots, p_m; g_0, \xi) = Z_G^{n/2} Z_Q^{m/2} G_r(q, p; g_\mu, \xi_\mu)$$

(24.1)

Then the Callan-Symanzik RG equation is

$$\left[ \mu \frac{\partial}{\partial \mu} + \mu \frac{d g_\mu}{d \mu} \frac{\partial}{\partial g} + \mu \frac{d \xi_\mu}{d \mu} \frac{\partial}{\partial \xi} + \frac{n}{2} \mu \frac{d \ln Z_G}{d \mu} + \frac{m}{2} \mu \frac{d \ln Z_Q}{d \mu} \right] G_r = 0.$$  

(24.2)

Renormalization has removed all divergences, so including the scale $\mu$ we may now apply ordinary dimensional analysis to define:

$$\mu \frac{d g_\mu}{d \mu} = \beta(g_\mu, \xi_\mu), \quad \mu \frac{d \xi_\mu}{d \mu} = \delta(g_\mu, \xi_\mu)$$  

(24.3)

$$\frac{1}{2} \mu \frac{d \ln Z_G}{d \mu} = -\gamma_G(g_\mu, \xi_\mu), \quad \frac{1}{2} \mu \frac{d \ln Z_Q}{d \mu} = -\gamma_q(g_\mu, \xi_\mu)$$  

(24.4)
Now we can regard the parameters $\mu, g, \xi$ in $G_r$ as independent provided $G_r$ is subject to the constraint

$$
\left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \delta \frac{\partial}{\partial \xi} - n\gamma_G - m\gamma_q \right] G^{n,m}_r = 0.
$$

(24.5)

One of the Ward identities says that the longitudinal component of the gluon propagator is not corrected by interactions:

$$
q^\mu D'_\mu = q^\mu D_{\mu\nu}^{\text{bare}} = -\frac{i}{q^2} \xi q^\nu.
$$

(24.6)

Inserting this in the CS equation for $G^{2,0}_r$ yields the relation

$$
\delta(g, \xi) - \frac{i q^\nu}{q^2} \frac{-i q^\nu}{q^2} = 2 \gamma_G(g, \xi) \frac{-i q^\nu}{q^2}
$$

(24.7)

giving $\delta = 2 \xi \gamma_G$. This is a situation where Landau gauge ($\xi = 0$) simplifies life since then $\delta = 0$ and the $\xi$ term is absent from the CS equation.

### 24.2 Calculation of $\beta, \gamma_g, \gamma_q$ to one loop

The coefficient functions can be extracted from the RG equation by expanding, say $G^{2,0}_r, G^{0,2}_r, G^{3,0}_r$ in perturbation theory. In doing so keep in mind that they will differ from one calculation scheme to another depending on what one takes as a measure of $g$. Notice the following:

1) Coupling renormalization occurs only at order $g^3$ and higher, because any bare vertex is independent of $\mu$.

2) Thus $\delta, \gamma_G, \gamma_Q$, which start at order $g^2$ can be obtained in lowest order from $G^{2,0}_r$ and $G^{0,2}_r$. From calculations in $\xi$ gauge we obtain for $SU(N)$

$$
\gamma_G = \frac{g^2}{16\pi^2} \left[ \left( \frac{13}{6} - \frac{\xi}{2} \right) N_c - \frac{4N_f}{6} \right]
$$

(24.8)

$$
\gamma_q = -\xi \frac{N g^2}{16\pi^2}
$$

(24.9)

(The $\delta \partial / \partial \xi$ term is of order $g^4$)

To compute $\beta$ we examine $G^{1,2}$ the quark gluon vertex, obtaining

$$
\beta = -\frac{N g^3}{16\pi^2} \left( \frac{11}{3} - \frac{2N_f}{3N_c} \right) + O(g^5)
$$

(24.10)

[Our earlier calculation was in Landau gauge ($\xi = 0$). Although for general $\xi$ the $\gamma$’s depend on $\xi$, that dependence cancels the $\xi$ dependence of the vertex correction in the determination of $\beta$.]
24.2.1 Non universality of $\beta, \alpha$

Suppose we consider an alternate renormalization scheme. We assume that the two prescriptions agree at lowest order. Then

\begin{align*}
g'(g) &= g + a_1 g^3 + a_2 g^5 + \cdots \quad (24.11) \\
\beta(g) &= -b g^3 - c g^5 + d g^7 + \cdots \quad (24.12)
\end{align*}

Then we calculates

\begin{align*}
\beta'(g') &\equiv \mu \frac{dg'}{d\mu} = \beta(g) \left[ 1 + 3a_1 g^2 + 5a_2 g^4 + \cdots \right] \\
&= -b g^3 - c g^5 - (3a_1 g^5 + O(g^7)) = -b(g^3 + 3a_1 g^5) - c g^5 + O(g^7) \\
&= -b(g + a_1 g^3)^3 - c g^5 + O(g^7) = -b g^3 - c g^5 + O(g^7) \quad (24.13)
\end{align*}

So the first two terms of $\beta$ are the same in any scheme. As an exercise you can show that the terms beyond the first two are different.

24.2.2 The Nature of the Callan-Symanzik Equation

Let us examine the CS equation for an observable $G(p, g, \mu)$ with $\gamma = 0$ that depends on only one momentum. Its perturbation expansion would have the form

\begin{equation}
G = \sum_{n=0}^{\infty} g^{2n} \sum_{m=0}^{n} a_m^n \left( \ln \frac{p}{\mu} \right)^m \quad (24.14)
\end{equation}

Plugging this expansion in the CS equation and equating to zero the net coefficient of each distinct power of $g$ and $\ln(p/\mu)$ relates all of the $a_m^n$ with $m \neq 0$ to $a_0^n$. For example suppose $\beta(g) = -b g^3$ exactly. Then one finds derives a recursion relation $ma_m^n = -2(n-1) ba_{m-1}^{n-1}$ with solution

\begin{equation}
a_m^n = (-2b)^m \binom{n-1}{m} a_0^{n-m} \quad (24.15)
\end{equation}
Inserting this formula in the double expansion then gives

\[ G = \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} g^{2n} a_0^{n-m} \binom{n-1}{m} \left(-2b \ln \frac{p}{\mu}\right)^m \]

\[ G = \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} g^{2n} a_0^{n-m} \binom{n-1}{m} \left(-2b \ln \frac{p}{\mu}\right)^m \]

\[ G = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} g^{2(k+m)} a_0^k \binom{k+m-1}{m} \left(-2b \ln \frac{p}{\mu}\right)^m \]

\[ G = \sum_{k=1}^{\infty} a_0^k \left(1 + 2bg^2 \ln(p/\mu)\right)^k \equiv \sum_{k=1}^{\infty} a_0^k (g^2(p^2))^k \]

where we have defined the running coupling \( g^2(p^2) = g^2/[1 + 2bg^2 \ln(p/\mu)] \). So in this simplified example the renormalization group sums the powers of \( \ln \)'s by replacing the expansion parameter \( g^2 \) with the running coupling \( g^2(p) \). Here \( b > 0 \) corresponds to asymptotic freedom.

### 24.3 High momentum behavior

Now let us return to the RG equation for a general (renormalized) QCD Green’s function (assuming masses are negligible) \( G^{n,m}(q_i, p_i, g, \xi, \mu) \) with \( n \) gluon legs and \( m \) quark legs:

\[ \left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + 2\xi \gamma_G \frac{\partial}{\partial \xi} - n\gamma_G - m\gamma_q \right) G^{n,m} = 0 \]

For dimensional reasons \( \beta \) and \( \gamma \) can only depend on \( g \) and \( \xi \). \( G^{n,m} \) is of course the F.T. of a correlation function of the gluon and quark fields. Its physical mass dimension is (in \( D \) spacetime dimensions)

\[ d_G = n \frac{D-2}{2} + m \frac{D-1}{2} - D(n+m) + D \]

because the gluon field \( A \) has dimension \( (D-2)/2 \), the quark field has dimension \( (D-1)/2 \). The F.T. of each field adds a \( d^Dx \) and we remove the momentum conserving delta function from the definition of \( G \). Since we have removed all infinities by renormalization, at the expense of introducing the fake parameter \( \mu \) we can apply standard dimensional analysis to scale out the overall \( \mu \) dependence. First of all introduce an overall momentum scale \( Q \) by writing \( q_i = Q\tilde{q}_i, p_i = Q\tilde{p}_i \). Then we can write

\[ G^{n,m} = \mu^{d_G} \tilde{G}^{n,m}(\tilde{q}_i, \tilde{p}_i; g, \xi, Q/\mu). \]

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So one can replace derivatives w.r.t. $\mu$ by derivatives w.r.t. $Q$:

$$\left(-Q \frac{\partial}{\partial Q} + d_G + \beta \frac{\partial}{\partial g} + 2\xi \gamma_G \frac{\partial}{\partial \xi} - n\gamma_G - m\gamma_q\right) \tilde{G}^{n,m} = 0$$

(24.20)

Next define a new variable $t$ by the differential equations;

$$\frac{dg}{dt} = \beta(g(t), \xi(t)), \quad g(0) = g$$

$$\frac{d\xi}{dt} = 2\xi(t)\gamma_G(g(t), \xi(t)), \quad \xi(0) = \xi.$$  

(24.21)

Then the RG equation becomes

$$\left(-Q \frac{\partial}{\partial Q} + \frac{\partial}{\partial t} + d_G - n\gamma_G - m\gamma_q\right) \tilde{G}^{n,m} = 0$$

(24.22)

which has the general solution

$$\tilde{G}^{n,m} = F(t + \ln(Q/\mu)) \exp \left\{ \int_0^t dt' [-d_G + n\gamma_G + m\gamma_q] \right\}$$

(24.23)

The function $F$ is of a single variable and can be evaluated by setting $t = 0$: $F(\ln(Q/\mu) = \tilde{G}^{n,m}(Q/\mu, g, \xi)$, so

$$\tilde{G}^{n,m}(Q/\mu, g(t), \xi(t)) = \tilde{G}^{n,m}(Qe^t/\mu, g, \xi) \exp \left\{ \int_0^t dt' [-d_G + n\gamma_G + m\gamma_q] \right\}$$

Finally set $\mu = Q$ to obtain

$$\tilde{G}^{n,m}(e^t, g, \xi) = \tilde{G}^{n,m}(1, g(t), \xi(t)) \exp \left\{ \int_0^t dt' [d_G - n\gamma_G - m\gamma_q] \right\}$$

(24.24)

from which we get

$$G^{n,m}(Q/\mu, g, \xi) = \mu^{d_G} \tilde{G}^{n,m}(1, g(\ln Q/\mu), \xi(\ln Q/\mu)) \times \exp \left\{ \int_0^{\ln Q/\mu} dt' [d_G - n\gamma_G(t') - m\gamma_q(t')] \right\}$$

$$= Q^{d_G} \tilde{G}^{n,m}(1, g(\ln Q/\mu), \xi(\ln Q/\mu)) \times \exp \left\{ - \int_0^{\ln Q/\mu} dt' [n\gamma_G(t') + m\gamma_q(t')] \right\}$$

(24.25)

This formula shows how to calculate the large $Q$ behavior of a Green’s function with knowledge of the large $t$ behaviors of $g(t), \xi(t)$, knowledge of $\tilde{G}(1, g, \xi)$ and $\gamma_G(g, \xi), \gamma_q(g, \xi)$. The only approximations made in this formula are the neglect of quark masses. With asymptotic
freedom \((b > 0)\) we can use perturbation theory to analyze the large \(Q\) behavior of Green’s functions, because then \(g(\ln Q) \to 0\) as \(Q \to \infty\).

To see how this works in a simplified context, suppose that \(g = g(0) \ll 1\). Then \(g(t)\) for \(t > 0\) is even smaller. If perturbation theory is valid at \(t = 0\) it is even more valid as \(t\) increases. In this situation we should be able to just keep the first few terms in the RG functions. Let’s truncate at two terms for \(\beta\):

\[
\beta(g) = -bg^3 - cg^5 + O(g^7) \quad (24.26)
\]

Since the first two terms are universal we expect \(b, c\) to be independent of \(\xi\). Then we can attempt to solve the equation for \(g\) with this truncated \(\beta\):

\[
dt = -\frac{dg}{bg^3 + cg^5} = -\frac{dg}{g^3(b + cg)^2}
\]

\[
t = \frac{1}{2b} \left[ \frac{1}{g^2(t)} - \frac{1}{g^2(0)} \right] - \frac{c}{2b^2} \ln \frac{g^2(t) + c}{g^2(0) + c}
\]

\[
\frac{1}{g^2(t)} = 2bt + \frac{1}{g^2(0)} + \frac{c}{b} \ln \frac{g^2(t)}{g^2(0)} + \frac{c}{b} \ln \frac{b + cg^2(t)}{b + cg^2(0)} \quad (24.27)
\]

To leading order we have

\[
g^2(\ln(Q/\mu)) \sim \frac{1}{2b \ln(Q/\mu) + 1/g^2} \equiv \frac{1}{2b \ln(Q/\Lambda)} \quad (24.28)
\]

where \(\Lambda = \mu^2 e^{1/(2bg^2)}\) is an RG invariant characteristic scale of QCD. That scale replaces the coupling \(g\) as a parameter. This is sometimes known as dimensional transmutation. Putting in the QCD values for \(b\) and \(c\) we find

\[
\frac{Ng^2}{4\pi^2} \to \frac{6}{(11 - 2n_f/N)} \ln Q/\Lambda_{QCD} + O\left(\frac{\ln \ln Q}{(\ln Q)^2}\right), \quad Q \to \infty \quad (24.29)
\]

To complete the evaluation of \(G^{m,m}\) at large \(Q\) we can use the lowest order results for the anomalous dimensions.

\[
\gamma_G \approx dg^2, \quad \gamma_q \approx f\xi g^2 \quad (24.30)
\]

to evaluate

\[
\int_0^{\ln Q/\mu} dt' g^2(t') = \int dgg' \frac{g^2}{\beta(g')} \approx -\frac{1}{b} \int \frac{dg'}{g'} \approx -\frac{1}{2b} \ln \frac{g^2(\ln(Q/\mu))}{g^2}
\]

Then in Landau gauge we can write

\[
G^{m,m}(Q/\mu, g, 0) = Q^{dG} \left[ \frac{g^2(\ln Q/\mu)}{g^2} \right]^{nd/(2b)} \tilde{G}^{m,m}(1, g(\ln Q/\mu), 0) \quad (24.31)
\]
24.4 Composite Operators

The physics of the color $SU(3)$ part of the standard model, the strong interactions is generally not tractable analytically, because our tools are limited to perturbation theory. We have learned how to calculate renormalized correlation functions of quark and gluon fields, and to use the renormalization group plus asymptotic freedom to learn something about their high momentum scaling behavior. We can not directly translate these results to experiments because we can’t form beams of quarks or beams of gluons. We can only form beams of protons, electrons, and indirectly beams of unstable particles like the muon or neutron.

To describe hadron scattering, using the reduction formalism, instead of quark and gluon operators, we must use composite operators that connect the one hadron state to the vacuum. For example the proton is thought to be a composite containing two up quarks and one down quark. This suggests that the composite field operator $\psi_{\text{proton}} + \epsilon_{\alpha \beta \gamma} u^\alpha(x) u^\beta(x) d^\gamma(x)$ would be appropriate for the reduction formalism. We would then assume that

$$\langle 0 | \psi_{\text{proton}}(0) | \text{proton} \rangle \sqrt{(2\pi)^3 2 E_{\text{proton}}} \equiv U_\lambda(p) \sqrt{Z_{\text{proton}}} \neq 0 \quad (24.32)$$

and use the reduction formalism to define the proton proton $S$-matrix. Notice that we take all fields to be at the same space-time point. This is not only for simplicity, but also to maintain gauge invariance. The requirement that the three quark field connects the proton state to the vacuum does not in any way imply that the proton has only three quarks: its state vector could also contain quark anti quark pairs and gluons. One could use a 6 quark operator to define scattering of deuterons, say. By using different flavor combinations one could describe other baryons, e.g. the $\Omega^{-}$ would couple to $\epsilon_{\alpha \beta \gamma} s^\alpha s^\beta s^\gamma$.

To include mesons in the scattering process, we could use operators like $\bar{q} f_1(x) \Gamma q f_2(x)$. For instance

$$\langle 0 | \bar{d}(0) \gamma_5 u(0) | \pi^+ \rangle \neq 0 \quad (24.33)$$

would enable the inclusion of the $\pi^+$ meson in the scattering process. In order to include high spin mesons in the scattering we could employ derivatives of operators such as $\bar{q} D_{\mu_1} \cdots D_{\mu_n} q$ in the reduction formalism. In this choice we use covariant derivatives so that the operator retains its gauge invariance. Each operator we use requires its own wave function renormalization factor $Z_\Omega$ which, in the renormalization group equation means its own anomalous dimension $\gamma_\Omega(g)$. These factors for actual hadronic on shell states would require finding say the proton pole in the F.T. of $\langle 0 | T \Omega(x) \Omega^\dagger(0) | 0 \rangle$, a task beyond our tools in perturbation theory. But as far as handling the UV divergences is concerned one can choose to renormalization at any scale $\mu$.

Among all of the operators we might choose to describe mesons, conserved currents are special. One advantage is that the normalization of the operator is fixed by the conserved charge of the state examined. Thus there can be no wave function renormalization factor, i.e. the anomalous dimension is zero.

It is particularly fruitful to study hadronic physics through their interaction with leptons. Consider for instance scattering of electrons by protons, which involves hadronic matrix
elements of the electromagnetic current operator. Suppose, for instance we are interested in
the electromagnetic properties of the proton. So we are interested in the matrix elements
of the quark parts of the current operator $j^\mu = e_0 \sum_f Q_f \bar{q} f \gamma^\mu q_f$. The quark model suggests
that the proton is a bound system of two $u$ quarks and one $d$ quark. But we can’t calculate
the wave functions since the quarks interact strongly. But we can nonetheless use symmetry
principles to narrow our ignorance. Lorentz invariance and current conservation tell us that
\begin{equation}
\langle p' | j^\mu(0) | p \rangle = \bar{u}_\lambda \left[ \gamma^\mu F_1(q^2) + \frac{[\gamma^\mu, \gamma \cdot q]}{4m_p} F_2(q^2) \right] u_\lambda
\end{equation}
Here $q$ is the momentum transfer $q = p' - p$. So the two “form factors” $F_1, F_2$ summarize our
ignorance. We can’t compute these form factors, but we can measure them in electron and
photon scattering, because the current matrix element is just the proton vertex for (virtual)
photon emission. Such measurements show that the form factor falls exponentially with $q$
for a while, but at very high $q$ becomes power behaved. We can interpret this matrix element
as the F.T. of the M.E. of the current operator:
\begin{equation}
\int d^4 x e^{ix \cdot q} \langle p' | j^\mu(x) | p \rangle = (2\pi)^4 \delta(p' - p - q) \langle p' | j^\mu(0) | p \rangle
\end{equation}
In the limit $q \to 0$ the left side for $\mu = 0$ is proportional to the matrix element of the total
charge which is just 1 in units of $e$. From which we learn that $F_1(0) = 1$. Similar consideration
relate $F_2(0) = (g_p - 2)/2$ to the magnetic moment of the proton $\mu_p = g_p e/(2m_p)$. For many
important EW processes $q$ is so small that it is accurate to simply replace the form factors
by their values at $q = 0$.

One might hope to use perturbation theory at very high $q$ since the QCD coupling is weak
then. However, according to the RG, strict perturbation theory is only valid if all momenta
are large and space-like, which is not true of the proton momenta which are on the proton
mass shell. If we think of the proton as 3 quarks in a bag (e. g. MIT bag model) a high
momentum virtual photon will strike one of the quarks hard giving it large momentum. The
only way the proton can recoil is for this momentum to be shared by all three quarks which
requires at least two gluon exchanges. In this way one can make plausible assertions of the
power behavior of the form factors at high $q$, but one gets no information on the coefficient
of that behavior.

More hopeful is the process called deep inelastic electroproduction, in which a high energy
electron is scattered with large angle off a proton, and one doesn’t observe the final hadrons:
in this sense it is a total cross section related by the optical theorem to the imaginary part
of the F.T. of the two current matrix element: $\langle p | T j^\mu(x) j^{\mu}(0) | p \rangle$. An even simpler example
is the process $e^+ e^- \to$ hadrons.

### 24.4.1 $e^+ e^- \to$ Hadrons

We can relate the total cross section of this process via the optical theorem to the F.T. of
$\langle 0 | T j_{EM}^\mu(x) j_{EM}^{\mu}(0) | 0 \rangle$ While we can’t calculate this VEV in perturbation theory at all $q$, we
can describe its large $q$ behavior in perturbation theory.
In this relatively simple case let’s consider how this works in practice. To be clear we will work only to lowest order in QED perturbation theory, but (in principle) to all orders in the strong interactions. Before looking at the asymptotics at large $q$, let’s recall that gauge invariance tells us that

$$i \int d^4xe^{iqx}\langle 0|T\bar{j}_EM^\mu(x)j_{EM}^\nu(0)|0\rangle = (q^\mu q^\nu - q^2 \eta^{\mu\nu})e_0^2 \Pi(q^2) + O(e_0^4)$$  \hspace{1cm} (24.36)

We factor out the $e_0^2$ from $\Pi$ and consider $\Pi$ to zeroth order in $e_0$. It therefore starts with the quark loop with no gluon exchange, and in QCD perturbation theory one adds diagrams with more gluons and quark loops.

**Renormalization Group Analysis**

$\Pi(q^2)$ requires a single subtraction to make it finite. Its normalization is fixed by the charge of the quarks (relative to the electron charge). For that reason there is no multiplicative renormalization, i.e. the anomalous dimension is zero. Thus we can write

$$\Pi(q^2) = \hat{\Pi}(q^2) + \Pi(q^2)|_{q^2=\mu^2} \; ; \; \hat{\Pi}(\mu^2) = 0$$  \hspace{1cm} (24.37)

where $\hat{\Pi}$ is finite when expressed in terms of $g(\mu)$. Since $\Pi(q^2)$ is independent of $\mu$ we have the RG equation

$$\mu \frac{d\Pi}{d\mu} = 0 = \mu \frac{\partial\hat{\Pi}}{\partial\mu} + \beta(g)\frac{\partial\hat{\Pi}}{\partial g} + \mu \frac{\partial\Pi(Q^2)}{\partial Q}\bigg|_{Q=\mu}$$  \hspace{1cm} (24.38)

where we wrote $q = Q\hat{q}$ where $\hat{q}^2 = 1$. By dimensional analysis we can write

$$\mu \frac{\partial\Pi(Q^2)}{\partial Q}\bigg|_{Q=\mu} = \epsilon(g)$$  \hspace{1cm} (24.39)

$$\mu \frac{\partial\hat{\Pi}}{\partial\mu} + \beta(g)\frac{\partial\hat{\Pi}}{\partial g} = -\epsilon(g)$$  \hspace{1cm} (24.40)

To compute $\epsilon$ we use the fact that $\hat{\Pi}(\mu^2) = 0$, so that $\epsilon = -\mu \partial\hat{\Pi}(q^2)/\partial\mu$ evaluated at $q^2 = \mu^2$.

By dimensional analysis we can replace $\partial/\partial\mu$ by $-\partial/\partial Q$. Introducing $t$ by $\beta(g(t)) = dg(t)/dt$ as before, we have

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \ln Q}\right)\left(\hat{\Pi}(Q,t) + \int_0^t dt'\epsilon(g(t'))\right) = 0$$

$$\hat{\Pi}(Q,t) + \int_0^t dt'\epsilon(g(t')) = F(t + \ln Q)$$  \hspace{1cm} (24.41)

Setting $t = 0$, giving $F(\ln Q) = \hat{\Pi}(Q/\mu, g)$ and then restoring it gives $F(t + \ln Q) = \hat{\Pi}(Qe^t/\mu, g)$. Then setting $Q = \mu$ and remembering $\hat{\Pi}(\mu, g(t)) = 0$ gives $\hat{\Pi}(e^t, g) = \int_0^t dt'\epsilon(g(t'))$ or

$$\hat{\Pi}(Q/\mu, g) = \int_0^{\ln Q/\mu} dt'\epsilon(g(t')).$$  \hspace{1cm} (24.42)

Asymptotic freedom then allows us to use perturbation theory to calculate $Q \to \infty$. 

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Perturbation Theory to 2 Loops

To get the first two terms in \( \epsilon \) we need to calculate \( \Pi \) to two loops. At one loop we simply multiply ordinary vacuum polarization by \( N \sum_i Q_i^2 \) to get

\[
\tilde{\Pi}_1 = -\frac{N}{12\pi^2} \sum_i Q_i^2 \ln \frac{q^2}{\mu^2} \tag{24.43}
\]

The two loop diagrams are just the corresponding QED versions (which we can read off from Jost and Luttinger) times \( \frac{N^2-1}{2} \sum_i Q_i^2 \), which is \( N \) times the sum of squared charges times the Casimir operator in the fundamental representation:

\[
\tilde{\Pi}_2 = -\frac{N^2-1}{2} \frac{g^2}{64\pi^4} \sum_i Q_i^2 \ln \frac{q^2}{\mu^2} \tag{24.44}
\]

Setting \( N = 3 \) for QCD this gives

\[
\Pi = \frac{1}{4\pi^2} \sum_i Q_i^2 \left( 1 + \frac{g^2}{4\pi^2} \right) \ln \frac{\mu^2}{q^2} + O(g^4) \tag{24.45}
\]

\[
\epsilon = -\mu \frac{\partial}{\partial \mu} \tilde{\Pi} = -\frac{1}{2\pi^2} \sum_i Q_i^2 \left( 1 + \frac{g^2}{4\pi^2} \right) + O(g^4) \tag{24.46}
\]

Therefore

\[
\tilde{\Pi} \left( \frac{Q_i \mu, g}{\mu} \right) = -\frac{1}{2\pi^2} \sum_i Q_i^2 \int_0^{\ln Q/\mu} dt' \left[ 1 + \frac{g^2(t')}{4\pi^2} + O(g^4) \right] \tag{24.47}
\]

Assuming \( \mu, Q \) are large enough so that \( g(t') \) is small over the whole integration range we can use the one loop running coupling

\[
\frac{g^2(t')}{4\pi^2} = \frac{2}{(11 - 2N_f/3)t' + 8\pi^2/g^2} \tag{24.48}
\]

\[
\int_0^t dt' \frac{g^2(t')}{4\pi^2} = \frac{2}{11 - 2N_f/3} \ln \frac{(11 - 2N_f/3)t + 8\pi^2/g^2}{8\pi^2/g^2} \tag{24.49}
\]

So finally

\[
\tilde{\Pi} \left( \frac{Q, \mu, g}{\mu} \right) \approx -\frac{1}{2\pi^2} \sum_i Q_i^2 \left[ \frac{1}{2} \ln \frac{Q^2}{\mu^2} + \frac{2}{11 - 2N_f/3} \ln \frac{g^2}{g^2(Q)} \right] \tag{24.50}
\]

valid for \( q^2 \to \infty \). This conclusion is solid for \( q \) large and spacelike. To apply calculations like this to an actual experiment, we must assume that analytically continuing the above
formula also gives the correct asymptotic behavior for large negative $Q^2$. By the optical theorem, the total cross section for $e^+e^- \to \text{hadrons}$ is proportional to the Imaginary part of $\tilde{\Pi}$ which is non zero with the above approximate formula for negative (timelike) $Q^2$. Putting $Q^2 \to |Q^2| e^{-i\pi}$ gives

$$\text{Im } \tilde{\Pi} = \frac{1}{2\pi^2} \sum_i Q_i^2 \left[ \frac{\pi}{2} - \frac{2}{11 - 2N_f/3} \text{Im } \ln \frac{g^2}{g^2(Q)} \right]$$

(24.51)

To evaluate the imaginary part of the second term we note that $\ln(A + iB) = \ln \sqrt{A^2 + B^2} + i \arctan(B/A)$, so its imaginary part is $\arctan(B/A)$.

$$\frac{g^2}{g^2(Q)} = \frac{(11 - 2N_f/3)(\ln |Q^2|/\mu^2 - i\pi) + 16\pi^2/g^2}{16\pi^2/g^2}$$

(24.52)

for which

$$\frac{B}{A} = -\frac{\pi(11 - 2N_f/3)}{(11 - 2N_f/3) \ln |Q^2|/\mu^2 + 16\pi^2/g^2}$$

$$\text{Im } \tilde{\Pi} \approx \frac{1}{4\pi} \sum_i Q_i^2 \left[ 1 + \frac{4}{(11 - 2N_f/3) \ln |Q^2|/\mu^2 + 16\pi^2/g^2} \right]$$

(24.53)

where we used $B \ll A$ to replace $\arctan(B/A)$ with $B/A$. For a quick normalization we recognize that the corresponding quantity for $e^+e^- \to \mu^+\mu^-$ is $1/12\pi$, so the ratio

$$R = \frac{\sigma_{\text{hadrons}}}{\sigma_{\mu^+\mu^-}} \approx 3 \sum_i Q_i^2 \left[ 1 + \frac{4}{(11 - 2N_f/3) \ln |Q^2|/\Lambda_{QCD}^2} \right]$$

(24.54)

$$\approx 3 \sum_i Q_i^2 \left[ 1 + \frac{g^2(|Q^2|)}{4\pi^2} \right]$$

(24.55)

where $\Lambda_{QCD} = \mu^2 e^{-16\pi^2/g^2/(11 - 2N_f/3)}$ is a fundamental scale of QCD. It is an RG invariant—meaning its value doesn’t depend on $\mu$! The first term in brackets is the parton model result and the second RG improved QCD corrections.

If we look back and keep the $N$ dependence, we find

$$R = N \sum_i Q_i^2 \left[ 1 + \frac{3}{8} \left( 1 - \frac{1}{N^2} \right) \frac{Ng^2(|Q^2|)}{4\pi^2} \right]$$

(24.56)

A couple of points to note. The rough agreement of $R$ with experiment requires $N = 3$ and so represents a measurement of the number of colors. A useful analysis tool of QCD is to study it in the large $N$ limit as suggested by ‘t Hooft: $N \to \infty$ with $Ng^2$ fixed. For $N = 3$, $R/N$ differs from the $N \to \infty$ limit by about 10%.

Another point is that the formula does not refer to quark masses at all: we have assumed $Q$ much larger than $\Lambda_{QCD}$ as well as all masses. In practice this is not really true. There is a
wide hierarchy of quark masses; $m_u,m_d$ are both much smaller than $\Lambda_{QCD}$, while $m_c,m_b,m_t$ are much larger. Perturbative QCD depends on $Q \gg \Lambda_{QCD}$, and so should be relevant even for $Q < m_c$. If we only include $u,d,s$, then $3 \sum Q_i^2 = 2$ is the zeroth order prediction for $R$. But adding $c$ increases this to $10/3$. So the prediction for $R$ depends on what thresholds have been exceeded. In effect at a given $Q$ one treats some masses as 0 but others as infinite.

Our calculation of the prediction for $R$ breaks into two steps. (1) Use asymptotic freedom to calculate $\tilde{\Pi}$ at large positive $Q^2$ (so $q$ is spacelike). (2) Continue the asymptotic result to $Q^2 < 0$ and calculate the imaginary part to get $R$. In retrospect it is clear that if one simply calculates the total cross section for $e^+e^- \rightarrow q\bar{q} + \text{gluons}$ with 0 or 1 additional gluon in the final state in perturbation theory, and then substitutes the running coupling $g^2(Q)$ for $\alpha_s$, one would obtain the same answer.

24.4.2 Interpreting naive perturbation theory in QCD

On the one hand we have a more or less solid way to connect the physics of asymptotic freedom with observed processes: first get asymptotics at large spacelike momenta and then continue the result to time like momenta which is where the physics happens. On the other hand this procedure produces the result more physically in terms of the production of quarks and gluons, in spite of the fact that it is hadrons not quarks and gluons that are actually produced. By focussing on inclusive processes (e.g. total cross sections) we avoid confronting quarks and gluons. This is the rule: don’t look to closely at the final state.

But there must be some truth to the quark and gluon picture of the final state. This leads to the idea that hadrons will be produced in “jets” that reflect their origin as a produced pair of quarks. Thus we say that a pair of high energy quarks are really produced but over time the quarks find antiquarks (or vice versa) and fragment into a shower of hadrons that remembers the parent quark’s momentum. If no gluons are produced with the quarks the hadrons would distribute into two back to back jets. If a hard gluon is produced along with the quarks, the final state would mimic three jets. Each produced quark or gluon with sufficient energy will fragment into a jet of hadrons. We summarize a typical sequence of events:

1. $q\bar{q}$ produced at a point with large momenta, so $\alpha_s(Q) \ll 1$
2. As $q\bar{q}$ separate $\alpha_s$ increases.
3. At $R \gtrsim 1/\Lambda_{QCD}$ perturbation theory ceases validity.
4. Each quark (or gluon) $\rightarrow$ a jet of hadrons
5. Jet momentum remembers momentum of parent quark or gluon.

Just as in QED, there are IR divergences in the emission of soft or collinear gluons individual contributions, but they cancel, leaving $O(\alpha_s)$ corrections. Non collinear high momentum (hard) gluons get interpreted as gluon jets. The dictionary is that jet scattering cross sections on average are the same as the cross sections of the parent quarks or gluons.
24.5 Operator product expansion (OPE)

Correlators of fundamental operators are made finite by wave function renormalization: \( \Gamma_{\text{bare}} = Z^n/2 \Gamma_{\text{Ren}} \). But a more complicated operator may introduce new infinities that require additional renormalization. For example inserting an operator \( \phi^2 \) is equivalent to the zero separation limit of two elementary \( \phi \)'s.

Our discussion of \( e^+e^- \rightarrow \text{hadrons} \) is an example of a more general technique, the operator product expansion (OPE). We related the desired cross section to the imaginary part of a two current correlation function, and then by analyzing \( Q \rightarrow \infty \) with use of the renormalization group obtained a prediction for the cross section at high energy. In coordinate space \( Q \rightarrow \infty \) corresponds to the separation of the current coordinates going to zero. K. Wilson proposed that in a general context one could make a short distance expansion of the product of two local operators of the form

\[
\Omega_1(x)\Omega_2(0) \sim \sum_n C_n(x)\Omega_n(0), \quad x \rightarrow 0
\]  

(24.57)

Here one imagines that \( n \) labels all the possible local operators. We expect the limit \( x \rightarrow 0 \) to be divergent, and the proposal asserts that all divergences are captured by the c-number coefficients \( C_n(x) \). Operators of higher dimension will have less singular coefficients. In our simple example we only used the VEV of two currents, rather than the actual operators, and since the expectation was in the vacuum, the \( Q \rightarrow \infty \) limit only involved the identity operator in the OPE. Wilson’s proposal is much more powerful because the same coefficient functions will appear no matter what matrix element is evaluated. Although physically plausible, the operator product expansion in this most powerful form is not completely proven.

Deep Inelastic Electroproduction

As we have discussed the simplest application of asymptotic freedom is \( e^+e^- \rightarrow \text{hadrons} \). To appreciate the utility of OPE, we consider briefly the inclusive process \( e^-p \rightarrow e^-X \) where one sums over all possible states \( X \). This can be regarded as virtual Compton scattering. Let \( p \) be the proton momentum and \( q \) the momentum of the virtual photon. The momentum of the system of hadrons \( X \) is then \( q + p \). Then kinematically \( m_X > m_p \) which means \(- (p + q)^2 > m_p^2 \) or \(- 2p \cdot p > q^2 \). In deep inelastic scattering \( q^2, -2p \cdot q \) are taken large at fixed ratio \( x = q^2 / (-2p \cdot q) \). The kinematic constraint translates to \( 0 < x < 1 \).

By the optical theorem the cross section is proportional to the imaginary part of the two current correlator

\[
\int d^4xe^{iq \cdot x}\langle p|\bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(0)\gamma^\nu \psi(0)|p\rangle
\]

(24.58)

Since we are interested in large \( q \), it is appropriate to use the OPE for the two currents.

\[
\int d^4xe^{iq \cdot x}\bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(0)\gamma^\nu \psi(0) \sim \sum_s C_s^{\mu_1\cdots\mu_s}(q)\bar{\psi}(0)\gamma^{\mu_1}D^{\mu_2}\cdots D^{\mu_s}\psi(0)
\]

To keep the discussion simple we have contracted the indices of the currents. The operators on the right have dimension \( 3 + s - 1 = s + 2 \) and anomalous dimension \( \gamma_s(g) \). Whereas
the left side has dimension $6 - 4 = 2$ and no anomalous dimension. It follows that the
coefficient function $C_s$ has dimension $-s$ and anomalous dimension $-\gamma_s(g)$. Here we have
included in the expansion operators of spin $s$ constructed from fundamental Dirac fields. For
simplicity we have not included other possible local operators. The OPE has factorized the
$q$ dependence from the state dependence. By Lorentz covariance we can write

$$C_{\mu_1 \cdots \mu_s}(q) = \frac{q^{\mu_1} \cdots q^{\mu_s}}{(q^2)^s} C_s,$$

$$\langle p | \bar{\psi}(0) \gamma^\mu D^{\mu_2} \cdots D^{\mu_s} \psi | p \rangle = D^{\mu_1} \cdots p^{\mu_s}$$

And the matrix element will have the structure

$$\sum \left( \frac{q \cdot p}{q^2} \right)^s A_s(q^2/\mu^2, g^2) \quad (24.59)$$

Then the terms are multiplied by $(q \cdot p/q^2)^s$ which is held fixed in the deep inelastic scattering
experiment. It is fashionable to define the twist of an operator as its dimension minus its
spin, so all of the operators we listed have twist 2. In the deep inelastic scaling limit they all
contribute, but higher twist operators get suppressed as $q^2 \to \infty$. There are then an infinite
number of operators that will contribute. For kinematic reasons $\frac{-2q \cdot p}{q^2} > 1$.

A few terms will dominate only in the unphysical domain $-2p \cdot q \ll q^2$, which can
be reached by analytic continuation via dispersion relations. The analysis is lengthy and
elaborate, so we defer more details to next semester.
Chapter 25

Spontaneous Global Symmetry
Breaking: Chiral Dynamics

25.1 Effective Action and Potential

The concept of effective action is a very useful tool to discuss symmetry breaking in quantum field theory. We start by constructing it in scalar quantum field theory, but it will become clear that its scope is much more general than that. So let's imagine a Lagrangian for any number of real scalar fields of the form

\[ \mathcal{L} = -\frac{1}{2} \sum_a (\partial \phi_a)^2 - V(\phi) \]  

(25.1)

where we put no limitations on \( V(\phi) \) for the moment. We first define a generating function for correlators of these fields:

\[ Z(J) = \int \mathcal{D}\phi e^{i \int d^4x (L + J_a(x)\phi_a(x))} \equiv e^{iW(J)}. \]  

(25.2)

Correlators of any number of \( \phi \)'s are obtained by taking that number of functional derivatives of \( Z \) w.r.t. the \( J \)'s. These correlators include disconnected diagrams as well as connected ones. Functional derivatives of \( W(J) \) give connected correlators.

Next we define an effective field \( \varphi \) by the formula

\[ \varphi_a(x) \equiv \frac{1}{iZ} \frac{\delta Z}{\delta J_a(x)} = \frac{\delta W}{\delta J_a(x)} \]  

(25.3)

. By this definition it is clear that \( \varphi \) has the interpretation of the expectation of the scalar field \( \phi_a \) in the presence of the source \( J \). In the limit \( J \to 0 \) this is simply the VEV of \( \phi_a \). The final step in the construction of the effective action is to perform a Legendre transformation, defining the effective action

\[ \Gamma(\varphi) \equiv W(J(\varphi)) - \int d^4x \sum_a \varphi_a J_a \]  

(25.4)
The point of the Legendre transform is to remove the implicit dependence of $\Gamma$ on $\varphi$ through $J$ so that

$$\frac{\delta \Gamma}{\delta \varphi_a(x)} = -J_a + \frac{\delta J_b}{\delta \varphi_a} \left[ \frac{\delta W}{\delta J_b} - \varphi_b \right] = -J_a(x)$$

(25.5)

We call $\Gamma(\varphi)$ the effective quantum action, because when the source $J_a = 0$, $\Gamma$ is stationary under variations of the effective field. Solutions of this action principle give possible values for the vacuum expectation value of the quantum field $\varphi_a = \langle 0 | \phi_a | 0 \rangle$.

Although we have gone through this construction using the scalar field theory as a crutch, it should be clear that every step of the construction goes through if the source is coupled to a composite local field, for example the $\bar{q}(I \pm \gamma_5)q_j$ in QCD. Then the interpretation of $\phi_{ij}^\pm$ would be $\langle 0 | \bar{q}(I \pm \gamma_5)q_j | 0 \rangle$.

For the case of scalar field theory, the effective action $\Gamma(\varphi_a)$ is the generating functional for the 1PIR Feynman diagrams. To see this, we first note the pair of relations

$$i \langle \phi_a(x)\phi_b(y) \rangle_c = \frac{\delta^2 W}{\delta J_a(x)\delta J_b(y)} = \frac{\delta \varphi_b(y)}{\delta J_a(x)} \tag{25.6}$$

$$\frac{\delta^2 \Gamma}{\delta \varphi_a(x)\delta \varphi_b(y)} = -\frac{\delta J_b(y)}{\delta \varphi_a(x)} \tag{25.7}$$

which together show that

$$\int d^4z \sum_c \frac{\delta^2 \Gamma}{\delta \varphi_a(x)\delta \varphi_c(z)} i \langle \phi_c(z)\phi_b(y) \rangle_c = -\delta_{ab}\delta(x - y). \tag{25.8}$$

This means that $-i\delta^2 \Gamma/\delta \varphi_a(x)\delta \varphi_b(y)$ is the inverse connected two point function, which is what we mean by the 1PIR two point function. Calculating higher functional derivatives of $\Gamma$, one finds that they are 1PIR. Carrying on this analysis to higher order shows that the connected diagrams are interpreted as trees, with the 1PIR diagrams as vertices and $\langle \phi_a(x)\phi_b(y) \rangle = -i\delta^2 W/\delta J_a(x)\delta J_b(y)$ as the propagator.

In this regard it is helpful to regard perturbation theory as a semiclassical approximation. To see this, restore the $\hbar$ dependence in the path integral, by multiplying the action by $1/\hbar$, so the limit $\hbar \to 0$ is a stationary phase approximation. On the other hand one can then rescale the fields by $\phi \to \sqrt{\hbar}\phi$, to make the quadratic derivative terms in the Lagrangian, which determine the perturbation propagator, independent of $\hbar$. After this rescaling, $V(\phi) \to V(\sqrt{\hbar}\phi)$, so an expansion in powers of $\hbar$ is also an expansion in the cubic and higher terms in the Lagrangian, a term $\phi^n \to \hbar^{(n-2)/2}$. Linear terms in $\varphi$ are removed by the stationary phase condition applied at leading order. This point of view shows that the zeroth order approximation to the effective action is just the classical action:

$$\frac{\delta}{\delta J_a(x)} \left( \int d^4x \mathcal{L}(\phi) + \int d^4x J_a(x)\phi_a(x) \right) = \phi_a(x) \tag{25.9}$$

so in the classical limit $\varphi_a = \phi_a$ so that

$$\Gamma = W - \varphi_a J_a = \int d^4x \mathcal{L} \tag{25.10}$$
precisely the classical action!

25.1.1 Effective potential

For the purposes of studying the vacuum, which is assumed to be Poincarè invariant, we should find that \( \langle \phi_a \rangle = \varphi_a = \text{constant} \). In that case the effective action is proportional to the space time volume, and we define the effective potential \( V(\varphi) \) by \( \Gamma = -(VT)V(\varphi) \). The value the VEV actually assumes must be a stationary point of \( V \)

\[
\frac{dV}{d\varphi} = 0 \tag{25.11}
\]

for \( \varphi_a = v_a \) implies that \( \langle \phi_a \rangle = v_a \).

The effective potential is the first term in an expansion of the effective action about derivatives of \( \varphi \):

\[
\Gamma(\varphi) = \int d^4x \left[ -V(\varphi) - Z(\varphi)(\partial^2\varphi)^2 + \cdots \right] \tag{25.12}
\]

More generally, if \( J_a(x) \) is time independent the path integral at large times projects onto the ground state of the source dependent Hamiltonian

\[
H_J = H_0 - \int d^3x J_a(x)\phi_a(x) \tag{25.13}
\]

and \( W(J) \to -2TE_G(J) \). If there is more than 1 stationary point of \( V \), the theory will choose the one that is the absolute minimum. In the classical limit this simply says that we should do perturbation theory with \( \phi = v + \hat{\phi} \) with \( v \) the minimum of the classical potential.

A final general point. By construction the effective action and effective potential will possess the same symmetries under transformations of \( \varphi_a \) as the quantum theory has under the corresponding transformations of \( \phi_a \). Since \( \phi_a \) does not have to be one of the fundamental fields of the theory, identifying these symmetries is not completely straightforward. Because of this the effective potential is the perfect language for the discussion of spontaneous symmetry breaking. This is especially so in quantum field theories whose UV divergences can spoil symmetries.

It is important to appreciate that the dependence of \( \Gamma \) on \( \varphi \) is obtained dynamically by the introduction of the source. One is not always guaranteed the existence of such a source for all values of \( \varphi \). This is seen by inspecting the construction in the classical approximation for the quartic potential

\[
V(\phi) = \frac{\lambda}{4!}(\phi^2 - a^2)^2 \tag{25.14}
\]

Then \( J = dV/d\phi = (\lambda/6)\phi(\phi^2 - a^2) \). Inverting this equation to get \( \phi(J) \) is ambiguous because it is multi-valued. The dynamics from the source is given by \( W(J) \) which can be reconstructed from \( V \):

\[
-W(J) = V(\phi(J)) - \phi(J)V'(\phi(J)) \tag{25.15}
\]
When $J$ varies from $-\infty \rightarrow 0$, $\phi(J)$ varies from $-\infty \rightarrow -a$, and when $J$ varies from $0 \rightarrow +\infty$, $\phi$ varies from $a \rightarrow +\infty$. These two curves span the whole range of $J$ following the true minimum of $-W(J)$. One can of course continue the relation $J(\phi)$ until the point where $V'' = 0$ which follows a curve above the ground state curve (this is analogous to supercooling or superheating in thermodynamics). There is a third branch connecting the inflection points on which the unstable $\phi = 0$ point lies.

### 25.1.2 Feynman diagrams for the effective potential

When the order parameter is the expectation of a fundamental scalar field, s systematic procedure to calculate the effective potential in perturbation theory is the expansion in number of loops. To set this up we first notice that

$$e^{i\Gamma[\phi]} = \int D\phi e^{i\int d^4x (L[\phi] + J(\phi - \phi))} = \int D\phi e^{i\int d^4x (L[\phi + \phi] + J \phi)}$$

(25.16)

Then we can expand $L[\phi + \phi]$ in a power series in $\phi$ with $\phi$ dependent coefficients and develop the logarithm of the path integral as a sum of connected 1PIR vacuum Feynman diagrams. The cancellation of the 1 particle reducible diagrams determines the choice of source $J$. But in practice one simply drops the source term together with all one particle reducible diagrams. For the effective potential, one choose $\phi =$constant:

$$L[\phi + \phi] + \phi J = -V(\phi) - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} V''(\phi) \phi^2 + \cdots$$

(25.17)

the linear term in $\phi$ is dropped because it would contribute a reducible diagram (or would be cancelled by choice of $J$). If we neglect the cubic and higher powers of $\phi$, The path integral is a Gaussian and leads to

$$-iVT V(\phi) = -iVT V(\phi) - \frac{1}{2} \text{Tr} \ln (-\partial^2 - V''(\phi)) + \cdots$$

(25.18)

To calculate the $\text{Tr} \ln$ we can go to momentum space with Wick rotation:

$$\text{Tr} \ln (-\partial^2 - V''(\phi)) = iVT \int \frac{d^4p}{(2\pi)^4} \ln(p^2 + m^2(\phi))$$

(25.19)

where we defined $m^2(\phi) \equiv V''(\phi)$. 

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\[ V(\phi) = \frac{1}{4!} (\phi^2 - a^2)^2 \]

\[ \mathcal{J} = \frac{dV}{d\varphi} = \frac{1}{6} \varphi \left( \varphi^2 - a^2 \right) \]

\[ -W(\mathcal{J}) = V(\varphi(\mathcal{J})) - \varphi(\mathcal{J})V'(\varphi(\mathcal{J})) \]
25.1.3 Symmetries of the effective action

Suppose the action and path integral measure are invariant under $\phi_a \rightarrow M_{ab} \phi_b$. We claim that this will imply that the effective quantum action is symmetric under $\phi_a \rightarrow M_{ab} \phi_b$. Carrying out such a transformation in the path integral with a source $J_a$ shows that

$$e^{iW(J)} = \int e^{iS+iJ \cdot \phi} = e^{iW(JM)}$$

so $W[J] = W[JM]$. Then

$$\Gamma[\phi] = W[J] - J_a \phi_a = W[JM] - (JM)_a (M^{-1} \phi)_a = \Gamma[M^{-1} \phi]$$

where we used the fact that $M^{-1} \phi$ is the effective field for $W[JM]$. To see this differentiate the relation $W[J] = W[JM]$:

$$\phi_a \equiv \frac{\delta W[J]}{\delta J_a} = M_{ab} \phi_b, \quad \phi_b^M = (M^{-1})_{ba} \phi_a.$$

In other words, at least for homogeneous linear symmetries of the fundamental action, the effective action has the same symmetries. In the case of constant $\phi_a$, this same symmetry extends to the effective potential.

25.1.4 Spontaneous Symmetry breaking

Symmetry plays a central role in our understanding of particle physics. In many cases the symmetry is only approximate because breaking terms in the Lagrangian happen to have small coefficients. But there are some symmetries that we take to be exact. These include spacetime symmetries such as Poincaré invariance. But the discrete spacetime symmetries like parity, time reversal and charge conjugation are broken in Nature. In gauge theories, gauge invariance is essential for self-consistency and must never be broken.

The apparent breaking of symmetry can be real, due to terms in the Quantum effective action that don’t respect it, or spontaneous, meaning that the quantum action is invariant but the vacuum state is not. Technically this means that the minimum of the quantum effective potential $\phi_a \neq 0$. We have already encountered spontaneous symmetry breaking in the Higgs sector of the standard model, without fully recognizing it. When we assumed that the scalar field had a nonzero vacuum expectation value, we were tacitly assuming that spontaneous breaking was taking place. In this case the relevant symmetry was the $SU(2) \times U(1)$ gauge symmetry. So let’s begin our study with that example.

The standard model Higgs field was taken to transform as a doublet under the $SU(2)$ factor and to have weak hypercharge $Y = -1$. Therefore the components of the doublet are complex. If the vacuum were invariant under the symmetry, we would have $\langle 0 | \phi | 0 \rangle = \langle 0 | U^\dagger \phi U | 0 \rangle$ which would imply that $\langle 0 | \phi | 0 \rangle = 0$ since $U^\dagger \phi U$ has different components than $\phi$. Let us consider the renormalizable Lagrangian for the scalar field

$$\mathcal{L} = -\partial^\dagger \phi \cdot \partial \phi - \frac{\lambda}{4} (\phi^\dagger \phi - v^2)^2$$

(25.23)
The zeroth order effective potential is

$$V(\phi) = \frac{\lambda}{4}(\phi^\dagger \phi - v^2)^2$$

(25.24)

which clearly is minimized when $\phi^\dagger \phi = v^2$. For definiteness consider the solution

$$\phi = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

(25.25)

with $v$ real. We then proceed by shifting $\phi_1 = v + \phi_1$ and $\phi_2 = \phi_2$, so that

$$\phi^\dagger \phi - v^2 = v(\phi_1 + \phi_1^\dagger) + \phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2$$

(25.26)

It follows that

$$V(\phi) = \frac{\lambda}{4} \left[ v(\phi_1 + \phi_1^\dagger) + \phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 \right]^2$$

(25.27)

$$= v^2 \lambda (\text{Re} \phi_1)^2 + \text{cubic and quartic}$$

(25.28)

This shows that the real part of $\phi_1$ gains a mass, while the imaginary part of $\phi_1$ and both real and imaginary parts of $\phi_2$ remain massless. To identify the mass of $\text{Re} \phi_1$, we note that the derivative terms involving $\phi$ can be written

$$-\partial \phi^\dagger \cdot \partial \phi = -(\partial \text{Re} \phi_1)^2 - (\partial \text{Re} \phi_2)^2 - (\partial \text{Im} \phi_1)^2 - (\partial \text{Im} \phi_2)^2$$

(25.29)

From which it follows that the mass squared of $\text{Re} \phi_1$ is $m_2^2 = v^2 \lambda$. This is the Higgs particle, whose measured mass of $m_h \approx 125\text{GeV}$ gives information about the quartic coupling $\lambda$. The three massless scalars play the role of the zero helicity states of the massive vector bosons of the standard model.

The existence of massless scalars whenever a continuous symmetry is spontaneously broken is the content of the Goldstone theorem and they are called Goldstone bosons, or more properly Nambu-Goldstone bosons. The reasoning behind the theorem is simple using the concept of the effective action and potential. In the scalar field example we just discussed, there are four real scalar fields $\phi_a$, which we can assemble into a 4 component vector and the potential is a function only of its squared length, $\sum_{a=1}^{4} \phi_a^2$. If it has a minimum with $\phi_0 \neq 0$, then any $O(4)$ rotation of $\phi_0$ is also a minimum. An infinitesimal rotation of $\phi_0$ defines directions perpendicular to $\phi_0$ along which the potential doesn’t change. Scalar fields in those directions will not have a mass term.

The Goldstone theorem requires a broken continuous symmetry. For example a single real scalar field could have a symmetry under $\phi \rightarrow -\phi$ for instance the potential $(\lambda/4!)(\phi^2 - a^2)$ has two minima at $\phi = \pm a$. Since those minima are discrete there is no implication of massless particles. In terms of the fluctuation field $\phi = \phi - v$ the potential has a mass term, a cubic term, and a quartic term. the content of the discrete symmetry is a specific relation between the three coefficients of these terms.

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25.2 Goldstone Bosons and Soft Pion Theorems

25.2.1 Goldstone’s Theorem

Suppose there is a vacuum state with \( \langle 0 | \phi_a | 0 \rangle = v^a \neq 0 \). Then \( \delta \Gamma / \delta \phi_a = 0 \) for \( \phi_a = v^a = \) constant. Suppose there is a continuous symmetry of \( \Gamma \) under \( \phi \to \phi - i\epsilon G \phi \). Then, restricting to constant fields it follows that

\[
-i\epsilon G_{ab} \phi_b \frac{\partial V}{\partial \phi_a} = 0
\]

for all \( \phi \). Differentiating w.r.t. \( \phi \) gives

\[
G_{ac} \frac{\partial V}{\partial \phi_c} + G_{ab} \phi_b \frac{\partial^2 V}{\partial \phi_c \partial \phi_a} = 0
\]

again for all \( \phi \). Setting \( \phi = v \) makes the first term zero, so we conclude that

\[
G_{ab} v^b \frac{\partial^2 V}{\partial \phi_c \partial \phi_a} \bigg|_{\phi=v} = 0
\]

Thus \( G_{ab} v^b \) is a zero eigenvector of the second derivative matrix of the effective potential, which is intuitively the mass term. More precisely, in terms of the effective action

\[
\frac{\partial^2 V}{\partial \phi_c \partial \phi_a} = \lim_{q \to 0} \int d^4xe^{iq \cdot x} \frac{\delta^2 \Gamma}{\delta \phi_c(x) \delta \phi_a(0)}
\]

which is the \( q \to 0 \) limit of the F.T. of the inverse propagator. The existence of a zero eigenvector of the latter implies that \( \int d^4xe^{iq \cdot x} \langle \phi_a(x) \phi_b(0) \rangle \) has a pole at \( q^2 = 0 \). That is, there are massless particles (Nambu-Goldstone bosons) coupling to the field \( \phi_a \). This conclusion depends on \( G_{ab} v^b \neq 0 \), i.e. that \( v^a \) is not invariant under the symmetry.

The number of NGB’s depends on the symmetry and the nature of the symmetry breaking. For example if the symmetry is \( O(N) \), with \( \phi_a \) transforming as a vector, There will be \( N - 1 \) NGB’s. We can count these as follows: there are \( N(N - 1)/2 \) generators of \( O(N) \). Of these the vector \( v \) is invariant under \( (N - 1)(N - 2)/2 \) generators. The difference \( N - 1 \) is the number of generators such that \( Gv \neq 0 \). More generally we can say that if \( G \) is the symmetry group and \( H \) is the subgroup of \( G \) that is unbroken, then the NGB’s are in 1-1 correspondence with the generators in \( G/H \).

25.2.2 SSB and matrix elements of operators.

By definition any order parameter \( \phi_a \) changes under the symmetry, so the corresponding operator satisfies

\[
[G_a, \phi_b] = (T^a)_{bc} \phi_c
\]

1 The meaning of the coset \( G/H \) is that two elements \( g_1, g_2 \) of \( G \) are identified if \( g_1 = g_2h \) with \( h \in H \).
But in field theory symmetry generators are integrals of time components of conserved currents, so we expect the local version
\[
[j^\mu_0(x,0), \phi_b(0)] = (T^a)_{bc} \phi_c(0) \delta(x).
\] (25.35)

And current conservation leads to
\[
q_\mu \int d^4x e^{-iq \cdot x} \langle 0| T j^\mu_a(x) \phi_b(0)|0 \rangle = -i \int d^4x e^{-iq \cdot x} \delta(t) \langle 0| [j^\mu_a(x), \phi_b(0)]|0 \rangle = -i (T^a)_{bc} \langle 0| \phi_c(0)|0 \rangle
\] (25.36)

If there were no massless particles the left side would go to 0 as \(q \to 0\), implying that \(\langle 0| \phi_c(0)|0 \rangle = 0\). Conversely if \(\langle 0| \phi_c(0)|0 \rangle \neq 0\), the left side must have singularities as \(q \to 0\), and hence massless particles (GB’s).

Recall our discussion of the reduction formalism where we considered single particle contributions to two point functions:
\[
\int d^4x e^{-iq \cdot x} \langle 0| T j^\mu_a(x) \phi_b(0)|0 \rangle = \sum \lambda \langle 0| j^\mu_a(0)|\lambda \rangle \langle \lambda| \phi_b(0)|0 \rangle - i(2\pi)^3 q^0 2 q^0 + m^2_\lambda
\] (25.37)

On general grounds we can write
\[
\langle 0| j^\mu_a(0)|\lambda \rangle = \frac{i q^\mu F_{a\lambda}}{(2\pi)^3/2 \sqrt{2q^0}}, \quad \langle \lambda| \phi_b(0)|0 \rangle = \frac{\sqrt{Z_{b\lambda}}}{(2\pi)^3/2 \sqrt{2q^0}}
\] (25.38)

and the left side of the conservation identity (25.36) reads
\[
\sum_\lambda \frac{q^2 F_{a\lambda} \sqrt{Z_{b\lambda}}}{q^2 + m^2_\lambda} \to \sum_{m_\lambda=0} F_{a\lambda} \sqrt{Z_{b\lambda}} = -i (T^a)_{bc} \psi^c
\] (25.39)
as \(q \to 0\). We see that the low energy couplings of the massless NGB’s are linked to the order parameter. Note that consistency demands that the NGB’s couple both to the order parameter and to the current for the symmetry. So one could choose either field to use in the reduction formalism to derive scattering amplitudes for the NGB’s.

Matrix elements of the current \(\langle \beta| j^\mu_a(0)|\alpha \rangle\), where \(|\beta\rangle, |\alpha\rangle\) are general outgoing or incoming multiparticle states, have massless poles in \(q^2 = (p_\beta - p_\alpha)^2\) of the form
\[
\frac{q^\mu}{q^2} F_{aB} \langle \beta, B|\alpha \rangle
\] (25.40)

where \(B\) is the NGB. Here \(-i \langle \beta, B|\alpha \rangle \equiv M_{\beta\alpha}^B\) has the interpretation as the S-matrix for NGB emission during the core process \(\alpha \to \beta\). We can write the current matrix element
\[
\langle \beta| j^\mu_a(0)|\alpha \rangle = \frac{i F_{aB} q^\mu}{q^2} M_{\beta\alpha}^B + N_{\beta\alpha}^{a\mu}
\] (25.41)
where $N$ has no pion pole. Then the conservation of the current implies

$$iF_{aB}M_{\beta\alpha}^B + q_\mu N^{\mu\alpha} = 0.$$  \hspace{1cm} (25.42)

The generic consequence of this equation is that NGB emission vanishes at low energy $q \to 0$. However there are exceptions to this statement: when the current attaches to an external line in a scattering amplitude the propagator between the attachment point and the rest of the diagram has a singularity as $q \to 0$.

$$\frac{1}{(q+p)^2 + m^2} = \frac{1}{q^2 + 2q \cdot p} \sim \frac{1}{2p \cdot q}.$$  \hspace{1cm} (25.43)

**25.2.3 Approximate symmetries**

We shall be considering approximate symmetries in QCD. So we need to develop a treatment of spontaneous breaking of an approximate symmetry. The upshot will be that the NG bosons gain a small mass, and are then called pseudo NGB’s. We use the conceptual framework of the effective potential. Write the latter as

$$V(\varphi) = V_0(\varphi) + V_1(\varphi)$$  \hspace{1cm} (25.44)

where $V_0$ is symmetric and $V_1$ breaks the symmetry, and we assume breaking is small. We assume $V_0' = 0$ for some nonzero $\varphi_0$. Then we search for a minimum of $V$ at $\varphi = \varphi_0 + \varphi_1$, with $\varphi_1 \ll \varphi_0$. Working to first order in the symmetry breaking we get the condition

$$\left. \frac{\partial^2 V_0}{\partial \varphi_a \partial \varphi_b} \right|_{\varphi_0} \varphi_1^b = \left. -\frac{\partial V_1}{\partial \varphi_a} \right|_{\varphi_0}$$  \hspace{1cm} (25.45)

This is an inhomogeneous linear equation for $\varphi_1$. But the matrix on the left has zero eigenvalues by the Goldstone theorem. The zero eigenvectors are just $T_{ab} \varphi_0^b$. Taking the inner product of these eigenvectors with the left and right side of the equation leads to the requirement that

$$\left. \frac{\partial V_1}{\partial \varphi_a} \right|_{\varphi_0} T_{ab} \varphi_0^b = 0.$$  \hspace{1cm} (25.46)

This is the vacuum alignment condition: of the many choices one can make for $\varphi_0$, one must pick the one that aligns with the symmetry breaking term in this sense. The manifold of possible vacua is given by the symmetry transformations applied to any particular vacuum $U\varphi_0$. The alignment condition can be interpreted as finding the $U$ which minimizes $V_1(U\varphi_0)$.

**25.2.4 (Approximate) Symmetries of QCD**

As the main application of these ideas we finally turn to QCD. Starting with the standard model with Higgs mechanism implemented, we can diagonalize the fermion mass terms,
generating the CKM flavor mixing matrix. We then isolate QCD by taking $M_W, M_Z \to \infty$. This just means we only consider QCD processes with energy and momenta much smaller than the vector boson masses. In that limit the interactions between leptons and quarks are purely electromagnetic, which we also neglect in first instance. That leaves us with “pure” QCD, interacting quarks and gluons, with dynamics governed by the Lagrangian

$$L_{\text{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_f \bar{Q}_f (-m_f + i\gamma \cdot D) Q_f$$

(25.47)

There is strong evidence that each quark has a different nonzero mass parameter which limits the symmetries to $U(1)^6$ generated by the number operators of each kind of quark. But the masses of the quarks range widely from the up and down quarks of mass a few MeV to the top quark with mass of 175 GeV! From asymptotic freedom we can define the basic hadronic scale as $\Lambda_{\text{QCD}}$ which from high momentum experiments is estimated to be several hundred MeV (depending on the renormalization scheme chosen). It is certainly a good approximation to set $m_u = m_d = 0$, and perhaps not so bad an approximation to also set $m_s = 0$. So we will study QCD with $n_f = 2$ or $n_f = 3$ massless quarks. We will also assume $P, C, T$ invariance. We have all together $N_f = 6$ flavors of quark, $n_f$ of which are treated initially as massless.

### 25.2.5 Hypothesis of chiral SSB

With $n_f$ massless quarks one can identify a matrix scalar order parameter $\sigma_{kl} = \langle 0|\bar{L}_k R_l|0 \rangle$ with $L$ and $R$ transforming under independent $n_f \times n_f$ unitary matrices $\sigma_{kl} \to (U^*_L \sigma U_R)_{kl}$. Thus QCD has an apparent $U(n_f) \times U(n_f)$ symmetry in this approximation. We can treat each $U(n_f)$ symmetry as $U(1) \times SU(n_f)$. We shall find that one combination of the $U(1)$’s is anomalous, meaning that quantization destroys that symmetry, and the other combination, namely $[U(1) \times U(1)]_{\text{diagonal}}$ is not spontaneously broken. So we will focus on the $SU(n_f) \times SU(n_f)$ chiral symmetry. The fundamental hypothesis is that, with $n_f$ massless quarks in QCD, $SU(n_f) \times SU(n_f)$ is spontaneously broken to $[SU(n_f) \times SU(n_f)]_{\text{diagonal}} = SU(n_f)$. This means that we assume that the effective potential in this approximation

$$V_0(\sigma) = V_0(U^*_L \sigma U_R)$$

(25.48)

has a minimum at

$$\sigma_{kl} = S\delta_{kl}, \quad S = S^* \neq 0$$

(25.49)

where the Kronecker delta ensures that $U^\dagger \sigma U = \sigma$ and the hypothesis that $\sigma$ is real is necessary if parity is not also spontaneously broken. Since the $n_f$ quarks actually have small masses, we should take them into account. The mass terms in the Lagrangian of QCD are

$$-\sum_f m_f \bar{q}_f q_f = -\sum_f m_f (\bar{L}_f R_f + \bar{R}_f L_f)$$

(25.50)
so we make the hypothesis that the symmetry breaking term in the Effective potential is

\[ \mathcal{V}_1 = \text{Tr}[\mathcal{M}\sigma + \sigma^i\mathcal{M}^i] \]  

By a chiral $SU(n_f) \times SU(n_f)$ transformation we can bring an arbitrary complex mass matrix $\mathcal{M}$ to diagonal form with positive entries times a common phase factor. So wolog we can take $\mathcal{M}_{kl} = e^{i\gamma}m_k\delta_{kl}$, which we shall do. For parity and charge conjugation invariance, $\gamma = 0, \pi$, i.e. the overall phase is just a sign.

Next we address the alignment problem. Before symmetry breaking the vacuum order parameter could be $SU_L^\dagger U_R \equiv SU$, and we should choose $U$ to minimize

\[ \mathcal{V}_1 = S \text{Tr}[\mathcal{M}U + U^\dagger\mathcal{M}] = Se^{i\gamma}\sum_k m_k(U + U^\dagger)_{kk} \]  

Where $U \in SU(n_f)$. If we assume $P$ and $C$ invariance, $Se^{i\gamma}$ is real but can be either positive or negative. If it is negative the minimum is evidently realized with $U = I$. If it is positive and $n_f$ is even, the minimum is realized with $U = -I$ because then $-I \in SU(n_f)$. However if $n_f$ is odd, $-I$ is not in $SU(n_f)$ and the alignment solution is more complicated! We shall assume that $Se^{i\gamma} < 0$ for QCD with 3 light quarks.

### 25.2.6 Masses of Nambu-Goldstone bosons in QCD.

From our general discussion the NGB’s of chiral SSB are determined by the symmetry transformations that change $\langle \sigma_{kl} \rangle$:

\[ \langle \sigma_{kl} \rangle \rightarrow \sigma(U_L^\dagger U_R)_{kl} \]  

Writing $U_L^\dagger U_R = U \rightarrow I - i\epsilon_aG_a$, the $G_a$ are the generators of $SU(n_f)$ which are $n_f^2 - 1$ in number, so there are that many NGB’s. For $n_f = 2$ (only $u, d$ massless) these are just the three pions $\pi^0, \pi^\pm$. For $n_f = 3$ ($s$ is also massless) add the $K^\pm$, $K^0$, $\bar{K}^0$, $\eta$ to the three pions.

In building the low energy effective action, we can use an $SU(n_f)$ matrix $U = e^{-i\pi_a\lambda^a/F}$ as the NGB effective field, transforming under $SU(n_f) \times SU(n_f)$ by $U \rightarrow U_L^\dagger U U_R$. To construct an invariant $\Gamma(U)$ we need at least 2 derivatives, which is uniquely.

\[ \Gamma_0(U) = -\frac{F^2}{4} \text{Tr} \partial_\mu U^\dagger \partial^\mu U = -\frac{F^2}{4} \text{Tr} \lambda_a \lambda_b \frac{\partial_\mu \pi_a \partial_\mu \pi_b}{F^2} + \cdots \]

\[ = -\frac{1}{2} \sum_a (\partial \pi_a)^2 + \cdots \]  

where the dots stand for terms cubic and higher in the NGB fields, $\pi_a$. Similarly the symmetry breaking mass term in the effective potential is

\[ \mathcal{V}_1 = -|S| \sum_k m_k(U + U^\dagger)_{kk} = -2|S| \sum_k m_k + \frac{|S|}{F^2} \sum_k m_k(\lambda_a \lambda_b)_{kk} \pi_a \pi_b + \cdots \]
In the case of $n_f = 2$ the $\lambda$'s are just the Pauli matrices $\tau_a$ and because $\pi_a \pi_b$ is symmetric we can replace

$$\tau_a \tau_b \rightarrow \frac{1}{2} \{\tau_a, \tau_b\} = \delta_{ab} \quad (25.56)$$

and the mass term becomes

$$-2|S| \sum_k m_k + \frac{|S|(m_u + m_d)}{F^2} \pi_a^2 \quad (25.57)$$

so all three pions get the same mass $m_\pi^2 = 2|S|(m_u + m_d)/F^2$.

The case $n_f = 3$ gets more complicated because the Gell-Mann matrices satisfy

$$\{\lambda_a, \lambda_b\} = \frac{4}{3} \delta_{ab} + 2d_{abc} \lambda_c \quad (25.58)$$

The only $\lambda$'s with diagonal entries are $\lambda_3, \lambda_8$ and we need

$$\sum_k m_k \lambda_{3kk} = m_u - m_d, \quad \sum_k m_k \lambda_{8kk} = \frac{1}{\sqrt{3}}(m_u + m_d - 2m_s) \quad (25.59)$$

Putting these into the symmetry breaking term gives

$$V_1 = -2|S| \sum_k m_k$$

$$+ \frac{|S|}{F^2} \left[ \frac{2}{3} \delta_{ab} \sum_k m_k + d_{ab3}(m_u - m_d) + d_{ab8} \frac{m_u + m_d - 2m_s}{\sqrt{3}} \right] \pi_a \pi_b \quad (25.60)$$

Looking up the $d$ coefficients we find

$$d_{ab3} \pi_a \pi_b = \frac{2}{\sqrt{3}} \pi_3 \pi_8 + \frac{1}{2}(\pi_4^2 + \pi_5^2 - \pi_6^2 - \pi_7^2) \quad (25.61)$$

$$d_{ab8} \pi_a \pi_b = \frac{1}{\sqrt{3}}(\pi_1^2 + \pi_2^2 + \pi_3^2 - \pi_8^2) - \frac{1}{2\sqrt{3}}(\pi_4^2 + \pi_5^2 + \pi_6^2 + \pi_7^2) \quad (25.62)$$

Collecting all the quadratic terms in the effective potential gives

$$V_1 = -\frac{m_u + m_d - 2m_s}{3} \left[ \frac{2}{\sqrt{3}} \pi_3 \pi_8 + \frac{1}{2}(\pi_4^2 + \pi_5^2 - \pi_6^2 - \pi_7^2) \right]$$

$$+ \frac{m_u + m_d - 2m_s}{3} \left[ (\pi_1^2 + \pi_2^2 + \pi_3^2 - \pi_8^2) - \frac{1}{2}(\pi_4^2 + \pi_5^2 + \pi_6^2 + \pi_7^2) \right]$$

$$= \left[ \frac{m_u + m_d}{3} (\pi_1^2 + \pi_2^2 + \pi_3^2) + \frac{m_u + m_d + 4m_s}{3} \pi_8^2 + (m_u - m_d) \frac{2}{\sqrt{3}} \pi_3 \pi_8 \right.$$}

$$+ (m_u + m_s)(\pi_4^2 + \pi_5^2) + (m_d + m_s)(\pi_6^2 + \pi_7^2) \right] \quad (25.63)$$
We identify \((\pi_1 \pm i\pi_2) / \sqrt{2}\) with \(\pi^\pm\), \((\pi_4 \pm i\pi_5) / \sqrt{2}\) with \(K^\pm\), and \((\pi_6 \pm i\pi_7) / \sqrt{2}\) with \(K^0, \bar{K}^0\). \(\pi_3\) is \(\pi_0\) and \(\pi_8\) is \(\eta\). Because of the mixing term \(\pi_3\pi_8\) these are not strictly mass eigenstates, but the effect of mixing is exceedingly small, of order \((m_u - m_d)^2 / m_s \sim m_\pi^4 / m_K^2\), so we neglect it in the following.

Neglecting this mixing effect we can simply read off the predictions for the meson masses:

\[
\begin{align*}
m_\pi^2 &= 2|S| F^2 (m_u + m_d), \quad m_K^2 = 2|S| F^2 (m_u + m_s) \\
m_{K^0}^2 &= 2|S| F^2 (m_d + m_s), \quad m_\eta^2 = 2|S| F^2 (m_u + m_d + 4m_s)
\end{align*}
\] (25.64)

These results predict one relation between the meson squared masses, namely

\[
2m_{K^0}^2 + 2m_{K^\pm}^2 - 3m_\eta^2 - m_\pi^2 = 0
\] (25.65)

In addition the calculations give estimates for the ratio of quark masses. For this purpose we take seriously the \(K^+ / K^0\) mass difference of a few MeV which is comparable to the \(\pi^+ / \pi^0\) mass difference, which the present physics does not allow for. It is plausible that these differences are due to EM corrections. If we add EM couplings to the QCD Lagrangian, in the limit that \(u, d, s\) are massless, we notice that because the \(d\) and \(s\) quarks have the same charge, which is half the \(u\) quark charge, there remains a chiral \((SU(2) \times U(1)) \times (SU(2) \times U(1))\) symmetry, which means that 4 NGB’s remain, namely \(\pi^0, \eta, K^0, \bar{K}^0\) will stay massless even after EM corrections. So only \(\pi^\pm, K^\pm\) can get a mass. But these mesons are related by the residual \(SU(2)\) so if no mass terms are included they would get a common mass. We don’t compute it but add it as a parameter \(\Delta = m_{\pi^0}^2 - m_{\pi^0}^2\). Then we add \(\Delta\) to the right side of the two formulas for \(m_{K^0}^2\) and \(m_{K^\pm}^2\). There is still only one relation because we are fitting one more mass with one new parameter:

\[
2m_{K^0}^2 + 2m_{K^\pm}^2 - 3m_\eta^2 - 2m_{\pi^0}^2 + m_{\pi^0}^2 = 0
\] (25.66)

Then we get, for example,

\[
\frac{m_u}{m_s} = \frac{2m_{\pi^0}^2 - m_{\pi^\pm}^2 + m_{K^0}^2 - m_{K^\pm}^2}{m_{K^0}^2 + m_{K^\pm}^2 - m_{\pi^\pm}^2} \approx 0.027
\] (25.67)

\[
\frac{m_d}{m_s} = \frac{m_{\pi^0}^2 + m_{K^0}^2 - m_{K^\pm}^2}{m_{K^0}^2 + m_{K^\pm}^2 - m_{\pi^\pm}^2} \approx 0.050
\] (25.68)

The existence of the relation, which is not experimentally perfect, means that equally valid expressions obtained by using it can give different estimates for these quark mass ratios.

### 25.2.7 Low energy pion scattering

We can use the effective action to calculate the low energy limit of pion pion scattering. Here we use the spontaneous breaking of chiral symmetry for the case \(n_f - 2\). We need to
expand the effective action to order $\pi_a^4$.

\[
U = I - \frac{i}{F} \pi_a^a - \frac{1}{2F^2} \pi_a^a + \frac{i\pi_a^a \tau_a}{6F^3} + \frac{(\pi_a^a)^2}{24F^4}
\]

\[
U + U^\dagger = 2I - \frac{1}{F} \pi_a^a + \frac{(\pi_a^a)^2}{12F^4}
\]

\[
\partial \mu U = \frac{i}{F} \partial \mu \pi_a^a - \frac{1}{F^2} \pi_a^a \partial \mu \pi_a - \frac{i\partial \mu \pi_a^a \tau_a}{6F^3} + \frac{i\pi_a^a \tau_a}{3F^3} \partial \mu \pi_a
\]

\[
\text{Tr}\partial \mu U^\dagger \partial \nu U = \frac{2}{F^2} (\partial \pi_a)^2 + \frac{2}{F^4} (\pi_a \partial \mu \pi_a)^2 - \frac{2}{3F^4} (\partial \mu \pi_a)^2 \pi_a^a - \frac{4}{3F^4} (\pi_a \partial \mu \pi_a)^2 \pi_a^a
\]

Putting everything together gives the effective action:

\[
\Gamma(\pi_a) = -\frac{1}{2} (\partial \pi_a)^2 - \frac{m_\pi^2}{2} \pi_a^a - \frac{1}{6F^2} (\pi_a \partial \mu \pi_a)^2 + \frac{1}{6F^2} (\partial \mu \pi_a)^2 \pi_a^a
\]

\[
+ \frac{m_\pi^2}{24F^2} (\pi_a^a)^2 + O(\pi^6)
\]  

(25.70)

The pion scattering amplitude predicted by this action is simply the sum of quartic vertices. Let the pion momenta and isospin labels be $p_i, a_i, i = 1, \cdots, 4$. Then the last quartic term produces

\[
\frac{i}{3F^2} \left[ \delta_{a_1a_2} \delta_{a_3a_4} (p_1 + p_2) \cdot i(p_3 + p_4) + \delta_{a_1a_3} \delta_{a_2a_4} i(p_1 + p_3) \cdot i(p_2 + p_4) + \delta_{a_1a_4} \delta_{a_2a_3} i(p_1 + p_4) \cdot i(p_2 + p_3) \right]
\]

(25.71)

The first quartic term produces

\[
\frac{i}{3F^2} \delta_{a_1a_2} \delta_{a_3a_4} i(p_1 + p_2) \cdot i(p_3 + p_4) + \delta_{a_1a_3} \delta_{a_2a_4} i(p_1 + p_3) \cdot i(p_2 + p_4) + \delta_{a_1a_4} \delta_{a_2a_3} i(p_1 + p_4) \cdot i(p_2 + p_3)
\]

(25.72)

Finally the second quartic vertex produces

\[
\frac{i}{3F^2} \left[ \delta_{a_1a_2} \delta_{a_3a_4} \mathbf{i}^2 (2p_1 \cdot p_2 + 2p_3 \cdot p_4) + \delta_{a_1a_3} \delta_{a_2a_4} i^2 (2p_1 \cdot p_3) + \delta_{a_1a_4} \delta_{a_2a_3} i^2 (2p_1 \cdot p_4) + \delta_{a_1a_3} \delta_{a_2a_4} i^2 (2p_1 \cdot p_3) + \delta_{a_1a_4} \delta_{a_2a_3} i^2 (2p_1 \cdot p_4) + \delta_{a_1a_3} \delta_{a_2a_4} i^2 (2p_1 \cdot p_3) + \delta_{a_1a_3} \delta_{a_2a_4} i^2 (2p_1 \cdot p_4) + \delta_{a_1a_3} \delta_{a_2a_4} i^2 (2p_1 \cdot p_3) + \delta_{a_1a_3} \delta_{a_2a_4} i^2 (2p_1 \cdot p_4)\right]
\]

(25.73)
The sum of these three vertices gives the pion-pion scattering amplitude:

\[
\mathcal{M} = \frac{i}{F^2} \times [\delta_{a_1a_2}\delta_{a_3a_4}(s - m_\pi^2) + \delta_{a_1a_3}\delta_{a_2a_4}(u - m_\pi^2) + \delta_{a_1a_4}\delta_{a_3a_2}(t - m_\pi^2)]
\]

(25.73)

\[
\rightarrow i\frac{m_\pi^2}{F^2} \times [3\delta_{a_1a_2}\delta_{a_3a_4} - \delta_{a_1a_3}\delta_{a_2a_4} - \delta_{a_1a_4}\delta_{a_3a_2}]
\]

(25.74)

where the last line gives the threshold limit in the s-channel \((s = 4m_\pi^2, u = t = 0)\). Let’s pause to review the validity of the conclusion that this formula gives he exact low energy pion scattering amplitude in pure QCD. Recall that \(\Gamma\) is the quantum effective action whose tree approximation gives the exact quantum action. We did not calculate \(\Gamma\) directly, but made the hypothesis that chiral SU(2) \(\times\) SU(2) was spontaneously broken to diagonal SU(2). Based on that hypothesis we then wrote the most general effective action involving the NGB’s of the broken symmetry (pions) with the minimum number \(2\) of derivatives. The last condition means its validity is low energies. Finally we introduced a mass term which broke the chiral symmetry, assuming it was linear in quark masses. These are the conditions for the validity of the prediction.

So far the parameter \(F\) seems to be an undetermined free parameter which could only be calculated if we could treat QCD nonperturbatively. While this is true, we can measure its value in a process independent of pion scattering, namely the weak decay \(\pi \rightarrow \mu + \bar{\nu}_\mu\). To understand this point recall the way the NGB effective field transforms under SU(2) \(\times\) SU(2):

\[
U(x) = e^{-ir^a\pi_a/F} \rightarrow U_L^\dagger U(x)U_R.
\]

(25.75)

Specifically consider the transformation generated by \(Q_5^a \equiv Q_R^a - Q_L^a = \int d^3x \sum_f \bar{q}_f \gamma^0 \gamma_5 (\tau_a/2)q_f\). These transformations satisfy \(U_L = U_R^{-1} = U_R^\dagger\). Parameterizing \(U_R = e^{-\xi_a\tau_a/2}\), the transformation of \(U(x)\) is

\[
e^{-ir^a\pi_a/F} \rightarrow e^{-\xi_a\tau_a/2}e^{-ir^a\pi_a/F}e^{-\xi_a\tau_a/2} = e^{-ir^a\pi_a/F - i\xi_a\tau_a + O(\pi_5)}
\]

(25.76)

by Baker-Hausdorff. This means that, among other effects, the chiral transformation shifts \(\pi_a \rightarrow \pi_a + F\xi_a\). But this shift has to be produced by \(-i[\pi_a, \xi_5 Q_5^a]\), which implies that \(A_5^0\) has a term linear in \(\pi_5\), namely

\[
A_5^0 = F\pi_a + O(\pi_5^2), \quad A^\mu = -F\partial_\mu \pi_a + O(\pi_5^2)
\]

(25.77)

This term produces a non zero matrix element

\[
\langle 0 | A^\mu_a (\pi_b, \mathbf{p}) | \rangle = -iFp^\mu \frac{\delta_{ab}}{(2\pi)^{3/2} \sqrt{2p^0}}
\]

(25.78)

The axial current \(A^\mu_a\) is a component of the current that couples to the electroweak bosons, and hence can be measured in electroweak processes. In particular the process \(\pi^- \rightarrow \mu^- + \bar{\nu}_\mu\) is controlled by \(F\), which experiment then fixes to be around 93 MeV. With this value low energy pion scattering is completely fixed!
25.2.8 Pion decay to 2 photons ($\pi^0 \rightarrow \gamma\gamma$)

The two photon decay of the neutral pion is important as the signature of the $\pi^0$ but also because its strength can only be explained by the existence of an anomaly in the neutral axial isospin current $j_3^\mu$. Such an anomaly is allowed when the axial current is not coupled to gauge fields. For example the quark part of the axial current is not coupled to a gauge field. (Parts of it contribute to the electroweak currents but those also include the leptons and their anomaly is cancelled between quarks and leptons.) Thus the neutral component

$$j_3^\mu = \frac{\tau_3}{2} \gamma_5 \gamma^\mu q = \frac{1}{2}(\bar{u}\gamma_5 \gamma^\mu u - \bar{d}\gamma_5 \gamma^\mu d)$$ (25.79)

of the axial isospin current for up and down quarks has an axial anomaly. In the approximation of massless quarks, we have found:

$$\partial_\mu j_{35}^\mu = N_c \left(\frac{4}{9} - \frac{1}{9}\right) \frac{\alpha_0}{8\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{N_c \alpha_0}{24\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},$$ (25.80)

where $F$ is the e.m. field strength. The numerical factors are explained as follows: The up and down quarks couple with opposite signs to $j_{35}$ and the contribution of each to the anomaly is the square of the charge. The overall factor of $N_c = 3$ is for the number of colors of each quark. This equation implies a nonvanishing matrix element of $j_{35}^\mu$ between the vacuum and a two photon state. Now neglect the weak interactions, keeping strong interactions to all orders. Since the strong and electromagnetic interactions conserve parity, this matrix element, $\langle 0 | j_{35}^\mu | \gamma\gamma \rangle$, must be a pseudo three index tensor $X_{\mu\rho\sigma}$ where $\rho$ and $\sigma$ are the Lorentz indices describing the polarization of the photons, which carry momenta $k_1, k_2$ respectively. EM gauge invariance then requires $k_{1\rho} X_{\mu\rho\sigma} = k_{2\sigma} X_{\mu\rho\sigma} = 0$, and Bose statistics for photons requires symmetry under $\rho, k_1 \leftrightarrow \sigma, k_2$. There is essentially only one such pseudotensor (up to terms proportional to $k_1^\rho$ or to $k_2^\rho$ which decouple from physical photons) that can be formed from the epsilon symbol and $k_1, k_2$ since $2^2$:

$$\epsilon^{\mu\rho\sigma\tau}(k_1 - k_2)_\tau + \frac{2(k_1 + k_2)\epsilon^{\mu\rho\sigma\tau} k_1, k_2\epsilon^{\rho\sigma\lambda\tau}}{(k_1 + k_2)^2} - \frac{2(k_1 + k_2)\epsilon^{\rho\sigma\lambda\tau} k_1, k_2\epsilon^{\mu\rho\lambda\tau}}{(k_1 + k_2)^2} = + \frac{2(k_1 + k_2)^\rho k_1, k_2\epsilon^{\rho\sigma\lambda\tau}}{(k_1 + k_2)^2}.$$ (25.83)

The following equation is a special case for $k_1^2 = k_2^2$ of the identity

$$\epsilon^{\rho\sigma\lambda\tau}(k_1 + k_2)_\tau = \epsilon^{\rho\sigma\lambda\tau}(k_1 - k_2)_\tau \frac{(k_1 + k_2)^2}{2}$$ (25.81)

$$+ \epsilon^{\rho\sigma\lambda\tau}(k_1 + k_2)_\tau \frac{(k_2^2 - k_1^2)}{2} + (k_1 + k_2)^\rho k_1, k_2\epsilon^{\rho\sigma\lambda\tau} - (k_1 + k_2)^\rho k_1, k_2\epsilon^{\mu\rho\lambda\tau}.$$ (25.82)

This identity can be proved by checking it in the center of mass frame of the two photons $k_1 = -k_2$. 287 ©1992, 2017 by Charles Thorn
Note that any contribution of $O(k)$ at small $k$ is of necessity nonanalytic at zero $k$. The anomaly equation fixes uniquely the coefficient of this nonanalytic term\(^3\). Thus we can write

$$\langle k_1, \lambda_1; k_2, \lambda_2 | j_{35}^{\mu} | 0 \rangle = i N_c \alpha_0 \frac{e^{\mu\nu\rho\sigma}}{24\pi} \langle k_1, \lambda_1; k_2, \lambda_2 | F_{\nu\rho} F_{\rho\sigma} | 0 \rangle \frac{(k_1 + k_2)^\mu}{(k_1 + k_2)^2},$$

(25.84)

where we stress we have approximated the quark masses as zero. The presence of the singularity at $(k_1 + k_2)^2 = 0$ is a striking consequence of the anomaly. The source of the singularity can be traced to the masslessness of the quarks. But if quarks are confined they can't be responsible for the singularity in the exact $m_u = m_d = 0$ amplitude. There are therefore two possibilities ('t Hooft): either some of the physical baryons are massless or there is a massless scalar (to be identified with a Goldstone boson) coupling to the axial isospin current. The latter possibility seems to be the one realized in Nature, with the pion playing the role of the Goldstone boson:

$$\langle q, \pi_0 | j_{35}^{\mu} | 0 \rangle = i q^\mu F_\pi \frac{(2\pi)^{3/2}}{\sqrt{2\omega}}.$$  

(25.85)

(The $\pi^0$ is related by strong isospin to the $\pi^\pm$. In the limit of exact isospin ($m_u = m_d$ and EM turned off) $F_{\pi^0} = F_{\pi^\pm}$, and the latter can be independently measured in the weak decay process $\pi^- \to \mu^- + \bar{\nu}_\mu$.) In that case the residue of the pole is $(2\pi)^{3/2} \sqrt{2\omega} F_\pi q^\mu$ times the transition amplitude for the $\pi_0$ to decay into two photons. In perturbation theory, this latter transition amplitude is $(-i)$ times the matrix element of the perturbation in the Hamiltonian density describing the electromagnetic interactions of hadrons, from which we conclude:

$$\langle k_1, \lambda_1; k_2, \lambda_2 | \mathcal{H}_{I}^{e.m.}(0) | q, \pi_0 \rangle \approx - \frac{1}{(2\pi)^{3/2} \sqrt{2\omega}} \frac{N_c \alpha_0}{24\pi F_\pi} e^{\mu\nu\rho\sigma} \times \langle k_1, \lambda_1; k_2, \lambda_2 | F_{\mu\nu} F_{\rho\sigma} | 0 \rangle,$$

(25.86)

where the approximation is due only to the fact that the quarks and the pion are not exactly massless. This corresponds to a term in the effective Lagrangian:

$$\frac{N_c \alpha_0}{24\pi F_\pi} \pi_0 e^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

(25.87)

The rate for this process was calculated in an exercise (Set 1, Problem 4) to be $\Gamma \approx m_\pi^3 G^2 / \pi$, where $G = N_C \alpha_0 / (24\pi F_\pi)$.

To the extent that this is a good approximation, we see that the anomaly controls the decay $\pi_0 \to 2\gamma$. In fact, this approximation with $N_c = 3$ for QCD, gives a good account of the experimental rate to within 20%. (Check this!) This success may be regarded as evidence for the three colors of quarks. Incidentally, the anomaly breaks the apparent chiral $U(1)$ invariance remaining after electromagnetic interactions have broken $SU(2) \times SU(2)$ by

\(^3\text{This is an essential aspect of the anomaly, reflecting the fact that it really can't be removed by polynomial adjustments to the definition of the current. If an analytic piece of the axial current could produce the anomaly, one could make an analytic adjustment to the definition of the current to remove it.}\)
virtue of unequal up and down charges. In particular the $\pi_0$ will have a small squared mass of order $\alpha^2$ even if all quark masses are zero. This shift however is very small compared to the order a shift given to the $\pi^+$ and does not disturb the derivation of the Gell-Mann Okubo relation in the $SU(3) \times SU(3)$ case.

### 25.3 Adler-Weisberger and Other Sum Rules

#### 25.3.1 Soft pion scattering off nucleons

Here we attempt to introduce baryons (3 quark hadrons) into the effective action formalism with the goal of deriving low energy amplitudes for NGB-baryon scattering. Spin 1/2 baryons fall into octets under $SU(3) \times SU(3)$ $(p, n, \Sigma^{\pm, 0}, \Lambda, \Xi^{0, -})$, which is not the fundamental irrep of $SU(3)$. In contrast, the proton and neutron are doublets under $SU(2) \times SU(2)$. Since $SU(2)$ doublets are the fundamental irrep of $SU(2)$, the construction of the effective action is simpler in that case, so we discuss it first.

We introduce an effective nucleon field as as an $SU(2)$ doublet $N = (p, n)$. Then the chiral group sends $N_L \to U_L N_L$ and $N_R \to U_R N_R$, so that $\bar{N}_L U(\pi) N_R$ and $\bar{N}_R U^\dagger(\pi) N_L$ are candidate chiral invariants to include in the effective action, where $U(\pi) = e^{i\gamma_5 \pi_a / F}$. We can write a parity invariant combination using $\gamma_5$:

$$\bar{N}_L U(\pi) N_R + \bar{N}_R U^\dagger(\pi) N_L = \bar{N} e^{i\gamma_5 \pi_a / F} N$$  \hspace{1cm} (25.88)

Additional chiral invariants include $\bar{N}i\gamma \cdot \partial N$, which we include with the canonical normalization and $\bar{N} \gamma^\mu (e^{-i\gamma_5 \pi_a / F} \partial_{\mu} e^{i\gamma_5 \pi_a / F}) N$. This exhausts the possibilities bilinear in $N$ and containing at most one derivative. Introduce the notation $U_5 = e^{i\gamma_5 \pi_a / F}$ so the complete low energy effective action is

$$\Gamma(U, N) = -\frac{F^2}{4} \text{Tr} \partial_{\mu} U^\dagger \partial_{\mu} U + |S| \text{Tr} M(U + U^\dagger)$$

$$+ \bar{N}i\gamma \cdot \partial N - m_N \bar{N} e^{i\gamma_5 \pi_a / F} N - iC \bar{N} \gamma^\mu (e^{-i\gamma_5 \pi_a / F} \partial_{\mu} e^{i\gamma_5 \pi_a / F}) N$$  \hspace{1cm} (25.89)

We shall analyze pion-nucleon scattering which involves up to 2 powers of $\pi_a$ multiplying the nucleon bilinear. So we expand

$$U_5 = I + \frac{i}{F} \gamma_5 \tau_a \pi_a \pi_a + \frac{1}{2 F^2} \pi_a^2 + O(\pi_a^3)$$  \hspace{1cm} (25.90)

$$U_5^\dagger \partial_{\mu} U_5 = \left(I - \frac{i}{F} \gamma_5 \tau_a \pi_a\right) \left(\frac{i}{F} \gamma_5 \tau_a \partial_{\mu} \pi_a - \frac{1}{2 F^2} \partial_{\mu} \pi_a^2\right)$$

$$= \frac{i}{F} \gamma_5 \tau_a \partial_{\mu} \pi_a - \frac{1}{2 F^2} \partial_{\mu} \pi_a^2 + \frac{1}{F^2} \tau_a \pi_a \tau_b \partial_{\mu} \pi_b$$

$$= \frac{i}{F} \gamma_5 \tau_a \partial_{\mu} \pi_a + \frac{1}{2 F^2} i\epsilon_{abc} \tau_c \pi_a \partial_{\mu} \pi_b$$  \hspace{1cm} (25.91)

To this order the nucleon bilinear in the effective action is

$$\bar{N} \left[i\gamma \cdot \partial - m_N - \frac{im_N}{F} \gamma_5 \tau_a \pi_a + C \gamma_5 \tau_a \partial_{\mu} \pi_a + \frac{m_N}{2 F^2} \pi_a^2 + \frac{C}{F^2} \epsilon_{abc} \tau_c \pi_a \gamma^\mu \partial_{\mu} \pi_b\right] N$$
Let us first discuss the pion-nucleon vertex implied by this action. It is given by

$$\bar{u}(p') \left( \frac{m_N}{F} \gamma_5 \tau_a - \frac{C}{F} \gamma^\mu \gamma_5 \tau_a (p' - p)_\mu \right) u(p) = \bar{u}(p') \left( \frac{m_N(1 + 2C)}{F} \gamma_5 \right) \tau_a$$

(25.92)

where we used the Dirac equation in momentum space. We infer the pion-nucleon coupling constant to be

$$G_{\pi N} = \frac{m_N(1 + 2C)}{F}.$$  

(25.93)

To relate \(C\) to a measurable property of the nucleon, consider the nucleon matrix element of the axial current \(A^\mu_a = \sum_f \bar{q}_f \gamma^\mu \gamma_5 t_a q_f\),

$$\langle N' | A^\mu_a | N \rangle = \bar{u}' t_a \left[ -\gamma^\mu \gamma_5 g_A (q^2) + q^\mu \gamma_5 g_2 (q^2) + i q_\nu [\gamma^\nu, \gamma^\mu] \gamma_5 g_3 (q^2) \right] u$$

(25.94)

The hypothesis of SSB for chiral symmetry says that in the limit \(m_u = m_d = 0\) this current is conserved, so in this limit we should have

$$\bar{u}' t_a \left[ -\gamma^\mu \gamma_5 g_A (q^2) + q^\mu \gamma_5 g_2 (q^2) \right] u = 0$$

(25.95)

Using the Dirac equation this requires \(-2m_N g_A (q^2) + q^2 g_2 (q^2) = 0\). Now \(g_A\) is measured in nuclear beta decay (for which \(q \approx 0\)) to be roughly 1.2, far from zero. So conservation requires the presence of the pion as a NGB, which contributes to the pion pole \(g_2\) term as \(g_2 (q^2) = 2G_{\pi N} F_\pi / q^2\). So we arrive at the famous Goldberger-Treiman relation

$$G_{\pi N} = \frac{m_N g_A}{F}, \quad \text{Goldberger} - \text{Treiman}$$

(25.96)

Comparing to our single soft pion relation we see that \(C = (g_A - 1)/2\).

We next turn to two soft pions so we can derive a prediction for low energy pion nucleon scattering. We just have to use our effective Lagrangian to calculate the tree diagrams for this process. First we look at the diagrams that don’t involve the \(C\) vertices:

$$\bar{u}' \frac{-i m_N^2}{F^2} \left[ \gamma_5 \tau_b \frac{m_N - \gamma \cdot (p + q)}{-m_\pi^2 + 2p \cdot q} \gamma_5 \tau_a + \gamma_5 \tau_a \frac{m_N - \gamma \cdot (p - q)}{-m_\pi^2 - 2p \cdot q} \gamma_5 \tau_b - \delta_{ab} \frac{1}{m_N} \right] u$$

$$= \bar{u}' \frac{-i m_N^2}{F^2} \left[ \tau_b \frac{-\gamma^0 m_\pi}{-m_\pi^2 - 2m_\pi m_N} \tau_a + \tau_a \frac{\gamma^0 m_\pi}{-m_\pi^2 - 2m_\pi m_N} \tau_b - \delta_{ab} \frac{1}{m_N} \right] u$$

(25.97)

The low energy limit is the limit of threshold, where all spatial components of momenta are zero and the time components are the masses:

$$\bar{u}' \frac{-i m_N^2}{F^2} \left[ \tau_b \frac{-\gamma^0 m_\pi}{-m_\pi^2} \tau_a + \tau_a \frac{\gamma^0 m_\pi}{-m_\pi^2} \tau_b - \delta_{ab} \frac{1}{m_N} \right] u$$

$$= \bar{u}' \frac{-i m_N}{F} \left[ \tau_b \frac{1}{2 + \frac{m_\pi}{m_N}} + \tau_a \frac{1}{2 - \frac{m_\pi}{m_N}} - \delta_{ab} \right] u$$

$$= \bar{u}' \frac{-i m_N}{F} \left[ \delta_{ab} \frac{m_\pi^2}{4 - m_\pi^2} + \epsilon_{bac} \tau_c \frac{-2m_\pi}{4 - m_\pi^2} \right] u$$

$$= \frac{m_\pi}{2F} \epsilon_{abc} \bar{u}' \tau_c u \left( 1 + O(m_\pi/m_N) \right)$$

(25.98)
As we shall see the $C$ vertices add corrections a factor of order $m_\pi/m_N$ smaller than this.
\[ G_{\pi N} = \frac{m_N (1 + 2c)}{F} \]
We next show that the total contribution of the vertices involving $C$ is smaller than these expression by a factor of $m_\pi/m_N$. To simplify the argument we evaluate the amplitudes at threshold. We first look at the diagrams of order $C^2$:

\[
\bar{u}' - iC^2 \frac{F^2}{F^2} (q \cdot \gamma) \gamma_5 \tau_b \frac{m_N - \gamma \cdot (p + q)}{-m_\pi^2 + 2p \cdot q} (-q \cdot \gamma \gamma_5 \tau_a u) \\
\rightarrow \bar{u}' \frac{iC^2}{F^2} (m_\pi \gamma_0) \gamma_5 \tau_b \frac{m_N + \gamma_0 (m_N + m_\pi)}{-m_\pi^2 - 2m_N m_\pi} (m_\pi \gamma_0 \gamma_5 \tau_a u) \\
= \bar{u}' - iC^2 \frac{F^2}{F^2} m_\pi \tau_b \frac{m_N - \gamma_0 (m_N + m_\pi)}{-m_\pi^2 - 2m_N m_\pi} m_\pi \tau_a u \\
= \bar{u}' - iC^2 \frac{F^2}{F^2} m_\pi \tau_b \frac{m_\pi^2}{m_\pi + 2m_N} \tau_a u
\]  

(25.99)

where we used the Dirac equation at threshold, namely $(\gamma^0 - 1)u = 0$. So this diagram and the one with meson legs crossed are down by a factor $m_\pi/m_N \approx 1/7$ as claimed. To complete the argument we examine the diagrams linear in $C$.

The four diagrams involving cubics are similar

\[
\bar{u}' \frac{im_\pi C}{F^2} (q \cdot \gamma) \gamma_5 \tau_b \frac{m_N - \gamma_0 (m_N + m_\pi)}{-m_\pi^2 + 2p \cdot q} (-q \cdot \gamma \gamma_5 \tau_a u) \\
\rightarrow \bar{u}' \frac{im_\pi C}{F^2} (m_\pi \gamma_0) \gamma_5 \tau_b \frac{m_N + \gamma_0 (m_N + m_\pi)}{-m_\pi^2 - 2m_N m_\pi} (m_\pi \gamma_0 \gamma_5 \tau_a u) \\
= \bar{u}' \frac{im_\pi C}{F^2} \gamma_5 \tau_a \frac{m_N - \gamma_0 (m_N + m_\pi)}{-m_\pi^2 - 2m_N m_\pi} m_\pi \tau_b u \\
= \bar{u}' \frac{im_\pi C}{F^2} \gamma_5 \tau_a \frac{m_\pi^2}{m_\pi + 2m_N} \tau_b u
\]

(25.100)

These four diagrams together give at threshold

\[
\frac{im_\pi C}{F^2} \bar{u}' [\tau_b, \tau_a] u (1 + O \left( \frac{m_\pi}{m_N} \right)) = 2 \frac{m_\pi C}{F^2} \epsilon_{abc} \bar{u}' \tau_c u (1 + O \left( \frac{m_\pi}{m_N} \right))
\]

(25.100)

This is still linear in $m_\pi$, but we have one more diagram linear in $C$, the four particle vertex

\[
\bar{u}' \frac{C}{F^2} \epsilon_{abc} \bar{u}' (iq' - iq) \cdot \gamma_5 \tau_a \to -2 \frac{m_\pi C}{F^2} \epsilon_{abc} \bar{u}' \tau_c u
\]

(25.101)

which cancels the terms linear in $m_\pi$ from the other four diagrams. The upshot is that $C = (g_A - 1)/2$ only enters low energy pion nucleon scattering at order $m_\pi^2$. So the $C = 0$ results are the leading contribution. To recapitulate, we have obtained

\[
\mathcal{M}_{\pi N} = \frac{m_\pi}{F^2} \epsilon_{abc} \bar{u}' \tau_c u (1 + O(m_\pi/m_N))
\]

(25.102)

This result was first obtained by Steven Weinberg. To interpret the isospin dependence, we recall that the nucleon is an iso doublet $I = 1/2$ and the pion is an isovector. The pion-nucleon system can then have $I = 3/2, 1/2$. We can then interpret $\epsilon_{abc} = i(T_c)_{ab}$, with $T$ the
isospin generator for $I = 1$. Then we have
\[
(T_c + t_c)^2 = 2 + \frac{3}{4} + 2T_c t_c = \begin{cases} 
15/4 & I = 3/2 \\
3/4 & I = 1/2 
\end{cases}
\]
so we can also cast the results as
\[
\mathcal{M}_{3/2} = -i\frac{m_\pi m_N}{F^2}, \quad \mathcal{M}_{1/2} = 2i\frac{m_\pi m_N}{F^2}
\]
(25.104)
The conventional $\pi N$ scattering amplitude $f$ is related to $\mathcal{M}$ by
\[
f = -i\frac{\mathcal{M}}{8\pi \sqrt{s}} \rightarrow -i\frac{\mathcal{M}}{8\pi (m_\pi + m_N)}
\]
at threshold. At 0 energy $f$ is also known as the scattering length $a$. With Weinberg’s prediction $a_{3/2} = -m_\pi/(8\pi F^2) \approx -0.075/m_\pi$ and $a_{1/2} = m_\pi/(4\pi F^2) \approx +0.15/m_\pi$ compared to $-0.101 \pm .004$ and $0.173 \pm .003$.

### 25.3.2 Dispersion relation for $\pi N$ scattering

Scattering amplitudes seem to be analytic functions of the energy. This is demonstrably true for non relativistic potential scattering and is also true in QFT order by order in perturbation theory. What is less rigorously known is the location of all possible singularities and the high energy behavior of scattering amplitudes. Experience with perturbation theory associates the energy behavior of scattering amplitudes. This is demonstrably true for non relativistic potential scattering and is also true in QFT order by order in perturbation theory. What is less rigorously known is the location of all possible singularities and the high energy behavior of scattering amplitudes. Experience with perturbation theory associates the energy behavior of scattering amplitudes.

We first specify the kinematics of the process. Let $p^\mu$, $q^\mu$ be the initial momenta of the nucleon and pion respectively. In the forward direction these are also the final momenta. The Mandelstam invariants are $s = -(p + q)^2$, $t = 0$, and $u = -(p - q)^2$. $s$ and $u$ can be expressed in terms of the invariant $p \cdot q \equiv -m_N v$:
\[
s = m_\pi^2 + m_N^2 + 2m_N v, \quad u = m_\pi^2 + m_N^2 - 2m_N v
\]
(25.105)
The energy variable $v$ is the energy of the pion in the nucleon rest frame. For simplicity we shall assume isospin symmetry is exact $m_u = m_d$. We use the Cartesian basis for the pion isospin $a_\alpha$, $a = 1, 2, 3$. Then $s$-$u$ crossing symmetry says that $\mathcal{M}_{a\alpha}(v) = \mathcal{M}_{a\alpha}(-v)$.

The threshold for $\pi N$ scattering in the $s$-channel is at $s = (m_\pi + m_N)^2$ which translates to $v = m_\pi$. Because of crossing symmetry this threshold singularity also appears in the $u$-channel $u = (m_\pi + m_N)^2$ or $v = -m_\pi$. Thus we cut the $v$ plane from $v = m_\pi$ to $v = \infty$ and from $v = -\infty$ to $v = -m_\pi$. The poles corresponding to the stable nucleon are at $s = m_N^2$ or at $v = -m_\pi^2/(2m_N)$ and at $u = m_N^2$ or $v = m_\pi^2/(2m_N)$.
Derivation of Adler–Weinberg Relation
\[ g_A^2 = 1 + \int_0^\infty \frac{d\nu}{\nu} \frac{s_{\nu,\nu} - s_{\nu,\nu} - \nu}{\nu^2} \approx 1.2 \]

\[ t = 0 \quad S = \nu \cdot (\nu + q) = m_N^2 + m_N^2 \quad F^2 m_N \nu \]
\[ u = -i (\nu - q) = m_N^2 + m_N^2 \quad q \cdot 2 m_N \nu \]
\[ m_N \nu = -\nu \cdot q \]
\[ \nu_0 = \nu_{threshold} = m_N \]

1) Analyticity: \( M_{ba}(\nu) \) analytic in cut complex \( \nu \) plane
2) Causality Symmetry: \( \nu \leq 0 \) and \( \nu \leq -\nu \):
\[ M_{0a}(\nu) = M_{ab}(\nu) \quad \text{for} \quad \nu \leq 0 \]

Nucleon pole:
\[ S = m_N^2 \quad \nu = \frac{m_N^2}{2m_N \nu} \]
\[ u = m_N^2 \quad \nu = \frac{m_N^2}{2m_N \nu} \]

Parametrize Pole Terms:
\[ (i\gamma^\mu)(M) \quad \tilde{u} \quad i\gamma^a T_b \quad -i \frac{(m_N - \nu)(\nu - \nu_0)}{m_N^2 - (\nu - \nu_0)^2} \quad i \gamma^a T_a \quad u \]

\[ = -i g_{\mu N}^2 \tilde{u} (p) T_b T_a \frac{m_N + (\nu - \nu_0)}{m_N^2 - (\nu - \nu_0)^2} u(p) \]
\[ = -i g_{\mu N}^2 \frac{1}{m_N^2 - (\nu - \nu_0)^2} R(p) P_T \tilde{u} (p) T_b T_a \tilde{u} (p) \]
\[ = -i g_{\mu N}^2 \frac{2m_N^2}{m_N^2 - (\nu - \nu_0)^2} T_b T_a \tilde{u} (p) \]
\[ = -i g_{\mu N}^2 \frac{\nu}{m_N^2 - (\nu - \nu_0)^2} T_b T_a \tilde{u} (p) \]
To parameterize the pole singularities we write down the Feynman amplitude for the tree process, with the physical values for the pion-nucleon coupling and nucleon and pion masses. The $s$-channel pole is

\[
(iG_{\pi N})^2 \bar{u} i\gamma_5 \tau_a \frac{-i(m_N - \gamma \cdot (p + q))}{m_N^2 + (p + q)^2} u = G_{\pi N}^2 \bar{u} i\gamma_5 \tau_a u
\]

\[
= \frac{-i(m_N + \gamma \cdot (p + q))}{m_N^2 - 2m_N \nu} \tau_b \tau_a u = \frac{iG_{\pi N}^2 \gamma \cdot q}{m_N^2 + 2m_N \nu} \tau_b \tau_a u,
\]

\[
= \frac{-iG_{\pi N}^2 2m_N \nu}{m_N^2 + 2m_N \nu} \tau_b \tau_a \rightarrow \frac{iG_{\pi N}^2 m_N^2}{m_N^2 + 2m_N \nu} \phi^\dagger \tau_b \tau_a \phi
\]

(25.106)

where in the last step we replaced $2m_N \nu$ in the numerator by its value $-m_N^2$ at the pole. In the previous step we used $\bar{u} \gamma^\mu u = 2\mu$ for the forward spinor matrix element. In the final expression $\phi$ is the isospin spinor for the initial nucleon. The $u$-channel nucleon pole is obtained by the substitution $\nu, a, b \rightarrow -\nu, b, a$. The sum of the two pole terms is

\[
\frac{iG_{\pi N}^2 m_N^2}{m_N^2 + 2m_N \nu} \phi^\dagger \tau_b \tau_a \phi + \frac{iG_{\pi N}^2 m_N^2}{m_N^2 - 2m_N \nu} \phi^\dagger \tau_a \tau_b \phi
\]

\[
= \frac{iG_{\pi N}^2 m_N^2}{m_N^4 - 4m_N^2 \nu^2} \phi^\dagger \left[ \tau_b \tau_a (m_N^2 - 2m_N \nu) + \tau_a \tau_b (m_N^2 + 2m_N \nu) \right] \phi
\]

\[
= \frac{2iG_{\pi N}^2 m_N^2}{m_N^4 - 4m_N^2 \nu^2} \phi^\dagger \left[ m_N^2 \delta_{ab} - m_N \nu [\tau_b, \tau_a] \right] \phi
\]

(25.107)

We next use Cauchy’s theorem to derive a dispersion relation. Pick a point $\nu$ in the complex $\nu$ plane and write

\[
\mathcal{M}_{ba}(\nu) = \frac{1}{2\pi i} \oint_C \frac{d\nu'}{\nu' - \nu} \mathcal{M}_{ba}(\nu')
\]

(25.108)

Here $C$ is a closed contour encircling $\nu$ in a counter-clockwise direction and entirely within the domain of analyticity of $\mathcal{M}$. Now deform $C$ until it hugs all the singularities outside of $C$. Assuming maximal analyticity, the final contour will consist of large semicircles at infinity, clockwise circles about the isolated poles, A contour just below the positive real axis from $+\infty$ to $m_\pi$ encircling $m_\pi$ and returning to $+\infty$ just above the real axis. And finally a similar contour running from $-\infty$ to $-m_\pi$ just above the real axis and back to $-\infty$ just below the real axis. Assuming the large semicircles give nothing we arrive at the formula

\[
\mathcal{M}_{ba}(\nu) = \frac{2iG_{\pi N}^2 m_N^2}{m_N^4 - 4m_N^2 \nu^2} \phi^\dagger \left[ m_N^2 \delta_{ab} - m_N \nu [\tau_b, \tau_a] \right] \phi
\]

\[
+ \frac{1}{2\pi i} \int_{+\infty}^{m_\pi} \frac{d\nu'}{\nu' - \nu} \left[ \mathcal{M}_{ba}(\nu' + i\epsilon) - \mathcal{M}_{ba}(\nu' - i\epsilon) \right]
\]

\[
+ \frac{1}{2\pi i} \int_{m_\pi}^{-m_\pi} \frac{d\nu'}{\nu' - \nu} \left[ \mathcal{M}_{ba}(\nu' + i\epsilon) - \mathcal{M}_{ba}(\nu' - i\epsilon) \right]
\]

(25.109)
We can change variables in the integral of the last line \( \nu' \to -\nu' \) so that both integrals go over the same region, and also use crossing symmetry \( M_{ba}(-\nu) = M_{ab}(\nu) \):

\[
M_{ba}(\nu) = \frac{2iG^{2}m_{N}^{2}}{m_{N}^{2} - 4m_{N}^{2}\nu^{2}} \phi^{\dagger} \left[ m_{\pi}^{2}\delta_{ab} - m_{N}\nu[\tau_{b}, \tau_{a}] \right] \phi - 2iG^{2}\pi N\phi^{\dagger}\delta_{ab}\phi \\
+ \frac{1}{2\pi i} \int_{m_{x}}^{\infty} \frac{d\nu'(\nu' + \nu)}{\nu'(\nu'^{2} - \nu^{2})} [M_{ba}(\nu' + i\epsilon) - M_{ba}(\nu' - i\epsilon)] \\
+ \frac{1}{2\pi i} \int_{m_{x}}^{\infty} \frac{d\nu'(\nu' - \nu)}{\nu'(\nu'^{2} - \nu^{2})} [-M_{ab}(\nu' - i\epsilon) + M_{ab}(\nu' + i\epsilon)]
\]

(25.110)

In obtaining this result we made an assumption that the semi-circular contours at infinity contributed nothing: we assumed \( M_{ba} \to 0 \) as \( \nu \to \infty \). What if this doesn’t hold? Then we can write a dispersion relation for \( \mathcal{M}/\nu^{n} \) instead, where \( n \) is large enough to drop the semicircular contours. The price paid for this extra convergence is a less powerful dispersion relation, because the new pole at \( \nu = 0 \) introduces terms involving \( \mathcal{M} \) and up to \( n - 1 \) derivatives of \( \mathcal{M} \). Experimental indications are that \( n = 1 \) should suffice. The resulting dispersion relation is referred to as a subtracted dispersion relation, because it amounts to starting with the unsubtracted dispersion relation and subtracting from both sides the first few terms of its Taylor series about \( \nu = 0 \).

Let's write the dispersion relation with 1 subtraction \( (n = 1) \):

\[
\mathcal{M}_{ba}(\nu) - \mathcal{M}_{ba}(0) = \frac{2iG^{2}\pi Nm_{\pi}^{2}}{m_{\pi}^{4} - 4m_{\pi}^{2}\nu^{2}} \phi^{\dagger} \left[ m_{\pi}^{2}\delta_{ab} - m_{N}\nu[\tau_{b}, \tau_{a}] \right] \phi - 2iG^{2}\pi N\phi^{\dagger}\delta_{ab}\phi \\
+ \frac{1}{2\pi i} \int_{m_{x}}^{\infty} \frac{d\nu'(\nu' + \nu)}{\nu'(\nu'^{2} - \nu^{2})} [M_{ba}(\nu' + i\epsilon) - M_{ba}(\nu' - i\epsilon)] \\
+ \frac{1}{2\pi i} \int_{m_{x}}^{\infty} \frac{d\nu'(\nu' - \nu)}{\nu'(\nu'^{2} - \nu^{2})} [M_{ab}(\nu' + i\epsilon) - M_{ab}(\nu' - i\epsilon)]
\]

(25.111)

It is noteworthy that one can write separate dispersion formulas for the parts of \( \mathcal{M} \) even and odd under \( \nu \to -\nu \).

\[
M_{ba}^{+}(\nu) = \mathcal{M}_{ba}(0) + \frac{2iG^{2}\pi Nm_{\pi}^{2}}{m_{\pi}^{4} - 4m_{\pi}^{2}\nu^{2}} \phi^{\dagger} \left[ m_{\pi}^{2}\delta_{ab} \right] \phi - 2iG^{2}\pi N\phi^{\dagger}\delta_{ab}\phi \\
+ \frac{1}{2\pi i} \int_{m_{x}}^{\infty} \frac{d\nu'(\nu'^{2} - \nu^{2})}{\nu'(\nu'^{2} - \nu^{2})} [M_{ba}(\nu' + i\epsilon) - M_{ba}(\nu' - i\epsilon)] \\
+ \frac{1}{2\pi i} \int_{m_{x}}^{\infty} \frac{d\nu'(\nu'^{2} - \nu^{2})}{\nu'(\nu'^{2} - \nu^{2})} [M_{ab}(\nu' + i\epsilon) - M_{ab}(\nu' - i\epsilon)]
\]

(25.112)

\[
M_{ba}^{-}(\nu) = \frac{2iG^{2}\pi Nm_{\pi}^{2}}{m_{\pi}^{4} - 4m_{\pi}^{2}\nu^{2}} \phi^{\dagger} \left[ -m_{N}\nu[\tau_{b}, \tau_{a}] \right] \phi \\
+ \frac{1}{2\pi i} \int_{m_{x}}^{\infty} \frac{d\nu'(\nu'^{2} - \nu^{2})}{\nu'^{2} - \nu^{2}} [M_{ba}(\nu' + i\epsilon) - M_{ba}(\nu' - i\epsilon)] \\
- \frac{1}{2\pi i} \int_{m_{x}}^{\infty} \frac{d\nu'(\nu'^{2} - \nu^{2})}{\nu'^{2} - \nu^{2}} [M_{ab}(\nu' + i\epsilon) - M_{ab}(\nu' - i\epsilon)]
\]

(25.113)
The final step is to use the optical theorem to relate the discontinuities $\mathcal{M}_{ba}(\nu' + i\epsilon) - \mathcal{M}_{ba}(\nu' - i\epsilon)$ to pion nucleon total cross sections. The optical theorem is a consequence of unitarity of the $S$-matrix. We write $S = I + M$ and impose $S^\dagger S = I$:

$$\langle f|M^\dagger|i \rangle + \langle f|M|i \rangle = -\langle f|M^\dagger M|i \rangle$$ (25.114)

The relation of $M$ to the Feynman amplitude is given by

$$\langle f|M|i \rangle = (2\pi)^4 \delta(P_f - P_i) \mathcal{M}_{fi} \prod_{k \in i, f} \sqrt{\frac{1}{(2\pi)^3 2E_k}}$$ (25.115)

Put this into the unitarity equation:

$$\mathcal{M}^*_i + \mathcal{M}_{ii} = -\sum_f \prod_{k \in f} \frac{1}{(2\pi)^3 2E_k} (2\pi)^4 \delta(P_f - P_i)|\mathcal{M}_{fi}|^2$$ (25.116)

$$= -\sigma_i^{\text{total}} 4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2| = -4|E_2\mathbf{p}_1 - E_1\mathbf{p}_2|\sigma_i^{\text{total}}$$

$$= -4|E_2\mathbf{p}_1 - E_1\mathbf{p}_2|\sigma_i^{\text{total}} \rightarrow -4m_2|\mathbf{p}_1|\sigma_i^{\text{total}}$$ (25.117)

where the last line is in the Lab frame. $\mathcal{M}$ has the property of being imaginary on the real axis between the branch points. This property extends to the condition $\mathcal{M}(\nu^*) = -\mathcal{M}(\nu^*)$ off the real axis, so one can write the discontinuity as

$$\mathcal{M}_{ii}(\nu' + i\epsilon) - \mathcal{M}_{ii}(\nu' - i\epsilon) = \mathcal{M}_{ii}(\nu' + i\epsilon) + \mathcal{M}_{ii}(\nu' + i\epsilon)^* = -4m_2|\mathbf{p}_1|\sigma_i^{\text{total}}$$ (25.118)

by the optical theorem. It is important for the optical theorem that the left side involves forward amplitudes. In the Cartesian basis that would mean $a = b$. But the charged pion state is $|\pi^\pm \rangle = (|\pi_1 \rangle \pm i|\pi_2 \rangle)/\sqrt{2}$ in the initial state, but $\langle \pi^\pm | = (|\pi_1 | \mp i|\pi_2 |)/\sqrt{2}$ in the final state. So the desired forward amplitude is actually $\mathcal{M}_{1+i,2+i2}/2$. The upper sign is the amplitude for $\pi^+N$ scattering and the lower sign for $\pi^-N$ scattering.

So we can now give the dispersion relation for $\pi^+N$ scattering by choosing $b = (1 - i2)/\sqrt{2}$ and $a = (1 + i2)/\sqrt{2}$ for which $\delta_{ab} = 1$ and $[\gamma_5, \gamma_0] = i([\gamma_1, \gamma_2] - [\gamma_2, \gamma_1])/2 = -2\gamma_3$.

$$\mathcal{M}^+_{\pi}(\nu) = \mathcal{M}_{\pi^+}(0) + \frac{2iG^2_{\pi N}m_\pi^2}{m_\pi^4 - 4m_N^2v^2}\phi^\dagger [m_\pi^2] \phi - 2iG^2_{\pi N}\phi^\dagger \phi$$

$$+ \frac{-4m_N}{2\pi i} \int_{m_\pi}^\infty \frac{dv'v^2}{v'} \sqrt{v^2 - m_\pi^2} [\sigma_{\pi^+} + \sigma_{\pi^-}]$$ (25.119)

$$\mathcal{M}^-_{\pi}(\nu) = \frac{2iG^2_{\pi N}m_\pi^2}{m_\pi^4 - 4m_N^2v^2}\phi^\dagger [2m_Nv\gamma_3] \phi$$

$$+ \frac{-4m_N}{2\pi i} \int_{m_\pi}^\infty \frac{dv'v\sqrt{v^2 - m_\pi^2}}{v^2 - v^2} [\sigma_{\pi^+} - \sigma_{\pi^-}]$$ (25.120)
25.3.3 Adler-Weisberger sum rule

Although the original derivation of the AW sum rule was somewhat different, it is illuminating, as stressed by S. Weinberg, to obtain it by combining the soft pion results with the dispersion relation for pion nucleon scattering. The former gave the threshold limit

\[ M_{ba} = \frac{m_{\pi}}{2F^2} \epsilon_{abc} \tilde{u}^* \tau_c u = \frac{m_{\pi}m_N}{F^2} \epsilon_{abc} \phi^\dagger \tau_c \phi = \frac{im_{\pi}m_N}{2F^2} \phi^\dagger [\tau_b, \tau_a] \phi \]  

(25.121)

which gives the low energy limit of \( M_{ba} \).

On the other hand the threshold limit of the dispersion relation is

\[ m_{\pi}^2 G_{\pi N}^2 \phi^\dagger [m_{\pi}^2 \delta_{ab}] \phi - 2iG_{\pi N}^2 \phi^\dagger \delta_{ab} \phi \]

\[ + \frac{1}{2\pi i} \int_{m_{\pi}}^\infty \frac{d
u'/m_{\pi}}{\nu' (\nu'^2 - m_{\pi}^2)} [M_{ba}(\nu' + i\epsilon) - M_{ba}(\nu' - i\epsilon)] \]

\[ + \frac{1}{2\pi i} \int_{m_{\pi}}^\infty \frac{d
u'/m_{\pi}}{\nu' (\nu'^2 - m_{\pi}^2)} [M_{ab}(\nu' + i\epsilon) - M_{ab}(\nu' - i\epsilon)] \]

(25.122)

\[ M_{ba}(m_{\pi}) = \frac{2iG_{\pi N}^2}{m_{\pi}^2 - 4m_N^2} \phi^\dagger [-m_N m_{\pi} [\tau_b, \tau_a]] \phi \]

\[ + \frac{1}{2\pi i} \int_{m_{\pi}}^\infty \frac{d
u'/m_{\pi}}{\nu' (\nu'^2 - m_{\pi}^2)} [M_{ba}(\nu' + i\epsilon) - M_{ba}(\nu' - i\epsilon)] \]

\[ - \frac{1}{2\pi i} \int_{m_{\pi}}^\infty \frac{d
u'/m_{\pi}}{\nu'^2 - m_{\pi}^2} [M_{ab}(\nu' + i\epsilon) - M_{ab}(\nu' - i\epsilon)] \]

(25.123)

the celebrated Adler-Weisberger sum rule follows by setting the left side of the second sum rule to its low energy prediction:

\[ \frac{im_{\pi}m_N}{2F^2} \phi^\dagger [\tau_b, \tau_a] \phi = \frac{ig_{\pi N}^2 m_{\pi} m_N}{2F^2} \phi^\dagger [\tau_b, \tau_a] \phi \]

\[ + \frac{1}{2\pi i} \int_{m_{\pi}}^\infty \frac{d
u'/m_{\pi}}{\nu'^2 - m_{\pi}^2} [M_{ba}(\nu' + i\epsilon) - M_{ba}(\nu' - i\epsilon)] \]

\[ - \frac{1}{2\pi i} \int_{m_{\pi}}^\infty \frac{d
u'/m_{\pi}}{\nu'^2 - m_{\pi}^2} [M_{ab}(\nu' + i\epsilon) - M_{ab}(\nu' - i\epsilon)] \]

(25.124)

where we neglected \( m_{\pi}^2 \) compared to \( m_N^2 \) and used the Goldberger-Treiman relation \( G_{\pi N} = g_A m_N / F \). Now apply the optical theorem to the case for \( \pi^+ N \) scattering leads to

\[ \frac{im_{\pi}m_N}{2F^2} \phi^\dagger (-2\tau_3) \phi = \frac{ig_{\pi N}^2 m_{\pi} m_N}{2F^2} \phi^\dagger (-2\tau_3) \phi \]

\[ + \frac{4m_N m_{\pi}}{2\pi i} \int_{m_{\pi}}^\infty \frac{d
u'}{\sqrt{\nu'^2 - m_{\pi}^2}} [\sigma_{\pi^+} - \sigma_{\pi^-}] \]

(25.125)

Or, rearranging,

\[ (g_A^2 - 1) \phi^\dagger \tau_3 \phi = \frac{2F^2}{\pi} \int_{m_{\pi}}^\infty \frac{d
u'}{\sqrt{\nu'^2 - m_{\pi}^2}} (\sigma_{\pi^+} - \sigma_{\pi^-}) \]

(25.126)
If the nucleon is a proton $\phi^+ \tau_3 \phi = +1$ so in that case we have

$$g_A^2 = 1 + \frac{2F^2}{\pi} \int_{m_\pi}^{\infty} \frac{dv'}{\sqrt{v'^2 - m_\pi^2}} (\sigma_{\pi^+} - \sigma_{\pi^-}) , \quad A - W \tag{25.127}$$

If the nucleon is a neutron $\phi^+ \tau_3 \phi = -1$ and the formula reads

$$g_A^2 = 1 - \frac{2F^2}{\pi} \int_{m_\pi}^{\infty} \frac{dv'}{\sqrt{v'^2 - m_\pi^2}} (\sigma_{\pi^+} + \sigma_{\pi^-}) , \quad A - W \tag{25.128}$$

Scattering on a neutron in isolation would be nearly impossible, but could be disentangled from neutron rich nuclei. In any case isospin invariance predicts (Exercise) that the replacement $\sigma_{\pi^+} - \sigma_{\pi^-} \rightarrow \sigma_{\pi^-} - \sigma_{\pi^+}$ on the right side would be equally valid.

## 25.4 Effective Lagrangians

### 25.4.1 Loop effects from chiral Lagrangian

By construction the tree diagrams of the exact effective action include all quantum effects due to loops. However in our treatment of spontaneous symmetry breakdown, we made use of a low energy approximation by expanding the effective action in powers of derivatives. However, this expansion would be technically invalid if the Green functions at low energies contains IR logarithms. In the presence of SSB of chiral symmetry these are called chiral logs. One can either abandon the derivative expansion idea or reinterpret it.

The idea of the reinterpretation is that such IR logs can be taken into account by expanding the Lagrangian in powers of derivatives, but treat the massless fields e.g. NGB’s as full quantum fields, including multiloop diagrams, which will supply the IR logs. That is we write

$$e^{i\Gamma(\phi)} = \int D\phi e^{i\int d^4x L_{\text{eff}}(\phi)} \tag{25.129}$$

where $L_{\text{eff}}$ is the most general power series in derivatives of $\phi$ consistent with the symmetries. Such a definition produces a nonrenormalizable QFT, with an infinite number of counterterms. But the IR logs are controlled by low energy couplings of massless particles and their leading behavior is not subject to the infinite ambiguity.

### 25.4.2 Baryons and $SU(3) \times SU(3)$.

We have devoted considerable time to the case of $n_f = 2$ massless quarks for which the lightest baryons (3 quark hadrons) are the proton and neutron with spin 1/2 which belong to a doublet under $SU(2)$, as do the $u,d$ quarks. This coincidence leads to a relatively simple effective Lagrangian for NGB’s interacting with baryons. In the case of $n_f = 3$
massless quarks, where the chiral symmetry is $SU(3) \times SU(3)$ the lightest spin 1/2 baryons fall into an octet of $SU(3)$:

$$J^P = (1/2)^+ : \ p, n, \Lambda^0, \Sigma^{\pm, 0}, \Xi^{-, 0}$$

(25.130)

The next heavier baryons have spin 3/2 and fall into a decuplet:

$$J^P = (3/2)^+ : \Delta^{+++, 0, -}, \Sigma^{\pm, 0*}, \Xi^{0, -}, \Omega$$

(25.131)

In the quark model the proton is $uud$ the neutron is $udd$, the $\Lambda^0$ is $uds$ the $\Sigma$’s are $uus, uds, dds$, the $\Xi$’s are $uss, dss$, the $\Delta$’s are $uuu, uud, udd, ddd$, the $\Sigma^*$’s are $uus, uds, dds$, the $\Xi^*$’s are $uss, dss$ and the $\Omega^-$ is $sss$. The masses of the baryons grow with the number of strange quarks due to the small ratio $m_{u,d}/m_s \ll 1$.

As we have seen in the meson sector the lightest odd parity spin zero mesons, which serve as the NGB’s, fall into an octet. In the quark model there are actually nine $\bar{q}q$ mesons, the ninth being an $SU(3)$ singlet. We have for spin 0 mesons

$$J^P = 0^- : \ K^{+, 0}, \pi^{\pm, 0}, \eta^0, \bar{K}^{-, 0}$$

(25.132)

with the singlet denoted $\eta'_0$. Its mass is significantly heavier than the octet masses, making it a poor NGB. We shall see that the $U_A(1)$ “symmetry” is actually broken by anomalies (quantum effects. For the vector mesons

$$J^P = 1^- : \ K^{+, 0*}, \rho^{\pm, 0}, \omega^0, \bar{K}^{-, 0*}$$

(25.133)

The ninth vector meson $\phi^0$ has properties consistent with an $\bar{s}s$ bound state, with the $\omega^0$ with properties as if made up only of up and down quarks.

In terms of quarks, the $K, K^*$ are $\bar{s}u, \bar{s}d$, the $\bar{K}, \bar{K}^*$ are $\bar{u}s, \bar{d}s$, the $\pi, \rho$ are $\bar{u}d, \bar{u}d - \bar{d}d, \bar{d}u$. However the $\eta, \eta'$ system is very different from the $\omega, \phi$ system: the $\eta'$ is nearly an $SU(3)$ singlet ($\bar{u}u + \bar{d}d + \bar{s}s$)), with the $\eta$ nearly a member of an $SU(3)$ octet ($\bar{u}u + \bar{d}d - 2\bar{s}s$)). In contrast the $\omega$ behaves like $\bar{u}u + \bar{d}d$ and the $\phi$ like $\bar{s}s$. 

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Meerss  \( I = \frac{1}{2} \), \( I = 1 \), \( I = 0 \), \( I = \frac{1}{2} \)

0\(^-\) K\(^{\pm}\), \( \pi^\pm \), \( \eta \), \( \eta' \) \( \text{octet} + \eta' \)

1\(^-\) \( K^* \), \( \rho^\pm \), \( \omega \), \( \phi \) \( \text{octet} + \phi \)

No \( U_L = U_R \) multiplets: \( SU(3) \times SU(3) \rightarrow SU(3) \)

8 quark generators \( \rightarrow 8 \) pseudoscalars, \( N - \phi \) bosons

\( = 0^- \), \( \text{octet} \),

Generators of \( SU(2) \)

\( t_1 = (1, 0) \), \( t_2 (0, -i) \), \( t_3 (i, 0) \)

Generators of \( SU(3) \)

\( \lambda_1, \lambda_2 \) \( \left( \begin{array}{c} \bar{u} \\ \bar{q} \\ \bar{d} \end{array} \right) \)

\( \lambda_1 = \left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right) \)

\( \lambda_3 = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \)

\( \lambda_5 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \)

\( \lambda_6 = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \)

\( \lambda_7 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \)

\( \lambda_8 = \frac{1}{\sqrt{3}} \left( \begin{array}{c} \bar{u} \\ 0 \\ -2 \bar{d} \end{array} \right) \) \( \alpha \gamma = \text{hypercharge} = \frac{S + \Delta}{2} \)

\( \lambda_4, \lambda_5, \frac{1}{2} (\lambda_3 + \sqrt{3} \lambda_8) \) \( U_5 \mu \nu \)

\( \lambda_6, \lambda_7, \frac{1}{2} (\lambda_3 - \sqrt{3} \lambda_8) \) \( V_5 \mu \nu \)

\[ \begin{array}{cccccc}
\eta^+ & 0 & 0 & 0 & 0 & 0 \\
0 & \eta & \eta' & 0 & 0 & 0 \\
0 & \eta' & \phi & 0 & 0 & 0 \\
0 & 0 & 0 & \text{octet} & \phi & 0 \\
0 & 0 & 0 & 0 & \text{octet} & \phi \\
\end{array} \]

\[ \frac{3}{2} \]
In the case $n_f = 2$ we introduced an isospin $1/2$ nucleon doublet field $N$ into the effective chiral lagrangian as in (25.89). For $n_f = 3$ we can represent the baryon octet fermion fields by traceless $3 \times 3$ matrices, $B_L, B_R$, with $L, R$ chiral transformations
\[ B_L \rightarrow U_L^\dagger B_L U_L, \quad B_R \rightarrow U_R^\dagger B_R U_R \] (25.134)
Then of course $\text{Tr} \bar{B}_L \gamma \cdot DB_L + \text{Tr} \bar{B}_R \gamma \cdot DB_R$ is invariant, but $\text{Tr} \bar{B}_L B_R$ is not. But recalling the transformation properties of our NGB effective field $U = e^{-i\lambda a \pi_a / F} \rightarrow U_L^\dagger U_R$, we find that $\text{Tr} U_L^\dagger \bar{B}_L U B_R$ is invariant. This last term gives an effective mass to the baryon even if the 3 quarks are exactly massless.

In considering the quark model of hadrons, we should keep in mind that if we consider the quarks as actual particles rather than as permanently confined constituents, the phenomenon of spontaneous breaking of chiral symmetry could give effective quark fields a mass, just like it gives the baryons mass. We could call this the constituent mass to distinguish it from the bare mass which is very small. If this constituent mass is of order $M_N/3$, these effective quarks could be treated nonrelativistically, and the nonrelativistic quark model could achieve some respectability.

### 25.5 Electroweak Processes involving NGB’s

We would like to extend our effective action description of NGB’s to include their electroweak decays. We start with the effective action for NGB’s discussed at the end of last semester.

\[ \mathcal{L}_{\text{eff}} = \frac{-F^2}{4} \text{Tr} \partial_\mu U L U^\dagger \partial^\mu U + \text{Tr} \mathcal{M}(U + U^\dagger) \] (25.135)

We need to express the axial and vector currents
\[ j_{a,\mu}^{a,L,R} = \bar{q} \gamma_\mu \lambda^a \frac{1 \pm \gamma_5}{2} q \] (25.136)
in terms of the NGB effective fields. We take guidance from the connection between the currents and chiral transformations. The trick is to make an $x$ dependent chiral transformation
\[ L, R \rightarrow L, R + i\epsilon_a L R (x) t^a L, R \] (25.137)
on the free quark Lagrangian
\[ \mathcal{L} = i(\bar{L} \gamma \cdot \partial L + \bar{R} \gamma \cdot \partial R) \rightarrow \mathcal{L} - (\bar{L} \gamma \cdot \partial t^a L + \bar{R} \gamma \cdot \partial t^a R) = \mathcal{L} - j_{a,\mu}^{a,L} \partial_\mu \epsilon_a^L - j_{a,\mu}^{a,R} \partial_\mu \epsilon_a^R \] (25.138)
The chiral order parameter is the expectation of the bilinear $\sigma^{lk} = \bar{L}^k R^l$ which transforms as
\[ \sigma^{lk} \rightarrow \sigma^{lk} - i\epsilon_L^{k} (t^a)_{k'l'} \sigma^{lk'} + i\epsilon_R^{l} (t^a)_{kl'} \sigma^{lk'} \]
\[ \sigma \rightarrow \sigma - i\sigma \epsilon_L^L + i\epsilon_R^R \sigma \] (25.139)
where $\epsilon^{L,R} = \epsilon^{R\ell a}$.

The effective field $U(x)$ for the NGB’s is obtained by postulating $\langle \sigma^{kl} \rangle = S_{dkl}$, and then doing a chiral transformation which changes $\langle \sigma^{kl} \rangle \to U_R \langle \sigma^{kl} \rangle U_L^\dagger = SU_R U_L^\dagger \equiv SU$. Letting $U$ depend on $x$ makes it the effective field. This identification shows that the chiral transformation on $U(x)$ is

$$U(x) \to U_R U(x) U_L^\dagger \to U(x) - iU\epsilon^L + i\epsilon^R U$$

(25.140)

for infinitesimal transformations. Then under $x$ dependent chiral transformations the effective Lagrangian

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}_{\text{eff}} - \frac{F^2}{4} \text{Tr} \left[ (U\partial_\mu U^\dagger - \partial_\mu U U^\dagger) i\partial^\mu \epsilon_R + (U^\dagger \partial_\mu U - \partial_\mu U^\dagger) i\partial^\mu \epsilon_L \right]$$

(25.141)

from which we read of the currents:

$$j^{a\mu}_L = \frac{F^2}{2} \text{Tr} U^\dagger \partial^\mu U^a$$

$$j^{a\mu}_R = -\frac{F^2}{2} \text{Tr} \partial^\mu U U^\dagger U^a$$

(25.142)

The matrix elements of these currents can then be used to calculate semileptonic processes involving any number of NGB’s. These processes could be decays or NGB scattering processes that include the emission or absorption of a lepton-antineutrino pair.

The calculation proceeds by expressing $U = e^{-i\sum_a \lambda_a \pi^a / F}$ and expanding the currents in powers of $\pi^a$. One then evaluates the tree approximation to the matrix element of this current using the effective Lagrangian.

$$U^\dagger \partial_\mu U = e^{i\lambda \pi / F} \partial_\mu e^{-i\lambda \pi / F} = \sum_{n=0}^\infty \frac{1}{(n+1)!} \Omega_n \left[ -i \frac{\lambda_a}{F} \partial_\mu \pi^a, -i \frac{\lambda_b}{F} \pi^b \right]$$

$$= -i \frac{\lambda_a}{F} \partial_\mu \pi^a + \frac{1}{2} \left[ -i \frac{\lambda_a}{F} \partial_\mu \pi^a, -i \frac{\lambda_b}{F} \pi^b \right] + \cdots$$

$$= -i \frac{\lambda_a}{F} \left( \partial_\mu \pi^a + f^{abc} \frac{1}{F} \partial_\mu \pi^b \pi^c \right) + \cdots$$

(25.143)

This quantity figures in the $j_L$. The corresponding quantity in $j_R$ is

$$-\partial_\mu U U^\dagger = U \partial_\mu U^\dagger$$

(25.144)

So once we have evaluated $j_L$ we get $j_R$ by the substitution $U \to U^\dagger$ or, what is the same thing, the substitution $\pi^a \to -\pi^a$. The first couple of terms in each are:

$$j^{a\mu}_L = \frac{F}{2} \left( \partial_\mu \pi^a + f^{abc} \frac{1}{F} \partial_\mu \pi^b \pi^c + \cdots \right)$$

$$j^{a\mu}_R = \frac{F}{2} \left( -\partial_\mu \pi^a + f^{abc} \frac{1}{F} \partial_\mu \pi^b \pi^c + \cdots \right)$$

(25.145)
The electroweak currents can be constructed as linear combinations of these. The linear
terms in \( \pi^a \) describe the processes \( \pi \to l\bar{\nu}_l \) and the quadratic terms describe \( \pi \to \pi l\bar{\nu}_l \). The
unwritten cubic terms would describe \( \pi \to \pi\pi l\bar{\nu}_l \) and so on.

### 25.5.1 The Problem of an Odd Number of NGB’s

Because the NGB’s are pseudoscalars, an amplitude involving an odd number of them has
to have a spacetime dependence linear in \( \epsilon^{\mu\nu\rho\sigma} \). For the case of chiral \( SU(2) \) we are able
to define a G-parity under which \( \pi^a \to -\pi^a \) which implies that all amplitudes with an
odd number of pions vanish. Thus the problem is avoided. But for chiral \( SU(3) \) this is
not possible: there are definitely strong interaction processes like \( KK \to \pi\pi\pi \) involving an
odd number of NGB’s. Unfortunately the effective Lagrangian we have been using has the
\( \pi \to -\pi \) symmetry, and there is no source for the epsilon tensor.

The escape from this problem was devised by Wess and Zumino who tried to build a
term with the structure:

\[
L_{WZ} = \epsilon^{\mu\nu\rho\sigma} \text{Tr} U \, \partial_\mu \, U \, U \, \partial_\nu \, U \, \partial_\rho \, U \, \partial_\sigma \, U \quad (25.146)
\]

Unfortunately, this term won’t do because it is a total derivative in even spacetime di-
mensions. If spacetime were odd dimensional the analogous structure would not be a total
derivative, and furthermore would produce terms with an odd number of NGB. The idea was
to extend the field \( U(x) \) to a 5 dimensional ball \( B_5 \) in such a way that the new term depended
only on the boundary values of the extended field. The Wess-Zumino term was the result,
but the full meaning of the construction was obscure until a clarification by Edward Witten.
The construction is commonly known as the WZW term, for Wess,Zumino,and Witten.

We start with spacetime continued to Euclidean space. If the point at infinity can be
adjoined, the space is \( S^4 \). This will be the case if \( U(x) \to U_0 \) independent of direction as
\( x \to \infty \). Then we can interpret \( U(x) \) as a mapping from \( S^4 \to SU(3) \). It is a topological fact
that \( \pi_4(SU(3)) = 0 \) meaning that the image of \( S^4 \) in \( SU(3) \) can be continuously deformed
to a point. This can be expressed by a function \( U(x,y) \) such that \( U(x,1) \) is the original field
on \( S^4 \) and \( U(x,0) = I \). The 5 dimensional space \( x,y \) has the topology of a disk \( D^5 \) with its
boundary \( S^4 \). Then the WZW term is the integral

\[
\Gamma_{WZW}^D = -\frac{i}{240\pi^2} \int_D d^5y \epsilon^{klmn} \text{Tr} U^{-1} \partial_j U \, U^{-1} \partial_k U \, U^{-1} \partial_l U \, U^{-1} \partial_m U \, U \, \partial_n U
\]

\[
\equiv \int_D d^5 y \omega(y) \quad (25.147)
\]

If the dimension were even, \( \omega \) would be a total derivative, and the integral would reduce to
one over the boundary of the disk. But, even though it is not a total derivative, its variation
under $U \rightarrow U + \delta U$ is a total derivative (Exercise), :

$$
\delta \Gamma_{WZW}^{D} = -\frac{i}{48\pi^2} \int_D d^5y e^{ijklmn} \partial_j (\text{Tr} U^{-1} \delta U U^{-1} \partial_k U U^{-1} \partial_l U U^{-1} \partial_m U U^{-1} \partial_n U)
$$

$$
= -\frac{i}{48\pi^2} \int_{S^4} d^4x e^{ijklmn} \text{Tr} U^{-1} \delta U U^{-1} \partial_k U U^{-1} \partial_l U U^{-1} \partial_m U U^{-1} \partial_n U
$$

(25.148)

The last line expresses the variation of the WZW term in terms of variations $\delta U$ restricted to $S^4$. In principle this variational equation could be used to define the WZW term referring only to 4 dimensional space time. The value of the 5 dimensional perspective is that it gives a way of writing down such a term explicitly.

Witten’s contribution was to recognize that, for topological reasons, $\Gamma_{WZW}$ must enter the path integral multiplied by an integer. The argument is subtle. The embedding of $S^4$ into $SU(3)$ can be extended into a 5 dimensional disk in at least two ways. Because $SU(3)$ is a compact manifold, $S^4$ can be shrunk to a point in the complement of $D^5$ forming a complementary disk $\bar{D}^5$. Gluing these two discs together at their boundary forms a topological sphere $S^5$. It is a known mathematical fact that $\pi_5(SU(3)) = Z$, meaning that topologically distinct mappings of $S^5$ into $SU(3)$ can be labelled by an integer winding number. This winding number is in fact

$$
w = \frac{1}{2\pi} \int_{S^5} d^5y \omega(y)
$$

$$
= \frac{1}{2\pi} \left[ \int_{D^5} d^5y \omega(y) - \int_{\bar{D}^5} d^5y \omega(y) \right] = \frac{1}{2\pi} \left[ \Gamma_{WZW}^{D} - \Gamma_{WZW}^{\bar{D}} \right]
$$

(25.149)

Thus if $\Gamma_{WZW}$ enters the path integrand as a factor

$$
e^{in\Gamma_{WZW}}
$$

(25.150)

with $n$ an integer the ambiguity in the choice of disks disappears, and the path integral is well-defined.

### 25.5.2 Electromagnetic couplings and the anomaly.

The charge matrix for the 3 light quarks is

$$
Q = \begin{pmatrix}
2/3 & 0 & 0 \\
0 & -1/3 & 0 \\
0 & 0 & -1/3
\end{pmatrix}
$$

(25.151)
Under gauge transformation $U \rightarrow U + i\epsilon [Q, U]$, we can exploit (25.148 to write
\[
\delta \Gamma_{WZW} = -\frac{i}{48\pi^2} \int d^4x \epsilon^{klmn} \text{Tr} U^{-1} i\epsilon [Q, U] U^{-1} \partial_k U U^{-1} \partial_l U U^{-1} \partial_m U U^{-1} \partial_n U
\]
\[
= -\frac{i}{48\pi^2} \int d^4x \epsilon^{klmn} \text{Tr} \epsilon (U^{-1} Q - QU^{-1}) \partial_k U U^{-1} \partial_l U U^{-1} \partial_m U U^{-1} \partial_n U
\]
\[
= \frac{1}{48\pi^2} \int d^4x \epsilon^{klmn} \epsilon(x) \partial_k \text{Tr} (U^{-1} Q + QU^{-1}) \partial_l U U^{-1} \partial_m U U^{-1} \partial_n U
\]
\[
= -\frac{1}{48\pi^2} \int d^4x \epsilon^{klmn} \partial_k \epsilon(x) \text{Tr} (U^{-1} Q + QU^{-1}) \partial_l U U^{-1} \partial_m U U^{-1} \partial_n U
\]

To prove the third line, simply carry out the derivative $\partial_k$ in that line, noticing that one only gets a contribution when the derivative hits one of the three $u^{-1}$ factors. The last two contributions cancel and the first just reproduces the second line.

Then to linear order in $A_\mu$ the construction
\[
\Gamma^1(A) = \Gamma_{WZW} - e \int d^4x A_\mu J^\mu
\]
(25.152)
\[
J^\mu = \frac{1}{48\pi^2} \epsilon^{klmn} \text{Tr} (U^{-1} Q + QU^{-1}) \partial_l UU^{-1} \partial_m U U^{-1} \partial_n U
\]
(25.153)
is gauge invariant under $A_\mu \rightarrow A_\mu - \partial_\mu \epsilon/e$ and $U \rightarrow U + i\epsilon [Q, U]$ to first order in $A$. To make it completely gauge invariant we need to cancel the change $\delta J^\mu$ under the gauge transform:
\[
\delta J^\mu = -\frac{1}{24\pi^2} \epsilon^{klmn} \partial_n \{ i \partial_l \epsilon \text{Tr} [Q^2 \partial_m UU^{-1} + Q^2 U^{-1} \partial_m U - QUQ \partial_m UU^{-1}] \}
\]
To see this first note that for constant $\epsilon$ the change vanishes because it amounts to the trace of a commutator. Thus one only gets a contribution when one of the three derivatives hit the $\epsilon$.

After an integration by parts, this gives then, after a lengthy bit of algebra,
\[
\delta \Gamma^1(A) = \frac{ie}{24\pi^2} \int d^4x \partial_\mu A_\mu \partial_\epsilon \epsilon^{nml} \text{Tr} [Q^2 \partial_m UU^{-1} + Q^2 U^{-1} \partial_m U + QUQ \partial_m UU^{-1}]
\]
(25.154)
It is easy to see that the trace on the last line is itself gauge invariant as is the factor $\partial_n A_\mu$ on the first line because of the epsilon symbol. Thus we can make the whole term gauge invariant by simply adding a term equal to the above expression with $e A_t$ substituted for $\partial_t \epsilon$.

\[
\Gamma_{WZW}(A) = \Gamma_{WZW} - e \int d^4x A_\mu J^\mu + \frac{ie^2}{24\pi^2} \int d^4x \partial_\mu A_\mu A_t \epsilon^{nml} \text{Tr} [Q^2 \partial_m UU^{-1} + Q^2 U^{-1} \partial_m U + QUQ \partial_m UU^{-1}]
\]
(25.155)
Then the complete chirally invariant effective action for NGB’s interacting with electromagnetism is (with $D_\mu U = \partial_\mu U - ieA_\mu [Q, U]$)

$$S_{\text{eff}} = -\frac{F^2}{4} \int d^4x \text{Tr}D_\mu U^\dagger D^\mu U + n\Gamma_{WZW}(A)$$ (25.156)

The quadratic term in $A$ in particular predicts the decay $\pi_0 \to \gamma\gamma$, and comparing it to our earlier calculation determines $n = N_c \to 3$ for the standard model. The WZW term represents the anomaly solely in terms of NGB’s.

To extract the amplitude for $\pi^0, \eta \to 2\gamma$, we just need to put $U = I + (1/F)\pi^a\lambda_a$, $\partial_m U = i\lambda_a\partial_m \pi^a$ and work out the linear terms in $\pi^a$. The trace is

$$\frac{3i}{F} \partial_m \text{Tr}Q^2 \pi^a\lambda_a = \frac{3i}{F} \partial_m \left[ \frac{4}{9} \left( \pi^0 + \frac{\eta}{\sqrt{3}} \right) + \frac{1}{9} \left( -\pi^0 + \frac{\eta}{\sqrt{3}} \right) - \frac{2\eta}{9\sqrt{3}} \right]$$

so the relevant term in the effective action is

$$\frac{e^2}{24\pi^2 F} \int d^4x \partial_n A_\mu \partial_m A_\rho \epsilon^{\mu\nu\rho\sigma} \text{Tr}F_{\mu\nu} F_{\rho\sigma}$$ (25.157)

According to Witten this term should be multiplied by an integer $n$, which by comparison with our anomaly calculation (25.87) should be $n = N_c$.

### 25.6 Anomalies, Instantons, and the $U(1)$ Problem

When we discussed the spontaneous breaking of chiral symmetry in QCD with massless quarks, we assumed that the symmetry was $SU(n_F) \times SU(n_F)$ for which the conserved currents are

$$j^\mu_a = \bar{q}\gamma^\mu \frac{\lambda_a}{2} q, \quad j^\mu_5 = \bar{q}\gamma^\mu \gamma^5 q$$ (25.159)

where $\lambda_a$ are the Gell-Mann $SU(3)$ matrices (for $n_F = 3$) or the Pauli matrices for $n_F = 2$. But there are also the $U(1)$ currents

$$j^\mu = \bar{q}\gamma^\mu q = \sum_{k=1}^{n_F} \bar{q}_k \gamma^\mu q_k, \quad j^\mu_5 = \bar{q}\gamma^\mu \gamma^5 q = \sum_{k=1}^{n_F} \bar{q}_k \gamma_\mu \gamma^5 q_k$$ (25.160)

which also appear to be conserved for massless quarks. The vector current is indeed conserved and $j^\mu_5/3$ can be identified as the current of baryon number. But we have learned that axial currents can have anomalies and in fact the anomaly for $j^\mu_5$ is given by

$$\partial_\mu j^\mu_5 = \frac{g_3^n n_F}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr}F_{\mu\nu} F_{\rho\sigma} = \frac{g_3^n n_F}{8\pi^2} \text{Tr}F_{\mu\nu} F^{\mu\nu} \equiv n_F A(x)$$ (25.161)
where $F_{\mu\nu} = \sum_a F_{a\mu\nu} \lambda^a / 2$ is the field strength of the color $SU(3)$ gauge group of QCD, and we have introduced the notation $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} / 2$. for the dual of $F_{\mu\nu}$. So the anomaly breaks this axial symmetry, which is good news since otherwise there would have to be a ninth NGB for which there is no evidence.

However, the consequences of this symmetry breaking are elusive because the right side of the equation can be expressed as the divergence of a four vector density $\partial_\mu K^\mu$. It is straightforward to show that

$$K^\mu = \frac{g_3^2 n}{4\pi^2} \epsilon^{\mu\rho\sigma\tau} \text{Tr}\left[A_\nu \partial_\rho A_\sigma - \frac{2i g_3}{3} A_\nu A_\rho A_\sigma\right]$$

(25.162)

satisfies this condition. So the anomaly equation can be written

$$\partial_\mu (J_5^\mu - K^\mu) = 0 \quad (25.163)$$

which makes it look like the chiral $U(1)$ invariance has been restored. But $K^\mu$ is not gauge invariant, so the "conserved current" $j_5^\mu + K^\mu$ is not an authentic physical observable.

The important physics point is that we don’t want a ninth NGB. To understand how this can be realized in spite of this unphysical conserved current, we recall the argument for NGB’s. Look at the anomaly equation in the context of a correlation function

$$-i q_\mu \int d^4 x e^{i q \cdot x} \langle j_5^\mu(x) \bar{q}_j(0)(1 \pm \gamma_5) q_i(0) \rangle \quad (25.164)$$

$$= n_F \int d^4 x e^{i q \cdot x} \langle A(x) \bar{q}_j(0)(1 \pm \gamma_5) q_i(0) \rangle \pm 2 \langle \bar{q}_j(0)(1 \pm \gamma_5) q_i(0) \rangle$$

The second term comes from the time derivative hitting the time ordering symbol producing the equal time commutator of $j_5^\mu(x)$ with $\bar{q}_j(0)(1 \pm \gamma_5) q_i(0)$. The essence of the Nambu-Goldstone theorem is that if the right side is not 0 in the $q \to 0$ limit, then there must be a massless scalar particle coupling both to $j_5^\mu$ and to $\bar{q}_j(1 \pm \gamma_5) q_i$. To prevent this conclusion the right side must vanish as $q \to 0$:

$$0 = 2 \langle \bar{q}_j(0)(1 \pm \gamma_5) q_i(0) \rangle \pm n_F \int d^4 x \langle A(x) \bar{q}_j(0)(1 \pm \gamma_5) q_i(0) \rangle$$

(25.165)

The first term on the right is non-zero if we make the hypothesis of spontaneous symmetry breaking of $SU(n_F) \times SU(n_F)$. So to truly resolve the $U(1)$ problem the second term must also be non-zero. This is elusive because, at least order by order in perturbation theory $\int d^4 x \text{Tr} F \tilde{F} = 0$ because it is the integral of a derivative. The proof that the second term is non-zero must involve a non-perturbative mechanism.

What this involves\footnote{The following discussion follows to a large extent the one by Sidney Coleman in his Erice lectures “The Uses of Instantons.”} can be analyzed first in classical gauge theory. Can we find classical solutions of the nonabelian gauge theory which have finite action $\int d^4 x F^2$ and at the same time $\int d^4 x F \tilde{F} \neq 0$? For clarity we continue to imaginary time so the fields are defined on
Euclidean 4 dimensional spacetime. In this case, finite action definitely implies that $F \rightarrow 0$ at infinity. However, the gauge fields need not vanish at infinity to achieve this; they are only required to approach a pure gauge there:

$$A_\mu \rightarrow \frac{i}{g} \frac{1}{4} \Omega^\dagger(x) \partial_\mu \Omega(x), \quad x \rightarrow \infty. \quad (25.166)$$

Then we need to find the large $x$ behavior of $K^\mu$ under the assumption that $F_{\mu\nu} \rightarrow 0$. The latter implies that

$$\partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow i g_3 [A_\mu, A_\nu].$$

Thus

$$K^\mu \rightarrow \frac{g_3^2 n_F}{4 \pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left[ i g_3 \frac{2}{3} A_\nu [A_\rho, A_\sigma] - \frac{2 i g_3}{3} A_\nu A_\rho A_\sigma \right] \rightarrow \frac{g_3^2 n_F}{12 \pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left[ \Omega^\dagger \partial_\nu \Omega \Omega^\dagger \partial_\rho \Omega \Omega^\dagger \partial_\sigma \Omega \right]. \quad (25.167)$$

Then we calculate

$$\frac{g_3^2 n_F}{16 \pi^2} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma} = \int d^4 x \partial_\mu K^\mu = \frac{n_F}{12 \pi^2} \int_{S_3} d\Omega_3 \epsilon^{rijk} \text{Tr} \left[ \Omega^\dagger \partial_i \Omega \Omega^\dagger \partial_j \Omega \Omega^\dagger \partial_k \Omega \right].$$

The expression on the right is a kind of winding number, counting the number of times the gauge function $\Omega(\theta)$ wraps the three sphere at infinity. Recall that in odd dimensions the variation of the integrand is a total derivative, so that its integral over the three sphere is a topological invariant. It’s a three dimensional analogue of the integral of a $U(1)$ group element $e^{i f(\theta)}$ on a circle $0 < \theta < 2\pi$. The requirement that $U$ is a well defined function on the circle $U(2\pi) = U(0)$ translates to $f(2\pi) - f(0) = 2\pi n$ where $n$ counts the net number of times the circle is wrapped. The formula for the winding number in this case is

$$\frac{1}{2 \pi i} \int_0^{2\pi} d\theta U^\dagger \partial_\theta U = \frac{1}{2 \pi i} \int_0^{2\pi} d\theta i f'(\theta) = \frac{f(2\pi) - f(0)}{2\pi} = n. \quad (25.169)$$

These winding number quantities help characterize topological aspects of the relevant group manifold. The concept of the homotopy group $\pi_n(M)$ of a manifold $M$ characterizes the maps of an $n$-dimensional sphere into the manifold. For the $U(1)$ case $\pi_1(U(1)) = Z$ the set of all integers. A simply connected manifold is characterized by the possibility of shrinking any closed curve to a point, in which case $\pi_1(M) = 0$.

For the $SU(N)$ group manifold ($N \geq 2$), the behavior of the gauge field at infinity is characterized by $\pi_3(SU(N)) = Z$. Let’s focus on the case $N = 2$, We can parameterize an element of $SU(2)$

$$\Omega(\theta) = \pm \sqrt{1 - \theta^2} I + i \theta \cdot \tau \quad (25.170)$$
where the + is the upper hemisphere of \( S_3 \), \( \theta^4 > 0 \) and the − the lower hemisphere. We calculate the integral for the upper hemisphere and multiply by 2 to get the complete integral. It is then straightforward to compute (Exercise)

\[
\int d\Omega_3 \epsilon^{ijk} \text{Tr} [\Omega^i \partial_4 \Omega^j \Omega^k \partial_4 \Omega] = \mp 2 \int d^3 \theta \frac{6}{\sqrt{1 - \theta^2}} = \mp 12\pi^2 \tag{25.171}
\]

Note that the measure \( d^3 \theta / \sqrt{1 - \theta^2} \) is \( O(4) \) invariant. This is the Euclidean version of the familiar zLorentz invariance of \( d^3 p / E(p) \). Finally we get

\[
\frac{g_3^2}{16\pi^2} \int d^4 x \epsilon^{\mu
u\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma} = \frac{1}{12\pi^2} \times 12\pi^2 \times 2 = 2 \tag{25.172}
\]

This is the result for unit winding number since the element

\[
\theta^4 I + i \theta \cdot \tau, \quad \theta^2 + \theta^2 = 1 \tag{25.173}
\]

wraps the \( S_3 \) sphere exactly once. For a general winding number \( \nu \) we use \( \Omega_\nu = [\Omega_1]^\nu \) and get

\[
\frac{g_3^2}{32\pi^2} \int d^4 x \epsilon^{\mu
u\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma} = \nu \in \mathbb{Z} \tag{25.174}
\]

Since small variations \( \delta A_\mu \) can’t change \( \nu \), field configurations for which the classical action \( \int d^4 x \text{Tr} F^2 / 2 \) is stationary, subject to the constraint that \( \nu \) is fixed, will solve the classical field equations. We are particularly interested in solutions which are the minima of the action in each topological sector \( \nu \). We can identify these by a trick. Consider

\[
\int d^4 x \text{Tr} (F_{\mu\nu} \pm \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}/2)^2 \geq 0 \tag{25.175}
\]

The inequality is true because spacetime is Euclidean and not Minkowski. Expanding the left side the inequality becomes

\[
2 \int d^4 x \text{Tr} F^2 \pm \int d^4 x \epsilon^{\mu
u\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma} \geq 0 \tag{25.176}
\]

Since the last term is fixed under the variation, the first term will be minimized when

\[
F_{\mu\nu} \pm \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = 0, \quad F_{\mu\nu} = \mp \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \equiv \mp \tilde{F}. \tag{25.177}
\]

This doesn’t yet give us an explicit solution but it reduces the problem from second order differential equations to first order differential equations. The equations state that the field

\[5\]Coleman’s argument: First show that with \( \Omega_3 = \Omega_1 \Omega_2, \nu_3 = \nu_1 + \nu_2 \). Since \( \nu \) is a topological invariant, its value doesn’t change if \( \Omega_1 \) and \( \Omega_2 \) are deformed so that one is the identity whenever the other is not the identity and vice versa. For these configurations, it is immediate that \( \nu_3 = \nu_1 + \nu_2 \.

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strength is equal to \( \mp \) its dual: in the first case we say the field strength is antidual and in the second case that it is self dual.

The Euclidean action for these solutions is

\[
S = \frac{1}{2} \int d^4x \text{Tr} F^2 = \mp \frac{1}{4} \int d^4x \epsilon^{\mu \nu \rho \sigma} \text{Tr} F_{\mu \nu} F_{\rho \sigma} = \mp \frac{8\pi^2 \nu}{g^2}
\]

(25.178)

The right side must be positive. The upper sign corresponds to an antidual \( F \) for which \( \nu < 0 \), while the lower sign corresponds to a selfdual \( F \) for which \( \nu > 0 \).

In fact explicit expressions for these instanton solutions have been obtained. We discuss the \( SU(2) \) case. We search for a solution such that

\[
A_\mu \to \frac{i}{g} \Omega^{-1} \partial_\mu \Omega, \quad x \to \infty
\]

(25.179)

with

\[
\Omega \sim \frac{1}{r} (x^4 + i \mathbf{x} \cdot \mathbf{\tau}), \quad r^2 = (x^4)^2 + \mathbf{x}^2
\]

(25.180)

Then

\[
\Omega^{-1} \partial_\mu \Omega \sim -\frac{x_\mu}{r^2} + \frac{1}{r^2} (x^4 - i \mathbf{x} \cdot \mathbf{\tau}) \begin{cases} 
1 & \mu = 4 \\
\frac{1}{r \tau^k} & \mu = k
\end{cases}
\]

\[
\sim \begin{cases} 
-\frac{i \mathbf{x} \cdot \mathbf{\tau}}{r^2} & \mu = 4 \\
\frac{i x^4 \tau^k + i (\mathbf{\tau} \times \mathbf{x})^k}{r^2} & \mu = k
\end{cases}
\]

(25.181)

This gives us the large \( r \) behavior of \( A_\mu \). However the singularity at \( r \to 0 \) makes it unsatisfactory at small \( r \). But replacing the \( r^2 \) in the denominator by \( r^2 + R^2 \) not only resolves the singularity, but also makes the ansatz satisfy the Euclidean field equations! The desired solution is then

\[
A_\mu = \begin{cases} 
-\frac{i \mathbf{x} \cdot \mathbf{\tau}}{r^2 + R^2} & \mu = 4 \\
\frac{i x^4 \tau^k + i (\mathbf{\tau} \times \mathbf{x})^k}{r^2 + R^2} & \mu = k
\end{cases}
\]

(25.182)

A good way to prove that this is a solution is to calculate the field strengths

\[
E_k = F_{4k} = \partial_4 A_k - \partial_k A_4 - ig [A_4, A_k]
\]

(25.183)

\[
B = \nabla \times \mathbf{A} - ig \mathbf{A} \times \mathbf{A}
\]

(25.184)

And show self duality or antiduality \( B = \pm E \), which I leave as an exercise for the reader.

Since \( SU(2) \) is a subgroup of any simple nonabelian group these solutions can be found for any such gauge group.\(^6\) The name instanton for these solutions comes from the feature

\(^6\)More is true. The mathematician R. Bott has shown that any map of \( S_3 \) into any simple group can be continuously deformed to a map into an \( SU(2) \) subgroup of \( G \). This assures that there are no new topological distinct maps than the ones we have identified.
that these are localized solutions in (Euclidean) spacetime. Localization in time makes it an instant. Among field configurations with winding number $\nu$ are ones with $\nu = \pm 1$ solutions. Since these solutions are stationary points of the action, they are valid saddle points for a semiclassical approximation to the path integral. One evaluates the action at the solution plus fluctuations $\delta A$, expands to second order in $\delta A$ and does the resulting Gaussian path integral. The resulting contribution for winding number $\nu$ has the form:

$$g^p e^{-8\pi^2 |\nu|/g^2}$$

(25.185)

where the power of $g$ in front comes from the Gaussian integral. The exponent of $-1/g^2$ has the character of classically forbidden WKB approximations in quantum mechanics. It represents a tunnelling from one approximate vacuum to another as we shall discuss in the next section. This exponential makes the contribution smaller than any power of $g$ and therefore smaller than any ordinary perturbative correction. This means that the contribution is meaningful only for a quantity that is zero to every order of perturbation theory.

The $U(1)$ chiral anomaly as $q \to 0$ is such a quantity. The instanton contributions make this limit nonzero and hence their presence shows that the unwanted $U(1)$ symmetry is not a problem in principle. In detail, the evasion of the NGB theorem requires the identity

$$0 = 2 \langle \bar{q}_j(1 \pm \gamma_5)q_k \rangle ± g^2 \int d^4x \langle \epsilon \text{Tr} FF \bar{q}_j(1 \pm \gamma_5)q_k \rangle$$

(25.186)

and the presence of instantons at least indicates that the second term is not 0. With $\nu$ an integer, we can only keep the condensate nonzero if $n_F = 1$. Then an instanton solution with winding number $\nu = \mp 1$ allows the $U(1)$ problem to be solved, because the order parameter can be nonzero without NGB’s.

However, if $n_F > 1$ the fact that $\nu$ is an integer means that the absence of NGB’s requires $\langle \bar{q}_j(1 \pm \gamma_5)q_k \rangle = 0$. This does not mean that the $U(1)$ symmetry is unbroken. If one considers the Ward identify for the correlator of the axial current with the more complicated order parameter

$$X_{n_F}^\pm = \langle (\bar{q}_j(1 \pm \gamma_5)q_k)^{n_F} \rangle$$

(25.187)

the factor of $(1 \pm n_F \nu)$ is replaced by $(n_F \pm n_F \nu) = n_F (1 \pm \nu)$, so that an instanton with $\nu = \mp 1$ allows $X_{n_F}^\pm \neq 0$ without NGB’s. This order parameter breaks $U(1)$ if non zero, It does not spontaneously break $SU(n_F) \times SU(n_F)$, which must rely on a separate nonperturbative dynamics beyond instantons.

The Atiyah-Singer index theorem (discussed in these notes in the chapter on anomalies) clarifies this difference between $n_F = 1$ and $n_F > 1$. According to the theorem $\nu = n_+ - n_-$ where $n_\pm$ is the number of zero modes of the Dirac operator with chirality $\pm 1$. With $n_F$ flavors of massless quarks, the vacuum path integral provides a factor $(\det [-i\gamma \cdot D])^{n_F}$, which will have at least $n_F \nu$ zeroes. To get a nonzero expectation of a correlator one needs a $\bar{q}(1 \pm \gamma_5)q$ insertion for each zero mode. When $n_F = 1$ there is one zero mode indicating $\langle \bar{q}q \rangle \neq 0$. Unfortunately for $n_F > 1$ this expectation will be zero! So some other mechanism is needed to produce chiral SSB for more than 1 massless flavor.
25.7 $\theta$ vacua and the Strong CP problem

25.7.1 The $\theta$ Vacuum

We first address the question of how to use instantons in the semiclassical approximation to the gauge theory path integral defined on Euclidean space time. One can use any classical solution as a saddle point of the path integral. We have identified a whole set of solutions which minimize the action under the constraint that the winding number $\nu$ is fixed. For example with $\nu = 1$ one has a single instanton centered at any space time point and with any size $R$. We can visualize a solution with $\nu = n$ as $n$ separated instantons centered at $n$ different space time points and with any sizes. In all cases though one has a family of solutions for fixed $\nu$ and all members of the family should be included in the path integral.

It is inconsistent with the clustering property of the vacuum path integral to constrain $\int d^4x \text{Tr } \tilde{F}\tilde{F}$ to a fixed value. Imagine a very large region of spacetime $R$ composed of two regions $R = R_1 + R_2$ each of which is also very large. Let $\nu, \nu_1, \nu_2$ be the respective winding numbers for the gauge fields in the respective regions. Then $\nu = \nu_1 + \nu_2$. For large enough regions $R, R_1, R_2$ we include all winding number sectors as a sum

$$\int_R dA e^{-S} = \sum_\nu f_\nu \int_{R,\nu} dA e^{-S}$$  \hspace{1cm} (25.188)

where $f_\nu$ is restricted by the clustering property

$$\int_R dA e^{-S} = \int_{R_1} dA e^{-S} \int_{R_2} dA e^{-S}, \quad R = R_1 + R_2, \ R_1, R_2 \to \infty$$

For this to hold, $f_\nu$ must satisfy $f_{\nu_1+\nu_2} = f_{\nu_1} f_{\nu_2}$, which implies that $f_\nu = e^{i\theta \nu}$. This solves the equation for any complex $\theta$. But since we must include all $-\infty < \nu < \infty$, we require that $\theta$ be real. After summing over all winding numbers, the constraint on $\int F\tilde{F}$ is relaxed and the exponent in the path integral has a new term:

$$-S \to -S + i\theta \frac{g^2}{16\pi^2} \int d^4x \text{Tr } \tilde{F}\tilde{F}$$  \hspace{1cm} (25.189)

With $\theta \neq 0$ we call the ground state of the QFT a $\theta$ vacuum.

In a semiclassical evaluation, the coefficient of $\theta$ is an integer, which implies that the physics is periodic under $\theta \to \theta + 2\pi$. For example inclusion of the instanton solutions with $\nu = \pm 1$ gives a contribution

$$(VT)K \cos \theta e^{-8\pi^2/g_\lambda^2}$$  \hspace{1cm} (25.190)

If the solutions with higher $\nu$ are approximated as a “dilute gas” of instantons this term goes into the exponent and then the vacuum energy density gets identified as $K \cos \theta e^{-8\pi^2/g_\lambda^2}$. The constant $K$ depends on the detailed form of the instanton solution and is in fact an infrared divergent integral over the instanton size $R$. The main physics point is that the vacuum energy is expected to depend periodically but nontrivially on $\theta$. 

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25.7.2 Massless Quarks

In the previous subsection we discussed the $\theta$-vacuum in the pure gauge theory, concluding that the vacuum energy is expected to depend on $\theta$. In this subsection we discuss the profound changes when the gauge fields are coupled to $n_F$ massless quarks. To start we return to the anomalous conservation law:

$$\partial_\mu j_5^\mu = \frac{g_3^2 n_F}{16\pi^2} \mathrm{Tr} \epsilon FF$$

Here we may think of the current as the matrix element

$$j_5^\mu(x) = \frac{\langle \text{out}|j_5^\mu(x)|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle}$$  \quad (25.192)

in the presence of an external gauge field $A_\mu$. Integrating both sides over $x$ we write

$$\int d^4x \partial j_5^\mu = \int d^3x j_0^\mu(x, \infty) - \int d^3x j_0^\mu(x, -\infty) = \frac{\langle \text{out}|Q_5(\infty) - Q_5(-\infty)|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle} = 2n_F \nu$$

next we define $\alpha$ states

$$\langle \text{out}, \alpha|\text{in}, \alpha \rangle = \langle \text{out}|e^{i\alpha Q_5(\infty)}e^{-i\alpha Q_5(-\infty)}|\text{in}\rangle$$ \quad (25.194)

which satisfy the differential equation

$$\frac{\partial}{\partial \alpha} \langle \text{out}, \alpha|\text{in}, \alpha \rangle = 2n_F \nu \langle \text{out}, \alpha|\text{in}, \alpha \rangle$$ \quad (25.195)

with the solution

$$\langle \text{out}, \alpha|\text{in}, \alpha \rangle = e^{2i n_F \nu \alpha} \langle \text{out}, 0|\text{in}, 0 \rangle$$ \quad (25.196)

If the matrix element on the right is calculated with Heisenberg quark operators $q_k(t)$ the one on the left is calculated with Heisenberg quark operators

$$e^{i\alpha Q_5(t)}q_k(t)e^{-i\alpha Q_5(t)} = e^{-i\alpha \gamma_5}q_k(t)$$ \quad (25.197)

To see this, remember that in general $\langle \text{out}|\text{in}\rangle = \lim_{t \to \infty} \langle \text{in}|U(t, -\infty)|\text{in}\rangle$, where $U(t, -\infty)$ is defined by the differential equation and boundary conditions

$$i\dot{U} = H_S(t)U, \quad U(-\infty, -\infty) = I.$$ \quad (25.198)

Then the phase of the Schrödinger picture fermion fields can be changed by the similarity transform $e^{i\alpha Q_5}q_k e^{-i\alpha Q_5} = e^{-i\alpha \gamma_5}q_k$. So the evolution operator for these rephased fields satisfies

$$i\dot{V} = \Omega^\dagger H_S(t)\Omega V, \quad V(-\infty, -\infty) = I.$$ \quad (25.199)
Comparing to the equation for $U$ we see that

$$V(t, -\infty) = \Omega^\dagger U(t, -\infty)\Omega = U(t, -\infty)\Omega^\dagger(t)\Omega(-\infty)$$  \hspace{1cm} (25.200)

where $\Omega^\dagger(t) = U^\dagger(t)\Omega$ is the Heisenberg picture $\Omega^\dagger$. Thus we have $\langle out, \alpha | in, \alpha \rangle = \langle out, 0 | \Omega^\dagger(t)\Omega(-\infty) | in, 0 \rangle$, where we chose $\Omega = e^{-i\alpha q_5}$.

To work in the $\theta$ vacuum, we add a term $i\theta \nu = i\theta \frac{g^2}{32\pi^2} \int d^4x \text{Tr} \epsilon FF$ to the exponent of the path integral. Then we can say that calculating with $e^{-i\alpha \gamma^5 q_5}$ and $\theta \neq 0$ is the same as calculating with $q_5$ and $\theta \rightarrow \theta + 2n_F \alpha$. Or alternatively calculating with $q_5$ and $\theta$ is the same as with $e^{i\theta \gamma^5/(2n_F)} q_5$ and $\theta = 0$.

For massless quarks physics is independent of the phases of the $L$ and $R$ components of the quark fields, because these fields enter the Hamiltonian only as $L^\dagger \gamma^0 \gamma L$ or $R^\dagger \gamma^0 \gamma R$. so for massless quarks the physics is independent of $\theta$. Technically, the reason the $\theta$ dependence disappears in instanton evaluations is that the index theorem ensures that there is at least one zero eigenvalue of the Dirac operator $-i\gamma \cdot D$ when $\nu \neq 0$, so that the quark path integral $\det(-i\gamma \cdot D) = 0$ for all $\nu \neq 0$: only the $\nu = 0$ term of the sum over winding numbers contributes. This $\theta$ independence holds even if only one quark is massless.

### 25.7.3 Massive Quarks

The situation just described is altered dramatically if all quarks are massive. Because then the Hamiltonian contains mass terms

$$\bar{L}_k m_{kl} R_l + \bar{R}_k m^*_{kl} L_l$$  \hspace{1cm} (25.202)

and these terms will acquire phases $e^{\pm i\theta/n_F}$ after the chiral $U(1)$ transformation has set $\theta = 0$.

We can use $SU(n_F) \times SU(n_F)$ transformations to diagonalize the mass matrix $m_{kl}$. We first find unitary matrices $U, V$ such that

$$U^\dagger m V = D$$  \hspace{1cm} (25.203)

where $D$ is diagonal with nonnegative entries. But $U$ and $\det V$ may not be unity, so $U, V$ may not be in $SU(n_F)$. However we can write $U = U_0 e^{i\phi_1}$ and $V = V_0 e^{i\phi_2}$ where $U_0, V_0 \in SU(n_F)$, so that

$$U_0^\dagger m V_0 = D e^{i(\phi_2 - \phi_1)} \equiv D e^{i\alpha}$$  \hspace{1cm} (25.204)

Note that $\det m = e^{i\alpha n_F}$. The field redefinitions $R \rightarrow V_0 R$, $L \rightarrow U_0 L$ diagonalize the mass matrix without disturbing $\theta$, but the diagonal entries have a common phase $m_k e^{i\alpha}$, $m_i \geq 0$. After this the mass terms are

$$\sum_k (\bar{L}_k R_k m_k e^{i\alpha} + \bar{R}_k L_k m_k e^{-i\alpha})$$  \hspace{1cm} (25.205)
If any of the quark masses, say $m_1 = 0$, this common phase can be removed by the redefinitions

$$
L_k \to L_k e^{i\alpha/2}, \quad R_k \to R_k e^{-i\alpha/2}, \quad k \neq 1
$$

$$
L_1 \to L_1 e^{-i\alpha(n_F-1)/2}, \quad R_1 \to R_1 e^{i\alpha(n_F-1)/2}
$$

The point is that since $m_1 = 0$, we can choose the phase change of $L_1, R_1$ so that the diagonal matrix of phase changes has unit determinant, which makes it an element of $SU(n_F)$. If all the masses $m_k \neq 0$ The anomalous chiral $U(1)$ transformation could remove the common phase, but only if $\theta$ is shifted an appropriate amount.

If we start out in a $\theta$ vacuum, we can use a chiral $U(1)$ transformation to set $\theta = 0$, with the mass term acquiring $\theta$ dependence

$$
\sum_k (\bar{L}_k R_k m_k e^{i(\alpha + \theta/n_F)} + \bar{R}_k L_k m_k e^{-i(\alpha + \theta/n_F)})
$$

The measurable phase is the combination $\bar{\theta} \equiv \theta + n_F \alpha$. The criterion for $CP$ invariance is thus $\bar{\theta} = 0$.

This is the strong CP problem. Instantons have taught us that in addition to quark masses and the strong coupling $g_3$, $\bar{\theta}$ is a bona fide free parameter in QCD, and if all quark masses are nonzero, physical processes will depend on $\bar{\theta}$. A nonzero $\bar{\theta}$ is a source of CP violation over and above a complex phase in the CKM matrix. The danger is that because it is in the strong interaction sector of the standard model, the CP violation is likely to be much larger than seen in experiment.

### 25.7.4 CP Violation from $\bar{\theta} \neq 0$

In this discussion we assume that $\bar{\theta} \ll 1$. Then we can locate the CP violation in the quark mass terms

$$
\sum_k (\bar{L}_k R_k m_k e^{i\bar{\theta}/n_F} + \bar{R}_k L_k m_k e^{-i\bar{\theta}/n_F})
$$

$$
= \sum_k m_k \bar{q}_k e^{i\bar{\theta} n_F/q_k} \approx \sum_k m_k \bar{q}_k \left(1 + i\gamma_5 \frac{\bar{\theta}}{n_F}\right) q_k
$$

It is convenient to introduce $SU(n_F)$ phases in such a way that the CP violation in the second term in parentheses is the same for all flavors of quark. This makes the CP violating term $d$ couple from the NGB’s due to spontaneous breaking of chiral symmetry. In other words, we can arrange the phases so that

$$
m_k + m_k i\gamma_5 \frac{\bar{\theta}}{n_F} \to m_k + iA\gamma_5 \bar{\theta}
$$
where $A$ can be determined by comparison of determinants:

\[
(1 + i\bar{\theta}\gamma_5) \prod_k m_k = \prod_k (m_k + iA\gamma_5\bar{\theta}) + O(\bar{\theta}^2)
\]

\[
= \prod_k m_k + iA\bar{\theta}\gamma_5 \sum_l \prod_{k\neq l} m_k + O(\bar{\theta}^2)
\]

\[
A = \frac{\prod_k m_k}{\sum_l \prod_{k\neq l} m_k} \approx \frac{m_u m_d + m_u m_s + m_d m_s}{m_u + m_d}, \quad n_F = 3
\]

(25.210)

where the last line is for QCD with 3 light flavors. Then the CP violating term in the QCD Lagrangian is

\[
\mathcal{L}_{CP} = \bar{\theta}(\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d + \bar{s}i\gamma_5 s) \frac{m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s}
\]

(25.211)

CP invariance implies that the electric dipole moment of the neutron is zero. This is because the electric dipole moment of the neutron at rest must be proportional to the neutron spin $\sigma/2$ by rotational invariance. But this operator is odd under time reversal, and hence by CPT is also odd under CP. But under CP, $e\mathbf{D}_n \rightarrow (-e)(-\mathbf{D}_n) = e\mathbf{D}_n$, so it is even under CP. Treating the CP violating term in perturbation theory leads, after a lengthy calculation, to the estimate $D_n \approx 5\bar{\theta} \cdot 10^{-16}$ e·cm. Experimentally no electric dipole moment has been detected, but an upper limit of $0.30 \times 10^{-25}$ e·cm has been established. This translates to an upper bound on $\bar{\theta} < 0.6 \times 10^{-10}$.

Although one can technically fine tune $\bar{\theta}$ to this accuracy, the tininess of the bound suggests that there should be some physics reason that $\bar{\theta} = 0$. If CP were a symmetry of nature we could simply invoke it. But its violation has been firmly established in weak decays and explained via the CKM matrix. Since the latter is the outcome of diagonalizing the fermion mass matrices, one would expect it to induce a nonzero value of $\bar{\theta}$ that had been set to zero for QCD. Unfortunately the expectation is a value considerably larger than the limit.

As we have already mentioned, setting the lightest quark mass $m_u$ to zero is one physics mechanism for $\bar{\theta}$ being exactly zero. This would conflict with the chiral $SU(3)$ value of $m_u/m_d \approx 1/2$. Since chiral $SU(3)$ has a long success record in explaining the physics of light hadrons, another mechanism would be desirable. Since none are available within the standard model, any new mechanism must involve some kind of extension of the standard model.

One of the most promising is the Peccei-Quinn mechanism, which introduces extra fields which enjoy a new and anomalous $U(1)$ symmetry. A simple extension involves adding a quark to the standard model along with a new complex scalar field. When this new symmetry is spontaneously broken an NGB, the axion $a$, appears. This axion couples to $\text{Tr}F\tilde{F}$ in the combination $\bar{\theta} + a/F_a$. Here $F_a$ is the axion analogue of the pion $F_{\pi}$. The dependence of the vacuum energy on $\bar{\theta}$ then becomes a nontrivial effective potential for the would be massless
NGB a. Assuming that the $\theta$ vacuum energy was a minimum at $\bar{\theta} = 0$, this effective potential will have a minimum at $\bar{\theta} + a/F_\pi = 0$, dynamically restoring CP invariance. Generically the effective potential will have a nonzero second derivative at its minimum so the axion will acquire a mass. This resolution of the strong CP problem is being actively investigated in current research, especially by P. Sikivie at the University of Florida. In addition to resolving the strong CP problem, the axion is a potential candidate for the dark matter in the universe.
Theorem on Symmetry Breaking Patterns of $SU(n_F) \times SU(n_F)$

\[ \left[ SU(n_F) \times SU(n_F) \right]_{\text{Diag}} \cong SU(n_F) \]

MUST Remain Unbroken \[ \left[ SU(n_F) \right]_{\text{regime}} \Rightarrow \text{all masses } \neq \text{full } SU(n_F) \]

- $m_i = m \neq 0 \quad \forall \, i = 1, \ldots, n_F$ Symmetry Breaking just $SU(n_F)$
- $S \subset \mathcal{B} \Rightarrow \text{NG bosons } \Rightarrow$ massless composite of massive constituents \( \Rightarrow \) stable, indistinguishable, persistent mass condition
- $m \to 0$

Vafa-Witten Argument

Fermion correlator (Euclidean path integral)

\[
\frac{1}{2} \int \frac{dA}{\text{Det}(\frac{i}{2} \Delta + M)} e^{-i S(A)}
\]

This argument FORMALLY proves $SU(n_F)_{\text{Diag}}$ Symmetry

But so unreal! The case $M = 0$ where symmetry is $SU(n_F) \times SU(n_F)$ and we want that $SYM$ to break spontaneously.

What is really needed is stability under $M \to M + \Delta M$, then $SU(n_F)$?

For $\Delta M \to$ symmetry breaking, consider $\mathcal{M}$ Tafts at $\mathcal{M}$ (conf)

But coeff could be $\frac{1}{\Delta M}$ (check this happens for $SU(n_F)$-chiral)
In Euclidean space, \( \left( \frac{1}{i} \delta D \right)^+ = -\left( \frac{1}{i} \delta D \right) \)

so elements of \( \frac{1}{i} \delta D + M \) are \( i \lambda + M \neq 0 \).

\[
\det \left( \frac{1}{i} \delta D + M \right) = \prod_{\lambda > 0} (H + i \lambda)
\]

\[
= \prod_{\lambda > 0} (H^2 + \lambda^2) > 0
\]

note if \( \lambda > 0 \) is eigenval of \( \frac{1}{i} \delta D \), so is \(-\lambda\).

In Euclidean space, \( e^{iS} \rightarrow e^{-\frac{i}{4} \int \sqrt{F^2}} > 0 \)

Bond on \( \langle \phi^2 \rangle \) as \( \frac{1}{4} \int \sqrt{F^2} \geq 0 \) and.

\( \langle \phi^2 \rangle = \langle \phi^2 \rangle = 1 \).

\( \phi \) as degeneracy \( \text{div}_{\text{tree}} \) dependence. Formal symmetry is:

in fact stable =) \( m \leq 3.5 \).

How about only appear symmetric

\[
\text{for } u, d \quad M_d - M_u = 0 \quad \text{or} \quad M_d = M_u
\]

\( M_5 \geq 3.5 m_u, m_d \)

\( \text{SU}(2), \text{SU}(1), \text{only appear} \quad g \quad \text{and only because } m_u, m_d, m_s \quad \text{are all related somehow} \). For sure, the U(1)'s are not broken: Q, B, S.

\( \text{Full } \text{SU}(N_F) \) would be included if \( m_u = m_d = m_s \), but this does not even appear here.

\( \text{U(1)} \) problem: Symmetry of \( \bar{q} i \chi \gamma^5 q \) is really \( \text{U}(N_F) \times \text{U}(1)_Y \)

\( \text{invariant} \); expect either parity dual to \( N = 0 \) or a light \( I = 0 \) pseudoscalar \( [m_s < m_u < \frac{1}{\sqrt{3}} \text{ mass of charged partitons}] \)

Ultimate resolution: Anomaly

\( R^0 \rightarrow \gamma \gamma \)
Chapter 26

Gauge Invariance, Ultraviolet Divergences and Gauge Anomalies II

26.1 Lattice Gauge Theory

In this chapter we have taken for granted that UV divergences do not ruin the path integral for the gauge fields themselves: we have concentrated on gauge anomalies found in the matter (non gauge field) sector of the theory. As far as anyone can tell this presumption is justified in perturbation theory, because of the existence of gauge invariant regulation procedures, especially dimensional regularization.

But at a deeper level, the path integral for the continuum theory must be regarded as a formal object. Replacing spacetime with a finite four dimensional lattice gives a concrete definition of the path integral as well as a systematic cutoff of the UV and IR divergences. The problem, solved by Ken Wilson, is to formulate a lattice path integral that is exactly gauge invariant. This involves a new description of the gauge degrees of freedom.

In our discussion of UV divergences and gauge invariance we made use of the concept of the path dependent group element

$$\Phi(x,y;C) = P \exp \left\{ i \int_{x,C}^y d\xi^\mu A_\mu \right\}$$  (26.1)

which is gauge covariant $\Phi \to U(y)\Phi U(x)$ under the gauge transformation $U(x)$. It follows that $\text{Tr}\Phi(x,x;C)$ is gauge invariant. Wilson’s idea was to choose lattice variables for the gauge field to be just such variables.

To adapt this idea to a $D$-dimensional rectangular lattice of points (called sites), imagine each nearest neighbor pair of sites connected by a straight line called a link. Then to each link $L$ of the lattice, we assign a group element $U_L$. The gauge transformations are then group elements $V_S$ assigned to each site. If $L$ is the link from site $S$ to site $S'$, then the gauge transformation is $U_L \to V_{S'}^L U_L V_S$. If $L$ is the reversed link from $S'$ to $S$, we define $U_L \equiv U_L^\dagger$, which is seen to be consistent.

The various matter fields in the theory will be assigned to the sites of the lattice: $\psi_S$. 

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Their gauge transformations are simply $\psi_S \rightarrow U_R \psi_S$, where $U_S(R)$ is the gauge transformation at site $S$ in the representation $R$ carried by $\psi$.

To define a lattice version of $\Psi$ pick two sites on the lattice, and an ordered sequence of links that make up a path $C$ from $S$ to $S'$. Then

$$\Psi \rightarrow \prod_{L \in C} U_L$$

where the $U_L$ are ordered from the one connected to $S$ to the one connected to $S'$. If $S = S'$ (the path $C$ is closed), then $\text{Tr} \Psi$ is gauge invariant.

The integration variables of the gauge path integral are chosen to be the $U_L$. A gauge group invariant Haar measure $dU$ is a standard concept in the theory of continuous groups. It is defined to satisfy $d(VU) = dU$ for an arbitrary group element $V$. In practice it is defined using the parameterization of each group element $G = \exp \{ i \theta^a L_a \}$, where $L_a$ is a basis for the Lie algebra, and taking the $\theta^a$ as integration variables. Taking the rotation group as an example, write $R = \exp \{ -i \theta \cdot J \}$, then the measure is $dR = \hat{d} \theta d\Omega_{\theta}$. Since we are mostly interested in compact gauge groups, the domain of integration for each $U_L$ is compact. If the lattice is also finite, no gauge fixing is necessary, since the integral over the gauge group is itself finite. The gauge group factor therefore rigorously cancels with no delicacy between the numerator and denominator of a correlation function of observables.

The guideline for choosing a lattice action is that it be a gauge invariant function of the link variables $U_L$, that it be real, and that it yields that standard Lagrangian $-F^2/4$ in the formal continuum limit. With a rectangular lattice, the simplest closed path is a square enclosed by four links. Such a square is called a plaquette. A gauge invariant variable for plaquette $P$ is then defined as

$$\text{Tr} U_{L_1} U_{L_2} U_{L_1}^\dagger U_{L_2}^\dagger \equiv \text{Tr} U_P$$

where $L_1'$ is translated one lattice step in the $L_2$ direction from $L_1$, and $L_2'$ is the link translated one lattice step in the $-L_1$ direction from $L_2$. Then a simple proposal for the lattice gauge action is

$$S = \sum_P \text{Tr} (U_P + U_P^\dagger)$$

where the second term in the summand can be regarded as associated with the plaquette $\bar{P}$ obtained from $P$ by reversing the direction of all the links in $P$. If we understand that the sum over $P$ includes $\bar{P}$ as well as $P$, the second term would be automatically included in the first. It is convenient to keep both terms explicitly in discussion of the formal continuum limit.

To study the continuum limit write each link variable $U_L = e^{iaA_L}$ where $a$ is the lattice spacing. Then

$$U_P = e^{iaA_1 + iaA_2 + iaA_2 - iaA_1 - iaA_2}$$

$$= e^{iaA_1 + iaA_2 - a^2[A_1,A_2]/2 + O(a^3)} e^{-iaA_1 - iaA_2 - a^2[A_1,A_2]/2 + O(a^3)}$$

$$= e^{iaA_1 + iaA_2 - a^2[A_1,A_2]/2 + a^2[A_1,A_2]/2 + O(a^3)}$$

$$= e^{iaA_1 + iA_2 - a^2[A_1,A_2]/2 + O(a^3)} + O(a^3)$$

$$= e^{ia^2(\partial_1 A_2 - \partial_2 A_1 + i[A_1,A_2])} + O(a^3)$$
To obtain the last line we used $A'_1 - A_1 = a \partial_2 A_1 + O(a^2)$, and $A'_2 - A_2 = -a \partial_1 A_2 + O(a^2)$. Notice that the quantity in parentheses in the exponent is precisely the nonabelian field strength! Since $U_P$ is a unitary matrix, the entire exponent is antihermitian including all orders in powers of $a$. Thus upon expanding $U_P + U_P^\dagger$ in powers of the exponent, all odd powers cancel between the two terms as the even terms add. Thus

$$\text{Tr}(U_P + U_P^\dagger) = 2\text{Tr}I - a^4 \text{Tr}(\partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2])^2 + O(a^5) \quad (26.6)$$

summing over all plaquettes then produces the formal continuum limit

$$\sum_P \text{Tr}(U_P + U_P^\dagger) \to -2 \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{constant.} \quad (26.7)$$

The lattice gauge theory path integral can then be written:

$$Z = \int \prod_L dU_L \exp \left\{ \frac{1}{4g_0^2} \sum_P \text{Tr} \left[ U_P + U_P^\dagger \right] \right\} \quad (26.8)$$

where $g_0$ is the gauge coupling parameter for the lattice. The lattice gauge theory path integral is formulated with a Euclidean space-time metric, related to the Minkowski metric by continuation to imaginary time: $it \to x^4$. Then we have to interpret evolution in imaginary time as $e^{-\Delta x^4}$. In other words we can read off energy eigenvalues by looking for exponential damping instead of oscillations.

Physical quantities can be accessed by inserting appropriate gauge invariant functions of the $U_L$ in front of the exponential and dividing the resulting integral by $Z$ itself, giving the expectation value of the physical quantity described by the inserted gauge invariant function. An important example is the Wilson loop variable $W(C)$, which plays a central role in Wilson’s criterion for quark confinement: Choose a lattice contour in the shape of an $L \times T$ rectangle. One can interpret its expectation value as the evolution of fixed fundamental and anti-fundamental charges separated by fixed distance $L$ for (imaginary) time $T$. It should therefore behave as $\sim e^{-E(L)T}$ at large $T$, where $E(L)$ is the energy of the fundamental charges in the gauge theory vacuum. Quark confinement means that $E(L) \sim kL$ at large $L$, and then the Wilson loop should behave as $e^{-kLT} = e^{-k\text{Area}}$, which is the Wilson criterion for confinement.

It is not hard to see that the criterion is met in lattice gauge theory in the strong coupling limit ($g_0 \to \infty$). What must be shown, however, is that it continues to be met when $g_0 \to 0$. Needless to say this is very difficult, and is still an open question for QCD.

### 26.2 Chiral Anomalies

The existence of a gauge invariant regularization scheme such as dimensional regularization assures the absence of anomalies (violations) of gauge invariance. Our careful analysis of the e.m. current showed that it is indeed possible to define $\langle \text{out} | \text{in} \rangle$ in a gauge invariant way for the case of the electromagnetic field. But our parallel discussion of the axial current $j_5^\mu = \ldots$
shown that it is not conserved for $m = 0$, contrary to what the $c$-number Dirac equation would lead us to believe. The significance of this is that whereas we can consistently couple the quantum Dirac field to electromagnetism through $j_\mu A_\mu$, the gauge coupling $j_5^\mu A_\mu$ violates gauge invariance, and would lead to inconsistencies. Our point splitting definition of the currents shows the presence of the anomaly and allows us to compute it. One can also understand why the popular regularization schemes I have described fail to forbid anomalies. The Pauli-Villars scheme requires the addition of massive fermions which explicitly violate chiral invariance, and dimensional regularization gives no method for defining $\epsilon^{\mu\nu\rho\sigma}$ or $\gamma_5$. The possibility of anomalies in axial gauge couplings puts constraints on viable theories of the weak interactions which violate parity conservation through just such couplings. The way parity violation enters the standard electroweak theory is by assigning left and right handed fermions to different representations of the electro-weak gauge group $SU(2) \times U(1)$. This is of course possible only if explicit mass terms are not included in the Hamiltonian. Thus the $I \pm \gamma_5$ projections of the Dirac field for each fermion couple in different ways to the gauge fields. What our discussion shows is that such a scheme would be inconsistent for a single fermion. The way the electroweak theory escapes this difficulty is by a cancellation of the anomalies between the contributions of different fermions.

We evaluated the anomaly explicitly in part I of our discussion of anomalies in Chapter 21 (see pp. 270).

### 26.2.1 Ambiguities

Having obtained the anomalous divergence law for a gauge invariant current, we now examine other definitions of the current useful in certain contexts. First consider the expansion of an unmodified current in powers of the gauge potential described by the series of diagrams.

\[
\frac{\langle \text{out}|j_\mu|\text{in} \rangle}{\langle \text{out}|\text{in} \rangle} = \quad (26.9)
\]

which have degrees of divergence $3, 2, 1, 0, -1$ respectively.

Note that the terms that are involved in the anomaly are parity violating containing an odd number of $\gamma_5$’s and start at 3rd order. Considering the l.h.s. of the divergence law order by order in $A$, we see that it is a sum of the divergence of the $n$th order term and, in the nonabelian case, terms with $A$ multiplying the $n - 1$th order term. Each of these diagrams naturally has a cyclic symmetry in the labels $a, \mu, x$ of the $n$ vertices. Our gauge
invariant construction of the current evidently does not reflect this symmetry in the terms with an odd number of \( \gamma_5 \)'s, because the anomaly is present in only one of the vertices. This difference reflects a fundamental ambiguity in the definition of divergent diagrams, which in four dimensions includes only those of order \( \leq 4 \). In momentum space, a diagram with degree of divergence \( D \) is ambiguous up to the addition of a polynomial in the external momenta of order \( D \). Accordingly the parity violating ambiguities are in the triangle and square diagrams and are \( \epsilon^{\mu\nu\rho\sigma}(\alpha p_1 + \beta p_2)_\rho \) and \( \alpha \epsilon^{\mu\nu\rho\sigma} \) respectively. Our construction, with the anomalies absent from all but one vertex, is related to the cyclic symmetric definition by the addition of such polynomials. A potentially confusing point is that the pentagon diagram is finite and unambiguous but the r.h.s. of the divergence law contains terms quartic in \( A \). This is explained by the fact that the l.h.s. contains a contribution from the square diagram. Changing the square diagram by a term proportional to \( \alpha \epsilon^{\mu\nu\rho\sigma} \) adds a term of exactly the structure of the quartic term on the r.h.s. of the divergence law. Thus depending on the resolution of the ambiguities in the triangle and square, there may or may not be a quartic term on the r.h.s. Similarly the cubic term on the r.h.s. is influenced by the ambiguity resolutions in both the square and triangle diagram.

### 26.2.2 Physical Consequences of the Chiral Anomaly

The existence of the chiral anomaly has two sorts of ramifications. The more fundamental is the constraints it puts on the sorts of gauge fields that can be consistently coupled to fermions. But even if the chiral current is not coupled to a gauge field, it is still an observable of the theory, which would be a conserved current for massless fermions in the absence of the anomaly. The anomaly breaks this conservation law in a way that becomes experimentally significant for very light fermions. The classic example is the decay \( \pi_0 \to \gamma + \gamma \) which would be oversuppressed by the small up and down quark masses were it not for the anomaly.

We first consider the limitations imposed on gauge couplings. The gauge fields mediating the weak interactions must couple differently to left and right handed fermions

\[
\psi_R \equiv \frac{I + \gamma_5}{2} \psi \quad \psi_L \equiv \frac{I - \gamma_5}{2} \psi.
\]

(26.10)

This is an experimental necessity. The standard electroweak theory, based on the nonabelian gauge group \( SU(2) \times U(1) \), achieves this requirement by assigning left handed fermions to doublets under \( SU(2) \) while the right-handed fermions are singlets under \( SU(2) \). The two types of fermion have different nonzero weak hypercharges \( y \) under the \( U(1) \). The ordinary electric charge is related to \( y \) and the weak isospin \( I_3 \), one of the three generators of the \( SU(2) \), according to the formula

\[
Q = I_3 + \frac{y}{2}.
\]

(26.11)

For example, the first generation of fermions consists of the electron \( e \), the electron neutrino \( \nu_e \) the up quark \( u \) and the down quark \( d \). The neutrino and left handed electron form a doublet
Since the neutrino is neutral with \( I_3 = 1/2 \) and the electron has charge \(-1\) with \( I_3 = -1/2 \), we have \( y_e = -1 \). In the standard model there is no right-handed neutrino and the right handed electron, being an \( SU(2) \) singlet has \( I_3 = 0 \) and hence \( y_{eR} = -2 \). The up and down quarks have charge \(+2/3\) and \(-1/3\) respectively. Their left handed components are an \( SU(2) \) doublet with \( I_3 = +1/2, -1/2 \) respectively, and accordingly carry weak hypercharge \( y_{qL} = +1/3 \). Their right handed components are singlets and hence have \( y_{uR} = +4/3 \) and \( y_{dR} = -2/3 \). There are at least two more generations which seem to repeat the pattern of the first only differing in masses, which of course cannot arise from explicit mass terms which would violate gauge invariance.

Now we consider the limitations imposed by the anomaly. Since the couplings to left and right handed fermions are different the currents that must be conserved for gauge invariance are \( \bar{\psi} \lambda_a (I \pm \gamma_5) \gamma_\mu \psi \), separately for \( \lambda_a = I \). When \( \lambda_a = \tau_a \in SU(2) \), only the left handed current couples to the gauge fields, and that’s the one that must be conserved. When defined in terms of the Green’s functions for the Dirac equation, these are just \( J^\mu_a \pm J^\mu_{5a} \). Since the gauge couplings conserve handedness for massless fermions, it is not necessary to include the \((I \pm \gamma_5)/2\) in the coupling of each gauge field. The anomaly is thus proportional to \( \text{Tr}[\lambda_a \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x)] \). The field strengths can be expanded in terms of the matrices.
\[ F = \sum_a F_a \lambda_a, \text{ so the vanishing of the anomaly requires} \]
\[ \sum \text{Tr} \lambda_a \{ \lambda_b, \lambda_c \} = 0 \quad (26.16) \]

where the sum is over the contribution of all fermions coupling to the gauge field under examination.

We first notice that when \( a, b, c \) all refer to the \( SU(2) \) matrices the contribution vanishes: \( \text{Tr} \tau_a \{ \tau_b, \tau_c \} = 2 \delta_{bc} \text{Tr} \tau_a = 0 \). When two refer to \( SU(2) \) and one to \( U(1) \) there is a potential anomaly proportional to \( \sum_L y_L \text{Tr} \tau_a \tau_b = 2 \delta_{ab} \sum_L y_L \) where the sum is over all the hypercharges of the left handed doublets. The contribution where only one index refers to \( SU(2) \) is clearly zero and we are left with the case where all indices refer to \( U(1) \). Then both left and right handed fermions contribute, but with opposite signs, so this contribution is proportional to

\[ \sum_R y_R^3 - \sum_L y_L^3 = 0. \quad (26.17) \]

It is fortunate that the fermion content of the standard model required by experiment satisfies the constraints on hypercharges we have just obtained. If we substitute the relation between electric charge and the weak hypercharges into the constraints, the first just requires that the charges of all the components of the left handed doublets sum to zero. Thus

\[ \lambda_{[b,\lambda_c,\lambda_d]} = \lambda_b[\lambda_c, \lambda_d] + \lambda_d[\lambda_b, \lambda_c] + \lambda_c[\lambda_d, \lambda_b] = 0, \quad (26.13) \]

where the indices enclosed in square brackets are completely antisymmetrized. However this condition is automatically satisfied if the first is:

\[ \lambda_{[b,\lambda_c,\lambda_d]} = \frac{1}{2} \{ \lambda_b, [\lambda_c, \lambda_d] \} + \frac{1}{2} \{ \lambda_d, [\lambda_b, \lambda_c] \} + \frac{1}{2} \{ \lambda_c, [\lambda_d, \lambda_b] \}, \quad (26.14) \]

by virtue of the Jacobi identity.
for the first generation this is realized because there are three "colors" for each quark: $-1 + 3(2/3 - 1/3) = 0$. The second generation consisting of the muon, muon neutrino, charmed quark and strange quark, has gauge couplings identical to the first and so the contribution to the anomaly from them also cancels. The third generation, follows the same pattern. the last member, the charged $2/3$ top quark, has recently (Spring 1995) been discovered at the TeVatron at Fermilab. There has long been evidence for the $\tau$ lepton, its neutrino, and the bottom quark (with charge $-1/3$). We can regard the required cancellation of anomalies as a prediction of the existence of the top quark, which has now been confirmed. The recent TeVatron experiments measure its mass to be $176 \pm 13$ GeV, almost two orders of magnitude larger than all other quarks and leptons.

We have yet to consider the "cubic" constraint from anomaly cancellation, $\sum_R y_R^3 - \sum_L y_L^3 = 0$. It is helpful to express this constraint also in terms of the ordinary electric charge.

$$\sum_R (2Q_R)^3 - \sum_L (2Q_L - 2I_3)^3 = 0 \quad (26.18)$$

$$8(\sum_R Q_R^3 - \sum_L Q_L^3) + 8 \sum_L (3Q_L^2 I_{3L} - 3Q_L I_{3L}^2 + I_{3L}^3) = 0. \quad (26.19)$$

The last term in the third sum vanishes within each doublet and the second term is proportional to $\sum_L Q_L$ because $(I_3)^2 = 1/4$, a constant for all terms. The contribution to the first term in the third sum from each doublet is $(Q_+^2 - (Q^- - 1)^2)/2$ which is just equal to the sum of the two charges of the doublet. Thus the whole third sum is proportional to $\sum_L Q_L$ which vanishes by the first "linear" constraint:

$$\sum_R y_R^3 - \sum_L y_L^3 = 8(\sum_R Q_R^3 - \sum_L Q_L^3) + 6 \sum_L Q_L. \quad (26.20)$$

Thus the new information in the "cubic" constraint reduces to

$$\sum_R (Q_R)^3 = \sum_L (Q_L)^3. \quad (26.21)$$

In the standard model each charged particle state has both a left and right handed component (this means it is possible for all charged particles to gain a (Dirac) mass), and this constraint is automatically satisfied. This left-right symmetry of nonzero charge assignments is an example of a vector-like $Q$. To define this "vector-like" property, first enumerate all of the fields according to their $L$ components. Thus we think of the right-handed fields as the charge conjugates of left handed fields: $R = L^c$, so that $Q_{L^c} = -Q_R$. In this new labeling the anomaly cancellation conditions read

$$\sum_{L \in \text{Doublet}} Q_L = 0 \quad (26.22)$$

$$\sum Q_L^2 = 0 \quad (26.23)$$
Then the charge operator $Q$ is vector-like if for each non-vanishing charge $Q > 0$ there are an equal number of left-handed fields with charge $Q$ and $-Q$. Thus the cubic equation above is automatically satisfied if $Q$ is vector-like. In the new notation the most general mass term is of the form

$$
\sum_{k,n} m_{k,n} L_k^T i \gamma^2 \gamma^0 L_n + h.c.
$$

(26.24)

If $Q$ is vector-like it is therefore possible to have a charge conserving mass for every field of non-zero charge. If $Q$ is not vector-like, there must remain at least one massless charged field.

But notice that the anomaly constraint could also have been satisfied in other more intricate ways, which would necessarily mean that $Q$ is not vector-like, entailing the prediction of massless charged fermions, which experiment strongly contradicts. The fact that the standard model fails to rule out such a possibility a priori, has led many theorists to the idea of grand unification: that the gauge group is a simple or semi-simple one (no $U(1)$ factors) and $SU(2) \times U(1)$ is just the remnant “low” energy group. One consequence of this hypothesis is the quantization of charge. In particular the hypercharge assignments will be determined by the representations to which each fermion is assigned. These will vary with the unification group and representation choice. Since the electric charge is proportional to a generator of a semi-simple group, its trace must be zero which means in our context that $\sum_L Q_L = 0$, a constraint satisfied in all unification schemes and satisfied by the fermions of the standard model (again because $Q$ is vector-like). It does not rule out massless charged particles completely, but it does put an additional restriction on how they can arise. But as pointed out by Alvarez-Gaume and Witten, this constraint also follows from the requirement that the fermions can consistently couple to (classical) gravity. We won’t show this, but there is a chiral anomaly for gravitons completely analogous to that for gauge particles. We don’t need its explicit form since the graviton couples “universally”. Thus this anomaly is just proportional to $\text{Tr} \lambda_a$ and vanishes for the $SU(2)$ current and for the $U(1)$ current gives $\sum_L y_L - \sum_R y_R = 2(\sum_L Q_L - \sum_R Q_R) = \sum L' Q_{L'} = 0$, where the last sum is over all left-handed fields in the new labeling scheme. Thus this argument for grand unification is weakened. Similarly, the consistent coupling of the electroweak $U(1)$ current to QCD requires an anomaly cancellation which holds if and only if $\sum_{L, \text{quarks}} Q_L - \sum_{L, \text{quarks}} Q_R = \sum_{\text{quarks}} Q_{L'} = 0$. These three constraints, viz. the cubic electroweak, gravitational, and QCD anomaly cancellation go some distance to forcing the vector-like character of $Q$.

In fact, if they are applied within a single “generation” $((\nu, e)_L, (u, d)_L, e_R, u_R, d_R)$, the vector-like character of $Q$ follows from the single further assumption that precisely one of the leptons is neutral. If the neutrino’s charge is fixed to be zero, the charge assignment to every other member of the generation is uniquely fixed by anomaly cancellation (Geng and Marshak, Minahan, Ramond and Warner) to be the standard one. More generally, with $Q_\nu \neq 0$ anomaly cancellation implies the following assignments:
Anomaly Free Charge Assignments: $Q_\nu \neq \frac{1}{2}$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$e_L$</th>
<th>$e_R$</th>
<th>$u_L$</th>
<th>$d_L$</th>
<th>$u_R$</th>
<th>$d_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_\nu$</td>
<td>$Q_\nu - 1$</td>
<td>$2Q_\nu - 1$</td>
<td>$\frac{1}{2} + \frac{1-2Q_\nu}{2N_c}$</td>
<td>$-\frac{1}{2} + \frac{1-2Q_\nu}{2N_c}$</td>
<td>$\pm \frac{1-2Q_\nu}{2} + \frac{1-2Q_\nu}{2N_c}$</td>
<td>$\mp \frac{1-2Q_\nu}{2} + \frac{1-2Q_\nu}{2N_c}$</td>
</tr>
</tbody>
</table>

Anomaly Free Charge Assignments: $Q_\nu = \frac{1}{2}$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$e_L$</th>
<th>$e_R$</th>
<th>$u_L$</th>
<th>$d_L$</th>
<th>$u_R$</th>
<th>$d_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$Q_{u_R}$</td>
<td>$-Q_{u_R}$</td>
</tr>
</tbody>
</table>

In the case $Q_\nu \neq 1/2, 0, 1$ none of the charge assignments allows a mass for even a single member of the generation: the entire generation must be massless! $Q_\nu = 0$ gives the standard assignments and $Q_\nu = 1$ the charge conjugate standard assignments, both cases allowing a $\Delta I_W = 1/2$ mass for $e, u,$ and $d$. The case $Q_\nu = 1/2$ forbids a $\Delta I_W = 1/2$ mass for the electron, but the up and down quarks can have a $\Delta I_W = 1/2$ mass if $Q_{u_R} = 1/2$. In this case a $\Delta I_W = 1$ mass for the $(\nu, e)$ doublet is possible. All of these possible mass terms violate the gauge symmetry and are forbidden unless the gauge symmetry is spontaneously broken (Higgs mechanism). Mass terms in the quark sector must be $\Delta I_W = 1/2$ (to be color singlets) and can arise if an $I_W = 1/2$ Higgs scalar develops a vacuum expectation value and has a Yukawa coupling to the quarks. In the cases $Q_\nu = 0, 1,$ a mass for the electron can arise from the same mechanism. However in the case $Q_\nu = 1/2$, one would also need an $I_W = 1$ Higgs scalar to give a mass to a lepton.

When an anomaly appears in the conservation of a global current (i.e. one that does not couple to gauge fields), it presents no inconsistency. It simply implies that a would be symmetry is broken by quantum effects. This can have important physical consequences, the most dramatic being its role in explaining $\pi_0$ decay into two photons, which we discussed earlier in these notes (see pp 355).

### 26.2.3 Mathematical Consequences of the Anomaly: Index Theorems

The chiral anomaly puts constraints on the eigenvalues of the Euclidean space Dirac operator $(1/i)\gamma \cdot D$ in the presence of sufficiently “nice” gauge fields. This differential operator is antihermitian, provided it acts on functions for which one can integrate by parts without keeping surface terms. We assume that $A$ is such that the Dirac operator has a complete set of eigenfunctions with this property. Then since it is antihermitian, the eigenvalues will be purely imaginary. For each nonzero eigenvalue $i\lambda_r$ there is another eigenvalue $-i\lambda_r$, because $\gamma_5$ anticommutes with $\gamma^\mu$: if $\psi_r$ is the eigenvector for $i\lambda_r$ then $\gamma_5\psi_r$ is the eigenvector for $-i\lambda_r$.  

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Now introduce the Euclidean Green function for \( m + \frac{1}{i} \gamma \cdot D \):

\[
(m + (1/i)\gamma \cdot D)S_E(x, y; A) = \delta(x - y).
\]

We can construct the gauge invariant \( \bar{S}_E \) in Euclidean space exactly as in Minkowski space. Then the quantity

\[
J_{5E}^\mu = -\text{Tr} \gamma_5 \gamma^\mu \lim_{y \to x} \bar{S}_E(x, y; A)
\]

has the anomalous conservation law

\[
\partial_\mu J_{5E}^\mu = 2im\text{Tr} \gamma_5 \lim \bar{S} - i\frac{g^2}{16\pi^2} \epsilon^{\mu \nu \rho \sigma} \text{Tr} F_{\mu \nu} F_{\rho \sigma}.
\]

The extra factor \(-i\) in the second term on the r.h.s. appears because we have continued the Minkowski result to Euclidean space: \( x^0 = -ix^4 \), and every contravariant time index gets this same \(-i\). We could have used a “Euclidean” \( \epsilon_E \) defined as the continuation of the Minkowski one so that \( \epsilon_E^{1234} \equiv i\epsilon^{1230} = -i \), and with such a definition no \(-i\) would appear. But the epsilon symbol is conventionally always understood to be real, and with the convention \( \epsilon^{1234} = +1 \) the \(-i\) must be explicitly included as written. Because \( m \neq 0 \) there are no infrared singularities so if we integrate both sides over \( x \), the l.h.s. will vanish and we get the identity

\[
2m \int d^4x \text{Tr} \gamma_5 \lim \bar{S} - \frac{g^2}{16\pi^2} \int d^4x \epsilon^{\mu \nu \rho \sigma} \text{Tr} F_{\mu \nu} F_{\rho \sigma} = 0.
\]

Now the quantity \( \text{Tr} \gamma_5 \bar{S} \) potentially has u.v. divergences, which would make the limit delicate. However the \( \gamma_5 \) requires multiplication by at least 4 gamma matrices to give a nonvanishing trace. To get this many one has to go to the order \( A^2 \) term in the weak field expansion which is a priori linearly divergent. But this is the term with 5 gamma matrices and that trace vanishes: the term with 4 gamma matrices is only log divergent. Finally to saturate the epsilon tensor one needs at least two vectors: one could be \( \epsilon \) but the other must be an external momentum, which gives one further power of convergence, enough to make it finite. Similarly, all higher terms are convergent. Thus \( \bar{S} \) can be replaced by \( S \), and the limit \( y \to x \) safely taken.

If the eigenfunctions of \((1/i)\gamma \cdot D\) are complete, we can represent the Green function as

\[
S_E(x, y; A) = \sum_r \frac{\psi_r(x)\psi_r^\dagger(y)}{m + i\lambda_r}
\]

and thus

\[
\int d^4x \text{Tr} \gamma_5 S = \sum_r \int d^4x \frac{\psi^\dagger_r(x)\gamma_5 \psi_r(x)}{m + i\lambda_r}
\]

But all terms for which \( \lambda_r \neq 0 \) vanish because \( \psi_r \) and \( \gamma_5 \psi_r \) then have different eigenvalues and so are orthogonal. Thus the sum is just over the values of \( r \) for which \( \lambda_r = 0 \). We
can organize the zero eigenfunctions according to the eigenvalues of $\gamma_5$ which are $+1$ and $-1$. Let $n_{\pm}$ be the number of zero eigenvalues with $\pm$ eigenvalue of $\gamma_5$. Then we have

$$\int d^4 x \text{Tr} \gamma_5 S = (n_+ - n_-)/m$$

and finally the Atiyah-Singer index theorem

$$\frac{g^2}{32\pi^2} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma} = (n_+ - n_-).$$

(26.31)

We have not been precise about the conditions on $A$ except to say that the Dirac operator must possess a complete set of eigenfunctions. It is not hard to show that $\epsilon F^2$ is a total derivative of a gauge noninvariant function. Thus if the $A$ falls off at infinity sufficiently rapidly we would expect the l.h.s. to vanish, so $n_+ = n_-$, with no conclusion about whether there are any zero eigenvalues of the Dirac operator. The proper condition is not that $A$ vanish at infinity, but rather that it approach a pure gauge there. Then the l.h.s. just measures the number of times this gauge function “winds around” the three-sphere at infinity. The manifold of $SU(2)$ is the three sphere, so nontrivial windings are possible in that case, but not in the $U(1)$ case, when the manifold is a circle. (You can’t lasso a sphere.) For such topologically nontrivial gauge fields the index theorem implies at least one vanishing eigenvalue for the Dirac operator. This has the nontrivial physical implication that the vacuum persistence amplitude, $\det \gamma \cdot D$, in the presence of such a field vanishes when the mass of the field vanishes.
Chapter 27

Higgs Mechanism and Custodial Symmetry

27.1 New Look at the Higgs Sector

In the standard model the Higgs field is a complex $SU(2)$ gauge doublet and $SU(3)$ gauge singlet,

$$ \phi = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}, \quad Y_\phi = -1 $$

$$ \phi_c = -i\sigma_2\phi^* = \begin{pmatrix} -\phi^-^* \\ \phi^0^* \end{pmatrix}, \quad Y_{\phi_c} = +1 $$

(27.1) (27.2)

and the Higgs mechanism is in effect when $\langle \phi^0 \rangle = v \equiv v / \sqrt{2} \neq 0$, and we may assume $v$ is real. The simplest scalar potential which dictates this nonzero VEV is

$$ V(\phi) = \lambda \frac{4}{4} (\phi^\dagger \phi - v^2)^2. $$

(27.3)

The symmetry of this potential is much bigger than the $SU(2) \times SU(2)$ gauge group of the electroweak theory. Indeed

$$ \phi^\dagger \phi = |\phi^-|^2 + |\phi^0|^2 = \phi_r^{\dagger -2} + \phi_i^{\dagger -2} + \phi_r^{02} + \phi_i^{02}, $$

(27.4)

where the subscripts $r, i$ denote real and imaginary parts, is invariant under $O(4)$ rotations! We are familiar with the isomorphism between $O(4)$ and $SU(2) \times SU(2)$, the latter of which is more convenient to employ in coupling the gauge fields of the standard model.

The $SU(2) \times SU(2)$ description of $O(4)$ is obtained by defining a $2 \times 2$ matrix scalar field

$$ \sqrt{2}\Phi \equiv I\phi^\dagger + i\sigma \cdot \phi = \begin{pmatrix} \phi^4 + i\phi^3 \\ -\phi^2 + i\phi^1 \\ \phi^2 + i\phi^1 \\ \phi^4 - i\phi^3 \end{pmatrix} $$

(27.5)

---

1In earlier chapters we defined $\langle \phi \rangle = v$ instead of the more standard $\langle \phi \rangle = v / \sqrt{2}$. In the following we shall use $v$ to denote the standard choice. We will continue to use both $v$ and $v \equiv v\sqrt{2}$ in our narrative.
where \((\phi^4, \phi)\) are the components of a 4-vector. Constraining these components to be real implies the constraint

\[
\Phi^* = \sigma_2 \Phi \sigma_2. \tag{27.6}
\]

Note that

\[
2 \det \Phi = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \tag{27.7}
\]
is the \(O(4)\) invariant form. Then the \(O(4)\) transformation is implemented as

\[
\Phi \to U_L \Phi U_R^\dagger \tag{27.8}
\]

where \(U_{L,R}\) are independent \(SU(2)\) matrices. Recall that the \(S\) of \(SU(2)\) means that the group elements are unitary \(2 \times 2\) matrices with unit determinant.

We can identify the \(U_L\) transformations with the gauge transformations under the \(SU(2)\) part of the electroweak gauge group if we identify the electroweak Higgs doublets as

\[
\begin{align*}
\phi &= \left( \begin{array}{c} \phi^0 \\ i\phi^- \end{array} \right) \\
\phi_c &= -i\sigma_2 \phi^* = \frac{1}{\sqrt{2}} \left( \begin{array}{c} i\phi^1 + \phi^2 \\ \phi^1 - i\phi^3 \end{array} \right).
\end{align*} \tag{27.9, 27.10}
\]

Then we can think of \(\Phi\) as a row vector with column vector entries

\[
\Phi = \left( \phi \ \phi_c \right) \tag{27.11}
\]

so that the transformation \(\Phi \to U(x)\Phi\) is just the electroweak \(SU(2)\) gauge transformation.

Next we consider the action of the electroweak \(U(1)\) gauge transformations. These multiply \(\phi\) and \(\phi_c\) by opposite phases. In the matrix language this can be done by a special \(SU(2)_R\) transformation:

\[
\Phi \to \Phi \begin{pmatrix} e^{-i\alpha(x)} & 0 \\ 0 & e^{i\alpha(x)} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \phi & e^{i\alpha} \phi_c \end{pmatrix} \tag{27.12}
\]

The diagonal matrix on the right is obviously unitary and has unit determinant so it is an element of \(SU(2)_R\). If \(\alpha(x)\) is the electroweak \(U(1)\) gauge transformation, it follows that the weak hypercharge is

\[
Y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{27.13}
\]

acting from the right.

The construction of the covariant derivative of \(\Phi\) is now evident:

\[
D_\mu \Phi = \partial_\mu \Phi - ig_2 W_\mu \Phi - ig_1 B_\mu \Phi \frac{Y}{2}, \tag{27.14}
\]
so we can proceed to the standard model Lagrangian. To express the Higgs potential in terms of $\Phi$ we note that

$$\Phi^\dagger \Phi = \left( \begin{array}{cc} \phi_c^\dagger \phi & \phi_c^\dagger \phi_c \\ \phi_c \phi & \phi_c \phi \end{array} \right) = I \phi^\dagger \phi$$  \hspace{1cm} (27.15)$$

where we used $\phi_c^\dagger \phi_c = 0$ and $\phi_c^\dagger \phi_c = \phi^\dagger \phi$. Taking the trace of both sides gives $\text{Tr} \Phi^\dagger \Phi = 2 \phi^\dagger \phi$.

Putting everything together, we have

$$L_{\text{higgs}} = -\frac{1}{2} \text{Tr}(D_\mu \Phi)^\dagger D^\mu \Phi - \frac{\lambda}{4} \left( \frac{1}{2} \text{Tr} \Phi^\dagger \Phi - v^2 \right)^2$$  \hspace{1cm} (27.16)$$

Now the symmetry of the Higgs potential under $SU(2)_L \times SU(2)_R$ is manifest. If $g_1 = 0$, the $SU(2)_R$ symmetry holds as a global symmetry of the derivative terms, because then $\Phi \to \Phi U$ is not obstructed by the presence of the $Y$ matrix. If $g_1 \neq 0$, the presence of $Y$ in the derivative term spoils this global symmetry. This global symmetry has been named a custodial symmetry, since by imposing it on generalizations of the standard model, some experimentally well-established relationships of the standard model will be maintained in the generalization.

### 27.2 Higgs Mechanism Revisited

Now let us see how the Higgs mechanism works in this new matrix language. Recall that in our original treatment, the VEV of the Higgs doublet was $(\phi^0, \phi^-) = (v, 0)$, and we were free to choose $v$ to be real. In terms of the matrix $\Phi$, the VEV would be $\Phi_0 = \langle \Phi \rangle = vI$. This VEV is obviously invariant under the diagonal $SU(2)$ where $U_L = U_R$. There are of course NGB’s described by the effective field $U_L \Phi_0 U_R = v U_L U_R^\dagger \equiv vU$. Notice that this description is parallel to our effective Lagrangian treatment of pions as NGB’s for spontaneously broken chiral symmetry.

We gain some insight into the workings of the Higgs mechanism by writing the effective Lagrangian for the NGB field:

$$D_\mu U = v \left[ \partial_\mu U - ig_2 W_\mu U - ig_1 B_\mu U \frac{Y}{2} \right]$$

$$= -ivU \left[ iU^\dagger \partial_\mu U + g_2 U^\dagger W_\mu U + g_1 B_\mu \frac{Y}{2} \right]$$

$$= -ig_2 vU \left[ W_\mu^\dagger + \frac{g_1}{g_2} B_\mu \frac{Y}{2} \right]$$  \hspace{1cm} (27.17)$$

But

$$W'^\mu \equiv U^\dagger W^\mu U + \frac{i}{g_2} U^\dagger \partial^\mu U$$  \hspace{1cm} (27.18)$$
is just an \(SU(2)\) gauge transformation on \(W^\mu\). This means that, if we define a new field \(W'\),
the matter and gauge part of the standard model Lagrangian will simply have \(W \rightarrow W'\) and
the NGB term reduces to

\[
-\frac{v^2}{2} \text{Tr}(D^\mu U)^\dagger D^\mu U = -\frac{g_2^2 v^2/2}{2} \left[ W'^2 + W'^2 + \frac{(W'_3 \cos \theta_W - B \sin \theta_W)^2}{\cos^2 \theta_W} \right]
\]

which provides the mass terms for \(W'\) and \(Z' = W'_3 \cos \theta_W - B \sin \theta_W\) with \(\tan \theta_W = g_1/g_2\).
Then \(M_W = g_2 v/2\) and, calling \(Z = W_3 \cos \theta_W - B \sin \theta_W\), \(M_Z = M_W/\cos \theta_W\). This is the
same mass term obtained by evaluating the Higgs part of the Lagrangian at \(\Phi_0\).

\[
\mathcal{L}_{higgs}(\Phi_0) = \frac{-v^2}{2} \text{Tr} \left( g_2 W^\dagger + g_1 B Y \right) \cdot \left( g_2 W + g_1 B Y^\dagger \right)
\]

\[
= -\frac{v^2}{4} \left[ g_2^2 (W_1^2 + W_2^2) + (g_2 W_3 - g_1 B)^2 \right]
\]

\[
= -\frac{g_2^2 v^2/2}{2} \left[ W'^2 + W'^2 + \frac{(W'_3 \cos \theta_W - B \sin \theta_W)^2}{\cos^2 \theta_W} \right] \tag{27.19}
\]

The fact that \(M_Z = M_W\) in the limit \(\theta_W \rightarrow 0\) can be understood as a consequence of custodial
\(SU(2)_R\), because for \(g_1 = 0\) \(SU(2)_R\) is an exact global symmetry of the Gauge-Higgs system.

To understand this statement we recall that \(W'\) is a combination of the original gauge field
and the NGB field \(U\). The construction of \(W'\) arranges the gauge transformation of \(W\) to
cancel that of \(U\), so that \(W'\) is “gauge invariant”. Although \(W\) was unaffected by the \(SU(2)_R\)
custodial symmetry, \(W'\) transforms under it as \(U_R W' U_R^\dagger\) because \(U \rightarrow U U_R\). Similarly, the
process of converting to \(W'\) involves the matter fields being redefined as \(F' = U^\dagger F\) so they
also transform under \(SU(2)_R\): \(F' \rightarrow U_R^\dagger F'\). Expressing the Lagrangian in terms of these
primed fields, which, by deleting the prime, can be thought of as choosing a gauge in which
the NGB field \(U = I\), one ends up with a gauge fixed (and therefore non gauge invariant)
Lagrangian which in the limit \(g_1 = 0\) and in the absence of the Yukawa couplings, enjoys a
global \(SU(2)_R\) symmetry which has important consequences.

We have seen that one consequence is \(M_W = M_Z\). Another important one is the equality
of couplings to the \(W\) and \(Z\) gauge bosons. Recall that the gauge couplings to fermions in
the standard model Lagrangian are

\[
g_2 f \bar{\gamma} \cdot (W_1 t_1 + W_2 t_2) \frac{I - \gamma_5}{2} f + \frac{g_2 Z}{\cos \theta_W} \left( t_3 \frac{I - \gamma_5}{2} - Q \sin^2 \theta_W \right) f \tag{27.20}
\]

The fact that the coefficient of the second term is 1 is a consequence of the custodial symmetry
when \(\theta_W = 0\). When studying the experimental consequences of the standard model, it is
useful to introduce a parameter called \(\rho\) which is a measure of the strength of neutral current
couplings relative to charged current couplings. In the effective low energy Lagrangian, the
interactions are parameterized by
\[
\mathcal{L}_{eff}' = \frac{G_F}{\sqrt{2}} \left[ \sum_{f_1, f_2} \bar{f}_1 t + \gamma (1 - \gamma_5) f_1 \cdot \bar{f}_2 t - \gamma (I - \gamma_5) f_2 \right. \\
- \left. \rho \left( \sum_f \bar{f}_3 t \gamma (I - \gamma_5) f_2 - 2J_{em} \sin^2 \theta_W \right)^2 \right]
\]  
(27.21)

where \( G_F/\sqrt{2} = g_2^2 / (8M_W^2) \). In writing this expression, we remember that since \( M_Z = M_W / \cos \theta_W \), \( \rho = 1 \) at tree level. However if electroweak radiative corrections are incorporated in \( \rho_{eff} \) the latter will only be 1 in the limit that the Yukawa couplings are zero and \( \theta_W = 0 \). Any physics beyond the standard model could also cause changes in \( \rho \). Experiments show that \( \Delta \rho = \rho - 1 \approx 0.01 \) to about 10%. This is consistent with the standard model violations of \( \rho = 1 \), the largest of which is due to the huge top quark mass, which contributes \( \Delta \rho = 3G_F m_t^2 / (8\pi^2 \sqrt{2}) \). This puts strong constraints on such new physics. Maintaining the custodial symmetry in generalizations helps satisfy these constraints.

### 27.3 Gauge-fixing the Standard Model v2

We now review the process of gauge-fixing using the matrix representation of the Higgs field. We focus on the electroweak part of the Lagrangian
\[
\mathcal{L}_{std} = \mathcal{L}_{Glue+\text{matter}} - \frac{1}{2} \text{Tr} F_W^2 - \frac{1}{4} F_B^2 - \frac{1}{2} \text{Tr} (D\Phi)^\dagger \cdot D\Phi - V(\Phi) 
\]  
(27.22)

\[
D_\mu \Phi = \partial_\mu \Phi - ig_2 W_\mu \Phi + ig_1 B_\mu \Phi t_3, \quad V(\Phi) = \frac{\lambda}{4} \text{Tr} \left( \frac{1}{2} \text{Tr} \Phi^\dagger \Phi - v^2 \right)^2
\]

where we used \( Y/2 = -t_3 \). Under an infinitesimal electroweak gauge transformation
\[
\begin{align*}
\Delta B_\mu &= -\partial_\mu \epsilon, \\
\Delta W_\mu &= -D_\mu G \\
\Delta \Phi &= -ig_2 G \Phi + ig_1 \epsilon \Phi t_3
\end{align*}
\]  
(27.23)

The Higgs potential is minimized for \( \Phi = vI \) so we change variables to \( \hat{\Phi} = \Phi - vI \), after which
\[
\begin{align*}
V(\Phi) &= \frac{\lambda v^2}{16} [\text{Tr}(\hat{\Phi} + \hat{\Phi}^\dagger)]^2 + \frac{\lambda v}{8} \text{Tr} \hat{\Phi}^\dagger \hat{\Phi} \text{Tr}(\hat{\Phi} + \hat{\Phi}^\dagger) + \frac{\lambda}{16} (\text{Tr} \hat{\Phi}^\dagger \hat{\Phi})^2 \\
-\frac{1}{2} \text{Tr}(D_\mu \Phi)^\dagger \cdot D^\mu \Phi &= -\frac{1}{2} \text{Tr}(D_\mu \hat{\Phi})^\dagger \cdot D^\mu \hat{\Phi} - \frac{v^2}{2} \text{Tr}(g_2 W - g_1 B t_3)^2 \\
-\frac{v}{2} \text{Tr}(D_\mu \hat{\Phi})^\dagger (-ig_2 W^\mu + ig_1 B^\mu t_3) \\
-\frac{v}{2} \text{Tr}(ig_2 W^\mu - ig_1 t_3 B^\mu) D_\mu \hat{\Phi}
\end{align*}
\]  
(27.24)
In the presence of SSB, 't Hooft invented a gauge that simplifies the quadratic terms in the Lagrangian. With strategic integration by parts the quadratic terms of the Higgs-gauge part of the Lagrangian are

\[-\text{Tr} \partial_\mu W_\nu \partial^\mu W^\nu - \frac{1}{2} \partial_\mu B_\nu \partial^\mu B^\nu - \frac{1}{2} \text{Tr}(\partial_\mu \hat{\Phi})^\dagger \partial^\mu \hat{\Phi} - \frac{\lambda v^2}{16} \text{Tr}(\hat{\Phi} + \hat{\Phi}^\dagger)^2 \]

+ \text{Tr}(\partial \cdot W)^2 + \frac{1}{2} (\partial \cdot B)^2 - \frac{v^2}{2} \text{Tr}(g_2 W - g_1 B t_3)^2

+ \frac{i v}{2} \text{Tr}(\hat{\Phi} - \hat{\Phi}^\dagger) (g_2 \partial \cdot W - g_1 \partial \cdot B t_3) \tag{27.25}\]

The awkward terms which make the kinetic terms non-diagonal are

\[+\text{Tr}(\partial \cdot W)^2 + \frac{1}{2} (\partial \cdot B)^2 + \frac{i v}{2} \text{Tr}(\hat{\Phi} - \hat{\Phi}^\dagger) (g_2 \partial \cdot W - g_1 \partial \cdot B t_3) \]

\[= \text{Tr} \left( \partial \cdot W + i g_2^v (\Phi - \Phi^\dagger) \right)^2 + \frac{1}{2} \left( \partial \cdot B - i g_1^v \frac{1}{2} \text{Tr} t_3 (\Phi - \Phi^\dagger) \right)^2 \]

\[+ \frac{g_2^v v^2}{16} \text{Tr}(\Phi - \Phi^\dagger)^2 + \frac{g_1^v v^2}{8} \left( \text{Tr} t_3 (\Phi - \Phi^\dagger) \right)^2 \tag{27.26}\]

The terms on the last line give masses to the NGB’s. ’t Hooft’s idea was to choose gauge fixing terms to cancel the terms on the middle line. Define gauge-fixing functions

\[F_B = \partial \cdot B - i \frac{g_1^v}{2} \text{Tr} t_3 (\Phi - \Phi^\dagger) \tag{27.27}\]

\[F_W = \partial \cdot W + \frac{i g_2^v}{4} (\Phi - \Phi^\dagger). \tag{27.28}\]

Then the ’t Hooft gauge is obtained by adding the terms $-\text{Tr} F_W^2 - (1/2) F_B^2$ to the Lagrangian. This arranges propagators with momentum independent numerators as in the Feynman gauge. A more general $\xi$ gauge is obtained by using the gauge-fixing functions

\[F_B^\xi = \partial \cdot B - i \xi \frac{g_1^v}{2} \text{Tr} t_3 (\Phi - \Phi^\dagger) \tag{27.29}\]

\[F_W^\xi = \partial \cdot W + i \xi \frac{g_2^v}{4} (\Phi - \Phi^\dagger) \tag{27.30}\]

and adding $-(1/\xi) \text{Tr} F_W^{\xi 2} - (1/(2\xi)) F_B^{\xi 2}$ to the Lagrangian. Then $\xi = 1$ reduces to the ’t Hooft-Feynman gauge and $\xi = 0$ to the ’t Hooft-Landau gauge, with transverse gauge propagators. For general $\xi$, the gauge propagator is determined by

\[[(q^2 + M^2) \eta_{\mu \rho} + (\xi^{-1} - 1) q_\mu q_\rho] \Delta^{\rho \nu} = -i \delta^{\rho \nu} \tag{27.31}\]

The scalar product of both sides with $q^\mu$ determines

\[q_\rho \Delta^{\rho \nu} = \frac{q^\nu}{M^2 + q^2 / \xi} \tag{27.32}\]

\[\Delta^{\mu \nu} = -i \frac{\eta^{\mu \nu} + (\xi - 1) q^{\mu} q^{\nu} / (\xi M^2 + q^2)}{q^2 + M^2} \tag{27.33}\]
This explicit form for the gauge propagator confirms that $\xi = 0$ gives the Landau gauge (transverse numerator), and $\xi = 1$ gives the 't Hooft-Feynman gauge ($q$ independent numerator). But there is one more interesting limit: $\xi \to \infty$. In this limit the gauge propagator becomes

$$-\frac{i \eta^{\mu\nu} + q^\mu q^\nu/M^2}{q^2 + M^2}$$

(27.34)

which is sometimes called the **unitary gauge.** The numerator has no singularities and projects, on-shell, onto the 3 physical polarizations of a massive vector particle. At the same time, as we shall see below, the masses of the NGB’s and of the FP ghosts go to infinity in this limit. Thus only the physical gauge and scalar degrees of freedom propagate. In this limit we have the situation described above in which the NGB’s have been “gauged away” and the gauge fields are “gauge invariant”. While this gauge is manifestly unitary it presents severe computational difficulties. The propagator has high momentum behavior $q^0$ rather than $q^{-2}$ scaling behavior of the other $\xi$ gauges. This wrecks the power counting for renormalization! So every gauge has flaws. For finite $\xi$ one has renormalizable power counting but unphysical propagating ghost particles which must decouple for unitarity. However if calculations respect gauge invariance the physics of all gauges is the same.

To obtain the ghost part of the Lagrangian we first subject the gauge-fixing functions to a gauge transformation:

$$\Delta F^\xi_B = -\partial^2 \epsilon - i\xi g_1 v^2 \text{Tr} t_3 (ig_2 G \Phi - ig_2 \Phi^\dagger G + ig_1 \Phi t_3 \epsilon + ig_1 t_3 \Phi^\dagger \epsilon)$$

(27.35)

$$\Delta F^\xi_W = -\partial \cdot DG + i\xi g_2 v^2 (ig_2 G \Phi - ig_2 \Phi^\dagger G + ig_1 \epsilon \Phi t_3 + ig_1 \epsilon t_3 \Phi^\dagger)$$

(27.36)

Then the ghost Lagrangian is obtained by replacing $G, \epsilon$ by $C_W, C_B$ respectively in The $\Delta F$’s and multiplying on left by $B_W, B_B$:

$$L_{gh} = \partial^\mu B_B \partial_\mu C_B + 2\partial^\mu B_W D_\mu C_W$$

$$-\xi g_1^2 v^2 \text{Tr} t_3 [(g_2 (C_W \Phi + \Phi^\dagger C_W) - g_1 (\Phi t_3 + t_3 \Phi^\dagger)C_B]$$

$$+\xi g_2^2 v^2 \text{Tr} B_W [g_2 (C_W \Phi + \Phi^\dagger C_W) - g_1 (\Phi t_3 + t_3 \Phi^\dagger)C_B]$$

(27.37)

We have already discussed the tree level mass terms for the gauge bosons, but we also have to consider the NGB’s in the Higgs sector, the FP ghosts and of course the Higgs particle. The mass terms for the NGB’s for general $\xi$ are

$$\xi \frac{g_2^2 v^2}{16} \text{Tr}(\Phi - \Phi^\dagger)^2 + \xi \frac{g_2^2 v^2}{8} (\text{Tr} t_3 (\Phi - \Phi^\dagger))^2$$

(27.38)

Recall $\Phi \sqrt{2} = I \phi_4 + 2i \mathbf{t} \cdot \phi$, which makes the derivative terms canonical

$$-\frac{1}{2} \text{Tr} \partial \Phi^\dagger \cdot \partial \Phi = -\frac{1}{2} ((\partial \phi_4)^2 + (\partial \phi)^2).$$

(27.39)
Each propagator is for the QED field, $B$ ghost has the mass of the neutral NGB in the ghost Lagrangian:

$$\phi$$

The NGB squared masses are proportional to $\xi$, and we see that for $\xi = 1$ the charged NGB’s $\phi_1 \pm i\phi_2$ have the same mass as the $W$ gauge bosons, and the neutral NGB $\phi_3$ has the same mass as the $Z$ gauge boson.

The masses assigned to the FP ghosts are obtained by setting $W = B = 0$ and $\Phi = vI$ in the ghost Lagrangian:

$$\partial^\mu B_B \partial_\mu C_B + 2\text{Tr} \partial^\mu B_W \partial_\mu C_W + \xi v^2 \text{Tr}(g_2 B_W - g_1 t_3 B_B)(g_2 C_W - g_1 t_3 C_B)$$

$$= \partial^\mu B_Z \partial_\mu C_Z + \partial^\mu B_A \partial_\mu C_A + 2\text{Tr} \partial^\mu B_{W_{1,2}} \partial_\mu C_{W_{1,2}}$$

$$+ \xi g_2^2 v^2 \text{Tr} B_{W_{1,2}} C_{W_{1,2}} + \xi \frac{g_2^2 v^2}{2 \cos^2 \theta_W} B_Z C_Z$$

where $B_Z = B_{W_3} \cos \theta_W - B_B \sin \theta_W$ and $C_Z = C_{W_3} \cos \theta_W - C_B \sin \theta_W$. So the charged ghosts have the same mass as the NGB’s which at $\xi = 1$ is that of the $W$ and the neutral ghost has the mass of the the neutral NGB which at $\xi = 1$ is the mass of the $Z$. The ghosts for the QED field, $B_A, C_A$, are not only massless but are also free (non-interacting). This means that in calculating correlations of the interacting fields, they contribute nothing and can be ignored.

In summary the gauge/higgs sector of the standard model includes gauge bosons, NGB’s, and FP ghosts. In the ’t Hooft-Feynman gauge ($\xi = 1$), the numerator of the propagators for these various particles is constant, and there are three tree level masses:

$$M_W^2 = \frac{g_2^2 v^2}{2} = \frac{g_2^2 v^2}{4} : W_{1,2}, \phi_{1,2}, B_{W_{1,2}}, C_{W_{1,2}}$$

$$M_Z^2 = \frac{M_W^2}{\cos^2 \theta_W} : Z^\mu, \phi_Z, B_Z, C_Z$$

$$M_y^2 = 0; A^\mu, B_A, C_A$$

Each propagator is $-i/(p^2 + M_i^2)$ for the spin 0 particles and this times $\eta_{\mu\nu}$ for the vector particles.

The mass eigen fields are

$$W_{1,2}, Z^\mu = W_3 \cos \theta_W - B^\mu \sin \theta_W, A^\mu = B^\mu \cos \theta_W + W_3^\mu \sin \theta_W$$

for the gauge bosons. For the higgs fields the mass eigenfields are linked to our matrix field $\Phi$ via $\Phi = (\phi_4 + 2it \cdot \phi)/\sqrt{2}$. The NGB’s are $\phi$ and masses come entirely from the covariant derivative terms of the Lagrangian. The higgs scalar is $h = \phi_4 - v\sqrt{2}$ and its mass is given by the quadratic term in the potential:

$$V(h, \phi) = \frac{\lambda v^2}{2} h^2 + \frac{\lambda v}{2\sqrt{2}} (h^2 + \phi^2) + \frac{\lambda}{16} (h^2 + \phi^2)^2$$
Thus the higgs mass is

\[ m_h^2 = \lambda v^2 = \frac{\lambda v^2}{2} = \frac{2\lambda}{g_2^2} M_W^2 \]

(27.47)

Since \( m_h \approx 125\text{GeV} \) and \( M_W \approx 80\text{GeV} \) this leads to an estimate of \( \lambda \approx 1.2g_2^2 \).
27.4 Fermion Contributions to the $\rho$ Parameter

Corrections to the gauge propagators which contribute to the $\rho$ parameter must be different for the $W$ boson and the $Z$ boson. In order to identify the contribution from leptons and quarks we need “vacuum polarization” diagrams. The generic such diagram will have different masses for the two fermion propagators and the gauge vertices $\gamma^\mu (1 + h_1 \gamma_5)$ and $\gamma^\nu (1 + h_2 \gamma_5)$. The calculation is very similar to the QED vacuum polarization we did last year, and I leave its evaluation as an exercise. The result, using dimensional regularization, is

$$\Pi_{\mu\nu}(k) = \eta_{\mu\nu} \Pi_0(k^2) + (k^2 \eta_{\mu\nu} - k_\mu k_\nu) \Pi(k^2)$$

where

$$\Pi(k^2) = \frac{1 + h_1 h_2}{2\pi^2} \int_0^1 dx x (1 - x) \ln \left\{ \frac{\Lambda^2}{A^2(x)} \right\}$$

$$\Pi_0(k^2) = \frac{1}{4\pi^2} \int_0^1 dx \left( (1 + h_1 h_2)[m_1^2 x + m_2^2 (1 - x)] - m_1 m_2 (1 - h_1 h_2) \right) \ln \left\{ \frac{\Lambda^2}{A^2(x)} \right\}$$

with $A^2 = m_1^2 x + m_2^2 (1 - x) + k^2 (1 - x)$, and the divergence as $d \to 4$ has been combined with the $\mu$ parameter that defines the scale in dimensional regularization to give a cutoff parameter $\Lambda^2$.

An easy check on this result is the QED case $h_1 = h_2 = 0$ and $m_1 = m_2 = m$, so $\Pi_0 \to 0$ and:

$$\Pi_{\mu\nu}(k^2) = \frac{1}{2\pi^2} (k^2 \eta_{\mu\nu} - k_\mu k_\nu) \int_0^1 dx x (1 - x) \ln \left\{ \frac{\Lambda^2}{m^2 + x (1 - x) k^2} \right\}$$

giving the running fine structure constant

$$\alpha(k^2) = \alpha + \frac{2\alpha^2}{\pi} \sum_j Q_j^2 \int_0^1 dx x (1 - x) \ln \left\{ \frac{m_j^2 + x (1 - x) k^2}{m_j^2} \right\}$$

(27.48)

Putting in the contribution of the known charged particles gives $\alpha(-M_Z^2) \approx 1/129$.

For the $W$ gauge boson, $h_1 = h_2 = -1$ and the two masses are the masses of the two members of each doublet. The product of the gauge couplings in this case is $g_2^2/8$ so

$$\Pi(k^2) = \frac{g_2^2}{16\pi^2} \int_0^1 dx x (1 - x) \ln \left\{ \frac{\Lambda^2}{m_1^2 x + m_2^2 (1 - x) + k^2} \right\}$$

$$\Pi_0(k^2) = \frac{g_2^2}{16\pi^2} \int_0^1 dx [m_1^2 x + m_2^2 (1 - x)] \ln \left\{ \frac{\Lambda^2}{m_1^2 x + m_2^2 (1 - x) + k^2} \right\}$$

For $k \ll m_i, M_W$ the correction factor to the $W$ boson propagator is simply

$$1 - \frac{g_2^2}{16\pi^2 M_W^2} \int_0^1 dx [m_1^2 x + m_2^2 (1 - x)] \ln \left\{ \frac{\Lambda^2}{m_1^2 x + m_2^2 (1 - x)} \right\}$$

(27.49)
By far the largest effect is due to the top quark, when one of the masses is \( m_t \) and the other is negligible:

\[
1 - \frac{3g_2^2m_t^2}{16\pi^2M_W^2} \int_0^1 dx x \ln \left\{ \frac{\Lambda^2}{m_t^2 x} \right\} \tag{27.50}
\]

For the neutral currents we examine the corrections to the \( Z \) propagator. We first write

\[
\Pi_{ZZ} = \Pi_{33} \cos^2 \theta_W + \Pi_{BB} \sin^2 \theta_W - 2 \sin \theta_W \cos \theta_W \Pi_{3B} \tag{27.51}
\]

and define

\[
I(m) = \int_0^1 dx m^2 \ln \frac{\Lambda^2}{m^2 + x(1-x)k^2} = m^2 \ln \frac{\Lambda^2}{m^2} + O(k^2) \tag{27.52}
\]

Then, neglecting the tiny \((\eta_{\mu\nu}k^2 - k_{\mu}k_{\nu})\) contribution,

\[
\begin{align*}
\Pi_{33} &= \frac{g_2^2}{32\pi^2}(I(m_U) + I(m_D)) \tag{27.53} \\
\Pi_{3B} &= \frac{g_1g_2}{32\pi^2}(I(m_U)(y_L - y_R^U) - I(m_D)(y_L - y_R^D)) \tag{27.54} \\
\Pi_{BB} &= \frac{g_1^2}{32\pi^2}[(y_L^2 + y_R^{U2})I(m_U) + (y_L^2 + y_R^{D2})I(m_D) - 2y_Ly_R^{U}I_U - 2y_Ly_R^{D}I_D] \\
&= \frac{g_1^2}{32\pi^2}[(y_L - y_R)^2I(m_U) + (y_L - y_R^D)^2I(m_D)] \tag{27.55}
\end{align*}
\]

Now notice that \( y_L - y_R = -2I_3 = -\tau_3 \), so these expressions reduce to

\[
\begin{align*}
\Pi_{33} &= \frac{g_2^2}{32\pi^2}(I(m_U) + I(m_D)) \tag{27.56} \\
\Pi_{3B} &= -\frac{g_1g_2}{32\pi^2}(I(m_U) + I(m_D)) \tag{27.57} \\
\Pi_{BB} &= \frac{g_1^2}{32\pi^2}[I(m_U) + I(m_D)] \tag{27.58}
\end{align*}
\]

so that

\[
\Pi_{ZZ} = \frac{(g_2 \cos \theta + g_1 \sin \theta_W)^2}{32\pi^2}[I(m_U) + I(m_D)] \tag{27.59}
\]

\[
= \frac{g_2^2}{32\pi^2 \cos^2 \theta_W}[I(m_U) + I(m_D)]
\]

\[
= \frac{g_2^2M_Z^2}{32\pi^2M_W^2} \left[ m_U^2 \ln \frac{\Lambda^2}{m_U^2} + m_D^2 \ln \frac{\Lambda^2}{m_D^2} \right] + O(k^2)
\]

In all these formulas \( U, D \) refer to the top and bottom members of an \( SU(2) \) doublet. Since the top quark is so much more massive than the leptons and other quarks, it is the only
contribution that matters in the contribution to $\rho$:

$$
\rho \approx \frac{\text{Neutral}}{\text{Charged}} = \frac{1 - 3g_2^2m_t^2/(32\pi^2M_W^2)\ln\Lambda^2/m_t^2}{1 - 3g_2^2m_t^2/(16\pi^2M_W^2)\int_0^1 dx x\ln(1/x)}
$$

$$
\rho - 1 \approx \frac{3g_2^2m_t^2}{16\pi^2M_W^2} \int_0^1 dx x\ln(1/x) \approx 0.01
$$

$$
\rho - 1 \approx \frac{3G_Fm_t^2}{8\pi^2\sqrt{2}} \approx 0.01
$$

(27.60)

to be compared with the experimental value $\Delta \rho = 0.011 \pm .0013 \pm .0018$. The factor of 3 in the above formula is just the number of quark colors $N_c = 3$. 

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Chapter 28

Electroweak Interactions of Leptons and Quarks

28.1 Vector Boson decay to Leptons and Quarks

The spin polarization of a massive vector boson can be described by a polarization vector \( \epsilon_\mu(k) \) satisfying \( k_\mu \epsilon_\mu = 0 \). In the rest frame of the boson this condition reads \( M_W \epsilon_0 = 0 \) so in the rest frame the polarization vector is purely spatial and its components correspond to the three \( S_z \) eigenstates \( 1, 0, -1 \) of a spin one particle. The rest frame normalization condition \( \epsilon^*_\lambda \cdot \epsilon_\lambda = \delta_{\lambda\lambda'} \) can be written \( \epsilon^*_\lambda \cdot \epsilon_\lambda = \delta_{\lambda\lambda'} \) as long as \( \epsilon \) transforms as a 4-vector.

28.1.1 \( W \) decay

The \( W \) boson can decay at lowest order into pairs of particles with total mass less than \( M_W \). Clearly the top quark at 175GeV and the Higgs particle at 125GeV are too massive to participate. This leaves all the leptons and the \( u,d,s,c,b \) quarks. Of course quarks in the final state eventually “hadronize and end up as baryons and mesons, but at this stage we treat them as though they are actually particles. Thus the final states in the decay of the \( W^- \) are

\[
l + \bar{\nu}_l, \quad l = e, \mu, \tau; \quad D + \bar{U}, \quad D = d, s, b
\]

For each of the \( D \) quarks, the corresponding \( \bar{U} \) is a linear combination of \( \bar{u} \) and \( \bar{c} \), due to the nontrivial \( CKM \) matrix. At this stage we neglect neutrino masses, so lepton flavor mixing will be absent.

The Feynman amplitude for the lepton processes is

\[
\mathcal{M} = \frac{g_2}{2\sqrt{2}} \bar{u}(p) \gamma \cdot \epsilon (1 - \gamma^5) v(p')
\]

\[
\sum_{\text{spins}} |\mathcal{M}|^2 = g_2^2 \left[ p \cdot \epsilon p' \cdot \epsilon^* + p \cdot \epsilon^* p' \cdot \epsilon - p \cdot p' + i \epsilon^{\alpha\lambda\mu\nu} p_\alpha \epsilon_{\lambda\mu} p'_\nu \epsilon^* \right]
\]
The details are in an exercise from last semester.

Let $Q = p + p'$ be the momentum of the $W$, so $Q \cdot \epsilon = 0$, and hence $p' \cdot \epsilon = -p \cdot \epsilon$. We work in the $W$ rest frame, $Q = 0$. Then we can write the last term as

$$i \epsilon^{\kappa \lambda \mu \nu} p_{\kappa} \epsilon_\lambda p'_{\mu} \epsilon^*_{\nu} = i \epsilon^{\kappa \lambda \mu \nu} p_{\kappa} \epsilon_\lambda Q_{\mu} \epsilon^*_{\nu} = -iM_W p \cdot (\epsilon \times \epsilon^*)$$  \hspace{1cm} (28.4)

and then

$$\sum_{\text{spins}} |\mathcal{M}|^2 = g_2^2 \left[ -2p \cdot \epsilon \cdot \epsilon^* + \frac{M_W^2 - (m_1^2 + m_2^2)}{2} - iM_W p \cdot (\epsilon \times \epsilon^*) \right]$$ \hspace{1cm} (28.5)

The total rate is given by

$$\Gamma = \int \frac{d\Omega}{(2\pi)^2} \frac{1}{8M_W} \sqrt{1 - \frac{(m_1 + m_2)^2}{M_W^2}} \sqrt{1 - \frac{(m_1 - m_2)^2}{M_W^2}} \langle |\mathcal{M}|^2 \rangle$$ \hspace{1cm} (28.6)

Here $p$ is determined from energy conservation

$$\frac{p}{M_W} = \frac{1}{2} \sqrt{1 - \frac{(m_1 + m_2)^2}{M_W^2}} \sqrt{1 - \frac{(m_1 - m_2)^2}{M_W^2}}$$ \hspace{1cm} (28.7)

For the lepton case, $p^2 = (M_W^2 - m_1^2)/4$

$$\int d\Omega \langle |\mathcal{M}|^2 \rangle = 4\pi g_2^2 \left[ \frac{M_W^2 - m_1^2}{2} - \frac{2}{3}p^2 \right]$$

$$= 4\pi g_2^2 (M_W^2 - m_1^2) \left[ \frac{1}{3} + \frac{m_1^2}{6M_W^2} \right]$$ \hspace{1cm} (28.8)

so the total rate is

$$\Gamma_{\nu_1} = \frac{G_F M_W^3}{6\pi \sqrt{2}} \left( 1 - \frac{m_1^2}{M_W^2} \right)^2 \left( 1 + \frac{m_1^2}{2M_W^2} \right)$$ \hspace{1cm} (28.9)

For the quarks, if we keep in the masses, $\langle |\mathcal{M}|^2 \rangle$ has an extra term proportional to $m_im_j$, the phase space is more complicated, there are three colors of each flavor, and a factor $|V_{ij}|^2$ where $V$ is the Kobayashi-Maskawa flavor changing matrix. These differences then lead to

$$\Gamma_{ij} = 3|V_{ij}|^2 \frac{G_F M_W^3}{6\pi \sqrt{2}} \sqrt{1 - \frac{(m_i + m_j)^2}{M_W^2}} \sqrt{1 - \frac{(m_i - m_j)^2}{M_W^2}} \left[ 1 - \frac{m_i^2 + m_j^2}{2M_W^2} - \frac{(m_i^2 - m_j^2)^2}{2M_W^2} \right]$$

We cannot observe the quark final states, since they are confined inside of hadrons. Instead the produced quarks will be 100% converted into hadrons. Assuming that the total rate for
decay into all possible hadronic states is given by the total rate calculation for all possible quark anti-quark pairs, we can estimate the ratio $\Gamma_{W\to\text{hadrons}}/\Gamma_{W\to\text{leptons}}$ and compare to the data. If we neglect masses, all of the decay rates simplify to

$$\Gamma_{\ell\nu} = \frac{g_2^2 M_W^3}{48\pi M_W^2} = \frac{G_F M_W^3}{16\pi^2} \quad \Gamma_{ij} = 3|V_{ij}|^2 \frac{G_F M_W^3}{6\pi\sqrt{2}}$$

(28.10)

The total leptonic rate is simply a factor of 3 times the first expression. The total hadronic rate is a bit more subtle since we sum over 3 down type quarks but only over two up type quarks. This involves the CKM matrix:

$$\sum_{j=u,c} \sum_{i=d,s,b} V_{ji}^* V_{ij} = \sum_{j=u,c} \delta_{jj} = 2$$

(28.11)

then

$$\frac{\Gamma_{\text{hadrons}}}{\Gamma_{\text{leptons}}} \approx 3 \cdot \frac{2}{3} = 2$$

(28.12)

Looking up the data, the branching fraction into hadrons is $0.6741 \pm 0.0027$ and that into leptons is $0.3086 \pm 0.0009$. The ratio is $0.6741/0.3086 \approx 2.18 \pm 0.02$. Pretty good agreement for such a simple model!

### 28.1.2 Z decay

The Feynman amplitude for the decay of the $Z$ in lowest order is

$$\mathcal{M} = \frac{g_2}{4\cos\theta_W} \bar{u}(p) \gamma \cdot \epsilon \left[ \tau_3 (1 - \gamma^5) - 4Q \sin^2 \theta_W \right] v(p')$$

(28.13)

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g_2^2}{16\cos^2\theta_W} \text{Tr}(m - \gamma \cdot p) \epsilon \cdot \gamma \left[ \tau_3 (1 - \gamma^5) - 4Q \sin^2 \theta_W \right]$$

$$(-m - \bar{p} \cdot \gamma) \gamma \cdot \epsilon^* \left[ \tau_3 (1 - \gamma^5) - 4Q \sin^2 \theta_W \right]$$

(28.14)

This expression should be summed over each fermion, where the mass, $\tau_3$, and $Q$ vary with the fermion. The trace can be evaluated without specifying these values. For simplicity we consider the $Z$ to be unpolarized, which is implemented by

$$\langle \epsilon_\mu \epsilon_{\nu^*} \rangle = \frac{1}{3} \left( \eta_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2} \right)$$

(28.15)

where $q = p + \bar{p}$ is the energy momentum of the $Z$. The trace of terms linear in $\gamma_5$ will then be zero because to be nonzero would require 4 different gamma matrices, and the average
over polarizations forces two of them to be the same. Thus

\[
\langle \sum_{\text{spins}} |\mathcal{M}|^2 \rangle = \frac{g_2^2}{48 \cos^2 \theta_W} \left( \eta_{\mu \nu} + \frac{q_\mu q_\nu}{M_Z^2} \right)
\]

\[
\left\{ [\tau_3 - 4Q \sin^2 \theta_W]^2 \text{Tr}(m - \gamma \cdot p)\gamma^\mu(-m - \bar{p} \cdot \gamma)\gamma^\nu + \text{Tr}(m - \gamma \cdot p)\gamma^\mu\gamma_5(-m - \bar{p} \cdot \gamma)\gamma^\nu\gamma_5 \right\}
\]

\[
= \frac{g_2^2}{48 \cos^2 \theta_W} \left( \eta_{\mu \nu} + \frac{q_\mu q_\nu}{M_Z^2} \right)
\]

\[
4 \left\{ m^2 \eta^{\mu \nu} ([\tau_3 - 4Q \sin^2 \theta_W]^2 - 1)
\right.
\]

\[
+ ([\tau_3 - 4Q \sin^2 \theta_W]^2 + 1)(-2p^\mu p^\nu - p \cdot \bar{p} \eta^{\mu \nu}) \right\}
\]

Now use

\[
p \cdot \bar{p} = m^2 - \frac{M_Z^2}{2}, \quad p \cdot q = -\frac{M_Z^2}{2}
\]

(28.16)

to get

\[
\langle \sum_{\text{spins}} |\mathcal{M}|^2 \rangle = \frac{g_2^2}{4 \cos^2 \theta_W} \left\{ m^2 ([\tau_3 - 4Q \sin^2 \theta_W]^2 - 1)
\right.
\]

\[
+ ([\tau_3 - 4Q \sin^2 \theta_W]^2 + 1) \left( \frac{2}{3} (m^2 - \frac{(p \cdot q)^2}{M_Z^2}) - p \cdot \bar{p} \right) \right\}
\]

(28.17)

The contribution of a single \( f \bar{f} \) pair to the decay rate is obtained by multiplying this result by \( p/(8\pi M_Z^2) \), where \( p = (M_Z/2)\sqrt{1 - 4m_f^2/M_Z^2} \). We can also write \( g^2/(8 \cos^2 \theta_W) =
\]
\( G_F M_W^2 / (\sqrt{2} \cos^2 \theta_W) = G_F^2 M_Z^2 / \sqrt{2}, \) so that

\[
\Gamma_f = \frac{G_F M_Z^3}{12\pi \sqrt{2}} \sqrt{1 - \frac{4m_f^2}{M_Z^2}} \left\{ 4(1 + \frac{2m^2}{M_Z^2})(2Q^2 \sin^4 \theta_W - \tau_3 Q \sin^2 \theta_W) + 1 - \frac{m_f^2}{M_Z^2} \right\} \tag{28.18}
\]

The total decay rate is the sum of this expression over all \( f \) for which \( 2m_f < M_Z \).

The contribution of all massless neutrinos is particularly simple:

\[
\Gamma_{\nu\bar{\nu}} = \frac{N_{\nu} G_F M_Z^3}{12\pi \sqrt{2}} \approx \frac{N_{\nu}}{3} 496\text{MeV} \tag{28.19}
\]

Compared to the experimental value from LEP of 499MeV. This measures the number of (massless) neutrinos to be 3. The number of observed generations. Of course neutral massless fermions that are completely neutral under the standard model gauge group (sterile neutrinos) would not be produced at LEP.

The partial rate into neutrinos is not measured directly. LEP was an \( e^+e^- \) collider running in the tunnel now used by the LHC. with \( E \sim M_Z \), the machine excites the \( Z \) resonance copiously. The total rate for \( Z \) decay determines the width of the Breit-Wigner resonance shape

\[
\text{amplitude} \propto \frac{\Gamma}{E - M_Z + i\Gamma/2} + \text{background} \tag{28.20}
\]

Thus \( \Gamma \) was inferred from careful measurement of the event shape in total energy. Then, separately, the branching fraction into visible final states, charged leptons, photons, and hadrons, was measured. Then the difference

\[
\Gamma - \Gamma_{\text{visible}} \approx 499\text{MeV} \tag{28.21}
\]

determines an upper bound on the missing final state contribution, including neutrinos and any other undetected particles. Then the calculated partial width into the known 3 neutrinos leaves very little room for other undetected particles.

It is a good first approximation to take the masses of all the contributing leptons and quarks to be zero. Then our lowest order calculation simplifies to

\[
\Gamma_f = \frac{G_F M_Z^3}{12\pi \sqrt{2}} \left\{ 4(2Q^2 \sin^4 \theta_W - \tau_3 Q \sin^2 \theta_W) + 1 \right\} \tag{28.22}
\]

The quantity inside the braces is assumes the values

\[
[1 + 8 \sin^4 \theta_W - 4 \sin^2 \theta_W] \times 3 \quad \text{for} \quad e, \mu, \tau \tag{28.23}
\]
\[
[1 + (8/9) \sin^4 \theta_W - (4/3) \sin^2 \theta_W] \times 9 \quad \text{for} \quad d, s, b \tag{28.24}
\]
\[
[1 + (32/9) \sin^4 \theta_W - (8/3) \sin^2 \theta_W] \times 6 \quad \text{for} \quad u, c \tag{28.25}
\]

where \( \sin^2 \theta_W \approx 0.22 \). With the assumption that the rate into hadrons can be approximated by the rate into quarks you can get a prediction for the total width of the \( Z \) to compare to the measured width 2.49GeV.
\[
\frac{g}{\sin \theta_W} f y^3 (\tilde{\ell}^-_L \cdot Q^+ \epsilon) \xi
\]
\[
g^\nu = -\nu \pm i \nu \\
g = t_3
\]
\[
\sum_{n} |m|^2 = \frac{g^2}{4 \cos^2 \theta_W} \sum_{n} \left( m_{t_3} - r_{\nu} \right) \epsilon \cdot \epsilon^* + \sum_{n} \left( q^2 + m^2 \right) \epsilon \cdot \epsilon^* + \left( q^2 + m^2 \right) \left( \epsilon^* \cdot \epsilon + q^2 \right)
\]
\[
\sum_{n} |m|^2 = \frac{g^2}{4 \cos^2 \theta_W} \sum_{n} \left( m_{t_3} - r_{\nu} \right) \epsilon \cdot \epsilon^* + \sum_{n} \left( q^2 + m^2 \right) \epsilon \cdot \epsilon^* + \left( q^2 + m^2 \right) \left( \epsilon^* \cdot \epsilon + q^2 \right)
\]
\[
2 \nu e V_{\mu}, \quad e = (0, \vec{e}) \\
\vec{p}_\nu = -\vec{p}_e, \quad \epsilon^* \cdot \epsilon = 1
\]
\[
E_\nu = E_e = \frac{m_e^2}{2} \\
\vec{p}_\nu \cdot \vec{p}_e = -\vec{p}_e^2 = E_e^2
\]
\[
\vec{p}_\nu \cdot \vec{v}_\nu = -\frac{m_e^2}{2} + m_e^2 \\
\vec{p}_\nu \cdot \vec{e} = -\vec{p}_e \cdot \vec{e} = \vec{p}_e \cdot \vec{e}
\]
\[
\theta (x+iy) \times (x-iy) = i (x \vec{e} \times -i \vec{e})
\]
\[
\sum_{\mu} |m|^2 = \frac{g^2}{4 \cos^2 \theta_W} \left\{ 4 m_{t_3}^2 (g^v - g^a) - 4 (g^v + g^a) \left[ m_e^2 - \frac{m_e^2}{2} \right] + 4 (g^v + g^a) \left[ \vec{p}_\nu \cdot \vec{p}_e \right] \right\}
\]
\[
\left( i \vec{e} \times \vec{e}^* \right) \cdot \epsilon^* \left\{ \begin{array}{c}
-\frac{m_e^2}{2} + \frac{m_e^2}{2} \\
+ 16 i g_\nu g_a E_e e^* \cdot \epsilon \cdot \epsilon^* \cdot \epsilon^* \\
\end{array} \right\}
\]
\[
Polarized Z = \frac{g^2}{4 \cos^2 \theta_W} \left\{ -8 m_e^2 g^2 + 2 \left( g^v_3 + g^a_3 \right) \vec{H}_R^2 - 8 \left( g^v_3 + g^a_3 \right) \vec{p}_e \cdot \vec{e}^* \left( \frac{m_e^2}{4} \right) \right\}
\]
\[
\left( \langle \epsilon \cdot \epsilon \rangle \right) = \frac{g^2}{4 \cos^2 \theta_W} \left\{ -8 m_e^2 g^2 + 2 \left( g^v_3 + g^a_3 \right) \vec{H}_R^2 \left[ 1 - \frac{1}{3} \left( \frac{m_e^2}{H_R^2} \right) \right] \right\}
\]
\[
\left( \langle \epsilon \cdot \epsilon \rangle \right) = \frac{g^2}{4 \cos^2 \theta_W} \left\{ -8 m_e^2 g^2 + 2 \left( g^v_3 + g^a_3 \right) \vec{H}_R^2 \left[ 1 - \frac{1}{3} \left( \frac{m_e^2}{H_R^2} \right) \right] \right\}
\]
2. \( \text{LEP} \) \( e^+ e^- \rightarrow \text{anything} \)

\[
\begin{align*}
\text{Width of } Z: & \quad \Gamma_Z = \frac{\pi}{\alpha} \Gamma_{\text{tot}} \frac{1}{(E-E_0)^2 + m_Z^2} \\
\Gamma_{\text{tot}} &= 2.495 \text{ GeV} \\
\Gamma_{\text{pre}} &= 83.99 \text{ MeV} \\
g_Y &= t_3 - 2 q_m q_{\nu}
\end{align*}
\]

\[
\sum |m|^2 = \frac{g^2}{4 \sin^2 \theta_W} \left| \bar{u} \gamma^\mu (g_Y + g_A \gamma_5) u \right|^2
\]

\[
\Gamma_{\text{tot}} = \frac{\pi}{\alpha} \frac{1}{(E-E_0)^2 + m_Z^2}
\]

\[
\begin{align*}
\Gamma_{\text{pre}} &= \text{Width of } \beta - \nu \\
\Gamma_{\text{pre}} &= \text{Cross section for detection}
\end{align*}
\]

\[
\begin{align*}
\Gamma_{\text{pre}} &= \frac{N_v G_F H_3^2}{8 \pi t_3} \\
\Gamma_{\text{pre}} &= 3 \frac{G_F H_3^2}{8 \pi t_3} \left[ 2 \left( \frac{1}{2} - 2 \sin^2 \theta_W \right)^2 + \frac{1}{2} \right]
\end{align*}
\]
28.2 Lepton or Quark Beta Decay

Muon Decay

The detailed study of the process $\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$ was crucial to establishing the current-current structure of the weak interactions, but in modern terms it represents the cleanest measurement of $G_F$. For this process, and indeed most of the processes which lay the experimental foundation for the standard model, the momentum and energy changes involved were much less than $M_W, M_Z$. This means that the gauge boson propagators are to an excellent approximation just $-i\eta_{\mu\nu}/M_2$. Then the tree approximation to all of these processes can be read off the effective Lagrangian

$$L'_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left[ \sum_{f_1, f_2} \bar{f}_1 t_+ \gamma(1 - \gamma_5) f_1 \cdot \bar{f}_2 t_- \gamma(I - \gamma_5) f_2 ight. \\
\left. + \rho \left( \sum_f \bar{f}_3 \gamma(I - \gamma_5) f - 2J_{em} \sin^2 \theta_W \right) \right] (28.26)$$

where $G_F/\sqrt{2} = g_2^2/(8M_W^2)$. The first term controls charged current processes, including $\beta$ decay. For $\mu \rightarrow e + \nu_\mu + \bar{\nu}_e$ we read off

$$\mathcal{M} = i \frac{G_F}{\sqrt{2}} \bar{u}_\nu_{\mu} \gamma^\lambda(1 - \gamma_5) u_\mu \bar{u}_e \gamma_\lambda(1 - \gamma_5) v_{\nu_e} (28.27)$$

Since the neutrinos are unobservable, it is convenient to rearrange the spinors in this expression so that the neutrino variables are in the same factor. This is accomplished by the Fierz rearrangement identity.

$$[\gamma^\mu(1 - \gamma_5)]_{\alpha\beta} [\gamma_\mu(1 - \gamma_5)]_{\gamma\delta} = -[\gamma^\mu(1 - \gamma_5)]_{\alpha\delta} [\gamma_\mu(1 - \gamma_5)]_{\gamma\beta}.$$ (28.28)

proved by using the fact that any $4 \times 4$ matrix can be written as a linear combination of the 16 matrices $I, \gamma_5, \gamma^\mu, \gamma^\mu \gamma_5, \sigma^{\mu\nu}$.

Using the Fierz rearrangement on the expression for $\mathcal{M}$ gives

$$\mathcal{M} = -i \frac{G_F}{\sqrt{2}} \bar{u}_\nu_{\mu} \gamma^\lambda(1 - \gamma_5) v_{\nu_e} \bar{u}_e \gamma_\lambda(1 - \gamma_5) u_\mu (28.29)$$

Writing the square of the neutrino matrix element as a trace, we see that the integral over neutrino phase space involves

$$\int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^6} (2\pi)^4 \delta^4(q_1 + q_2 + Q) \text{Tr} q_1 \cdot \gamma_\lambda(1 - \gamma_5) q_2 \cdot \gamma_\kappa(1 - \gamma_5)$$

$$= N[Q_\lambda Q_\kappa - \eta_{\mu\lambda} Q^2] (28.30)$$

where $Q = p_e - p_\mu$. 

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The integral is a Lorentz covariant tensor of rank 2, and multiplying it by \( Q^\lambda \) and summing \( \lambda \) gives zero (because \( q_1^2 = q_2^2 = 0 \). Thus it must be of the form \( N [ Q_\lambda Q_\kappa - \eta_{\lambda\kappa} Q^2 ] \). To evaluate \( N \) take the trace of both sides

\[
-3Q^2N = \int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^2} \delta^4(q_1 + q_2 + Q) 4Tr q_1 \cdot \gamma q_2 \cdot \gamma(1 - \gamma_5) \\
= -16 \int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^2} \delta^4(q_1 + q_2 + Q) q_1 \cdot q_2 \\
= = -8Q^2 \int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^2} \delta^4(q_1 + q_2 + Q) 
\]  
(28.31)

Evaluating the integral in a frame where \( Q = 0 \) gives

\[
3N = \frac{2}{(2\pi)^2} 4\pi \int_0^\infty dq_1 \delta(2q_1 + Q^0) = \frac{1}{\pi} \theta(-Q^0) 
\]  
(28.32)

The square of the lepton factor summed over electron spins, but not over muon spins, times the phase space integral of the neutrino squared amplitude gives the differential rate:

\[
d\Gamma = \frac{G_F^2}{2} \frac{d^3p_e}{(2\pi)^3 4E_e m_\mu} \bar{u}_\mu \gamma^\nu(1 - \gamma_5)(m_e - \gamma \cdot p_e) \gamma^\lambda(1 - \gamma_5) u_\mu \frac{Q_\kappa Q_\lambda + Q^2 \eta_{\kappa\lambda}}{3\pi} \\
= = -\frac{2G_F^2}{2} \frac{d^3p_e}{(2\pi)^3 4E_e m_\mu} \bar{u}_\mu \gamma^\nu \cdot p_e \gamma^\lambda(1 - \gamma_5) u_\mu \frac{Q_\kappa Q_\lambda + Q^2 \eta_{\kappa\lambda}}{3\pi} 
\]  
(28.33)

Next we use \( Q = p_e - p_\mu \) to rewrite

\[
\bar{u}_\mu Q \cdot \gamma \gamma \cdot p_e Q \cdot \gamma(1 - \gamma_5) u_\mu = Q^2 \bar{u}_\mu \gamma \cdot p_e (1 - \gamma_5) u_\mu \\
-2p_e \cdot Q u_\mu Q \cdot \gamma(1 - \gamma_5) u_\mu \\
-2Q^2 \bar{u}_\mu \gamma \cdot p_e (1 - \gamma_5) u_\mu
\]  
(28.34)

In the muon rest frame \( u_\mu = \sqrt{2m_\mu}(\phi, 0) \) and

\[
\bar{u}_\mu \gamma \cdot p_e (1 - \gamma_5) u_\mu = 2m_\mu(-p_e \cdot \phi^\dagger \sigma \phi - E_e) \\
\bar{u}_\mu \gamma \cdot Q (1 - \gamma_5) u_\mu = 2m_\mu(-p_e \cdot \phi^\dagger \sigma \phi - (E_e - m_\mu)) \\
Q^2 = -m_e^2 - m_\mu^2 + 2E_e m_\mu, \\
-2p_e \cdot Q = 2m_e^2 - 2E_e m_\mu
\]  
(28.36)

(28.37)

(28.38)

Inserting these results in \( d\Gamma \),

\[
d\Gamma = \frac{G_F^2}{48\pi^4 E_e} \left[ Q^2 (-p_e \cdot \phi^\dagger \sigma \phi - E_e) \\
+ + 2p_e \cdot Q (-p_e \cdot \phi^\dagger \sigma \phi - (E_e - m_\mu)) \right] 
\]  
(28.39)

\[
\frac{d^2\Gamma}{dp_e d\Omega} = \frac{G_F^2 p_e^2}{48\pi^4 E_e} \left[ 3m_e^2 E_e - 4m_\mu E_e^2 + 3m_e^2 E_e - 2m_\mu m_e^2 \\
- - p_e \cdot (\sigma)(4m_\mu E_e - m_\mu^2 - 3m_e^2) \right]
\]
where \( \langle \sigma \rangle = \bar{u}_\mu \Sigma u_\mu / \bar{u}_\mu u_\mu = \phi^\dagger \sigma \phi \) is the muon polarization in its rest frame.

The term \(-p_e \cdot \langle \sigma \rangle\), which indicates a preferred electron emission anti-parallel to the muon polarization, violates parity under which \( p_e \) changes sign but \( \langle \sigma \rangle \) remains invariant. For the charge conjugate process, the parity violating term has the opposite sign so positron emission is preferentially parallel to the anti-muon polarization. The result is invariant under the combined \( CP \) symmetry.

Integration over \( d\Omega \) kills the \( p_e \cdot \langle \sigma \rangle \) and multiplies the rest by \( 4\pi \):

\[
\Gamma = \int dp_e \frac{G_F^2 p_e}{12\pi^3} \left[ 3m^2_\mu p_e - 4m_\mu p_e^2 \right] + O(m_e^2/m_\mu^2) \tag{28.40}
\]

where we also set \( m_e = 0 \). The range of \( p_e \) is determined by \( p_e + q_1 + q_2 = m_\mu \) where \( p_e = -q_1 - q_2 \). It is maximal \( m_\mu/2 \) when the \( q \)'s are parallel and 0 when they are equal and opposite.

The total rate is for \( m_e = 0 \)

\[
\Gamma = \int_0^{m_\mu/2} dp_e \frac{G_F^2 p_e}{12\pi^3} \left[ 3m^2_\mu p_e - 4m_\mu p_e^2 \right] = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{28.41}
\]

\[
G_F = \sqrt{\frac{192\pi^2 \Gamma}{m_\mu^5}}
\]

We define the electron momentum distribution as

\[
P(p_e) = \frac{1}{\Gamma} \frac{d\Gamma}{dp_e} \tag{28.42}
\]

\[
\rightarrow \frac{16}{m_\mu} \left[ 3 \frac{p_e^2}{m_\mu^2} - 4 \frac{p_e^3}{m_\mu^3} \right], \quad m_e = 0 \tag{28.43}
\]

and plot it in the figure.
Other beta decay processes The description of $\tau$ decay into either $\mu\bar{\nu}_\mu$ or $e\bar{\nu}_e\nu_\tau$ is exactly parallel to $\mu$ beta decay. All of the neutrinos can be taken massless, but setting $m_\mu = 0$ is not as good an approximation as setting $m_e = 0$. But for a rough comparison one can estimate that the total rates for these processes would vary as the mass of the decaying lepton to the fifth power. But unlike the muon, the tau has many more important decay modes, so the total rates for these processes are not related to the total rate for all processes: you need to use the branching fractions to make a comparison.

The beta decay of a quark is actually observed as the beta decay of a nucleon or nucleus. If we pretend a quark could actually be isolated the process would be much like the decay of the muon, with the complication of the CKM matrix and with a nonnegligible mass for the quark in the final state which replaces the $\nu_\mu$ in muon decay. For example, for the decay $d \to u + e + \bar{\nu}_e$ the Feynman amplitude is

$$M = i \frac{G_F}{\sqrt{2}} V_{ud} \bar{u}_u \gamma^\lambda (1 - \gamma_5) u_d \bar{u}_e \gamma_\lambda (1 - \gamma_5) v_\nu_e$$

which after the Fierz rearrangement can be written

$$M = -i \frac{G_F}{\sqrt{2}} V_{ud} \bar{u}_u \gamma^\lambda (1 - \gamma_5) v_\nu_e \bar{u}_e \gamma_\lambda (1 - \gamma_5) u_d$$

If we want to focus on the electron distribution we need to integrate over the phase space of the $u\bar{\nu}_e$ system, which differs from the muon case because of the mass of the $u$ quark. Thus we need a generalization of the phase space integral formula

$$I(Q, m) = 2 \int \frac{d^3q_1 d^3q_2}{4\sqrt{q_1^2 + m^2 |q_2| (2\pi)^2}} \delta^4(q_1 + q_2 + Q) \text{Tr} q_1 \cdot \gamma_\lambda q_2 \cdot \gamma_\kappa (1 - \gamma_5)$$

$$= AQ_\lambda Q_\kappa + B \eta_\kappa\lambda$$

(28.46)
Taking the trace of both sides gives

\[ A Q^2 + 4B = \int \frac{d^3q_1 d^3q_2}{\sqrt{q_1^2 + m^2} |q_2| (2\pi)^6} (2\pi)^4 \delta^4(q_1 + q_2 + Q) \text{Tr}q_1 \cdot \gamma q_2 \cdot \gamma \]

\[ = -\frac{2}{\pi} \frac{q_1 (Q^2 + m^2)}{(q_1 + \sqrt{q_1^2 + m^2})} = -\frac{1}{\pi} \frac{(Q^2 + m^2)^2}{Q^2} \quad (28.47) \]

To get another relation multiply the phase space integral by \( Q^4 \lambda Q^\kappa \) to get

\[ Q^4 A + Q^2 B = 2 \int \frac{d^3q_1 d^3q_2}{4\sqrt{q_1^2 + m^2} |q_2| (2\pi)^2} \delta^4(q_1 + q_2 + Q) \text{Tr}q_1 \cdot \gamma q_1 \cdot \gamma q_2 \cdot \gamma q_1 \cdot \gamma \]

\[ = -\frac{m^2 (Q^2 + m^2)^2}{2\pi} \frac{1}{Q^2} \quad (28.48) \]

Solving for \( A, B \)

\[ -3Q^2 A = -\frac{1}{\pi} \frac{(Q^2 + m^2)^2}{Q^2} \left[ 1 - \frac{2m^2}{Q^2} \right] \]

\[ 3B = -\frac{1}{\pi} \frac{(Q^2 + m^2)^2}{Q^2} \left[ 1 - \frac{m^2}{2Q^2} \right] \quad (28.49) \]

Giving

\[ I(Q, m) = \frac{1}{3\pi} \frac{(Q^2 + m^2)^2}{Q^4} \]

\[ \times \left( Q_\kappa Q_\lambda \left[ 1 - \frac{2m^2}{Q^2} \right] - Q^2 \eta_{\kappa\lambda} \left[ 1 - \frac{m^2}{2Q^2} \right] \right) \quad (28.50) \]

### 28.3 Scattering Processes

In addition to the beta decay processes, the same basic vertices also describe two to two scattering processes involving neutrinos, charged leptons, and hadrons. Here we consider the purely leptonic processes. We begin with the neutrino scattering process \( \nu_\mu + e \rightarrow \nu_\mu + e \).

Assuming the energy is much less than \( M_{Z,W} \), the Feynman amplitude is

\[ \mathcal{M} = i \frac{G_F}{\sqrt{2}} \bar{u}_\nu \gamma^\mu (1 - \gamma_5) u_\nu \bar{u}_e \gamma_\mu (1 - \gamma_5 - 4 \sin^2 \theta_W) u_e \quad (28.51) \]

Squared and summed over the final electron spin and averaged over the initial electron spin gives

\[ \langle |\mathcal{M}|^2 \rangle = \frac{G_F^2}{16} \text{Tr}[q' \cdot \gamma \gamma^\mu (1 - \gamma_5) q \cdot \gamma^\nu (1 - \gamma_5)] \]

\[ \text{Tr}[(m - p' \cdot \gamma) \gamma_\mu (1 - \gamma_5 - 4 \sin^2 \theta_W)(m - p \cdot \gamma) \gamma_\nu (1 - \gamma_5) - 4 \sin^2 \theta_W)] \quad (28.52) \]
The trace on the first line is
\[ T_1 = 8[q^\mu q^\nu + q^\nu q^\mu - q^\cdot q\eta_q^\mu\nu + i\epsilon^{\kappa\lambda\mu\nu}q_\kappa q_\lambda] \] (28.53)

The trace on the last line is
\[ T_2 = -4\eta_{\mu\nu}m^2[(1 - 4\sin^2\theta_W)^2 - 1] \\
+4[p'_\mu p_\nu + p'_\nu p_\mu - p'\cdot p\eta_{\mu\nu}][(1 - 4\sin^2\theta_W)^2 + 1] \\
+8i\epsilon_{\kappa\lambda\mu\nu}p_\kappa p_\lambda[1 - 4\sin^2\theta_W] \] (28.54)

Then
\[ \langle |M|^2 \rangle = 4G_F^2 \left[ m^2 q\cdot q'[(1 - 4\sin^2\theta_W)^2 - 1] \\
+[(1 - 4\sin^2\theta_W)^2 + 1](p'\cdot q'p\cdot q + p'\cdot qp\cdot q') \\
+2(1 - 4\sin^2\theta_W)(p'\cdot q'p\cdot q - p'\cdot qp\cdot q') \right] \] (28.55)

Neglecting the electron mass we have in the center of mass frame
\[ p\cdot q = p'\cdot q' = -2E^2 = -s/2, \quad p\cdot q' = p'\cdot q = -E^2(1 + \cos\theta) \]
\[ \frac{d\sigma}{d\Omega} = \frac{G_F^2 E^2}{4\pi^2}[(1 - 2\sin^2\theta_W)^2 + \sin^4\theta_W(1 + \cos\theta)^2] \] (28.56)
\[ \sigma = \frac{G_F^2 E^2}{\pi}[(1 - 2\sin^2\theta_W)^2 + \frac{4}{3}\sin^4\theta_W] \] (28.57)

Neutrino scattering gives a handle on measuring \( \sin^2\theta_W \). In fact before the \( W \) and \( Z \) were produced in accelerators, neutrino scattering, especially off protons gave a very good measurement of \( \sin^2\theta_W \), and therefore an estimate of the relative masses of \( W \) and \( Z \).

If the muon neutrino is replaced by its anti neutrino, the amplitude is obtained from the one just discussed by the crossing \( q \leftrightarrow -q' \). Then by inspection one arrives at the result
\[ \frac{d\sigma}{d\Omega_{\bar{\nu}_\mu}} = \frac{G_F^2 E^2}{4\pi^2}[(1 - 2\sin^2\theta_W)^2 + \sin^4\theta_W(1 + \cos\theta)^2] + 4\sin^4\theta_W \] (28.58)
\[ \sigma = \frac{G_F^2 E^2}{\pi} \left[ \frac{1}{3}(1 - 2\sin^2\theta_W)^2 + 4\sin^4\theta_W \right] \] (28.59)

A simple example of scattering involving the charged current is \( \nu_\mu + e \rightarrow \mu + \nu_e \) which in lowest order doesn’t involve the \( Z \) and it is described by a single diagram: it is a crossed version of the same diagram that describes muon beta decay!

\[ \nu_\mu + e^- \rightarrow \mu^- + \nu_e \]
\[ \nu_e + e^- \rightarrow \mu^- + \nu_\mu. \]
\[
\frac{d\sigma}{d\Omega_{\nu_\mu\to\nu_e}} = \frac{G_F^2(s - m_\mu^2)^2}{4\pi^2s} \approx \frac{G_F^2s}{4\pi^2} \quad (28.60)
\]
\[
\frac{d\sigma}{d\Omega_{\bar{\nu}_e\to\bar{\nu}_\mu}} = \frac{G_F^2(s - m_\mu^2)(t - m_\mu^2)(t - m_e^2)}{4\pi^2s(s - m_e^2)} \approx \frac{G_F^2s}{4\pi^2} \left(1 + \cos\theta\right)^2 \quad (28.61)
\]
where \(s = (p_e + E_e)^2\) is the CM energy squared, and \(t = -2p_\mu(E_e + p_e \cos\theta) + m_e^2\) with \(\theta\) the angle between the outgoing neutrino direction and the incoming neutrino direction. The approximate forms are valid when the energy is much larger than either lepton mass.

Compare the angular dependences of the two reactions. Can you find a simple explanation for the dramatic difference for backward scattering?

Neutrinos are left handed whereas antineutrinos are right handed. In both reactions only the left handed components of the incoming and outgoing leptons participate. In the center of mass system the total spin along the lepton momentum is zero for the first reaction but \(\pm 1\) for the second reaction. In the second reaction angular momentum conservation thus forbids backward scattering which explains the factor \((1 + \cos\theta)^2\) which vanishes in the backward direction.

\[
\sigma_{\nu_\mu\to\nu_e} = \frac{G_F^2(s - m_\mu^2)^2}{\pi s} \approx \frac{G_F^2s}{\pi} \quad (28.62)
\]
\[
\sigma_{\bar{\nu}_e\to\bar{\nu}_\mu} \approx \frac{G_F^2s}{3\pi} \quad (28.63)
\]

We have been discussing scattering experiments involving neutrinos, which would not occur in QED. However a QED process like \(e^+ + e^- \to \mu^+ + \mu^-\) will also have contributions from Z exchange in addition to photon exchange. We consider this process in an exercise. In the center of mass frame and for energies small compared to \(M_Z\) but large compared to \(m_\mu\), including contributions from both the photon and the Z boson. In this kinematic region the Z boson term is small compared to the photon term, so in the squared amplitude you may drop the square of the Z term. We assume unpolarized e’s and unobserved spin of the \(\mu\)’s. Then

\[
\frac{d\sigma}{d\Omega} \approx \frac{e^4}{64\pi^2s} \left[ (1 + \cos^2\theta) (1 + \xi(1 - 4 \sin^2\theta_W)^2) + 2\xi \cos\theta \right] \quad (28.64)
\]
\[
\xi \equiv \frac{s}{8M_Z^2 \sin^2\theta_W \cos^2\theta_W} = -\frac{G_Fs}{4\pi\alpha\sqrt{2}} \quad (28.65)
\]

The total cross section \(\sigma\) and the front back asymmetry defined as

\[A_{FB} = \frac{1}{\sigma} \left[ \int_{\theta<\pi/2} d\Omega - \int_{\theta>\pi/2} d\Omega \right] \frac{d\sigma}{d\Omega} .\]

are

\[
\sigma \approx \frac{e^4}{12\pi s} (1 + \xi(1 - 4 \sin^2\theta_W)^2) \quad (28.66)
\]
\[
A_{FB} \approx \frac{3\xi}{4} = -\frac{3G_Fs}{16\pi\alpha\sqrt{2}} = -6.74 \times 10^{-5} \left[ \frac{s}{(GeV)^2} \right] \quad (28.67)
\]
Chapter 29

Electroweak Interactions of Hadrons

29.1 Hadron Spectroscopy

Baryons and mesons, collectively called hadrons, are color singlet composites of quarks and antiquarks, bound by the color SU(3) gauge force of the standard model. The hypothesis of quark confinement asserts that color nonsinglet states cannot be isolated. Just how this happens is an open question. But the hadrons can be produced and studied in accelerator experiments, and the wealth of data accumulated has been convincingly organized by symmetry principles.

These symmetries are associated with the fact that with the electroweak forces turned off ($g_1 = g_2 = 0$ and $M_W, M_Z \to \infty$), the quarks of different flavors differ only by their masses. As far as we know there are precisely 6 flavors of quark; $d, u, s, c, b, t$ ranging in mass from a few MeV to 175GeV. With all masses different the only symmetry we would have would be $U(1)^6$, corresponding to the different types of flavor numbers. None of the six masses are particularly close to one another relatively, but some are much lighter than others. Three in particular, $u, d, s$ are significantly smaller than the fundamental scale of QCD, $\Lambda_{QCD}$. This suggests that in QCD the symmetries obtained from setting $m_u = m_d = m_s = 0$ would be reasonable approximate symmetries for QCD. The symmetries gained by setting only $m_u = m_d = 0$ should be even better. In the first case the symmetry is $SU(3) \times SU(3)$ and in the second case $SU(2) \times SU(2)$.

We studied these chiral symmetries last semester, where we focussed on the spontaneous breaking of $SU(n_f) \times SU(n_f)$ to $SU(n_f)$, where $n_f = 2, 3$, with the associated pseudoscalar NGB’s. The particle spectrum should show the pattern of a weakly broken $SU(2)$ or $SU(3)$, the former being the most dramatic.

29.1.1 $n_f = 2$ Isospin and Charge Conjugation

If the $u, d$ quarks are massless, and the $SU(2) \times SU(2)$ symmetry is spontaneously broken to diagonal $SU(2)$, then all particles should fall into representations of that $SU(2)$, and their interactions should also respect that symmetry. This symmetry is called isospin and has been an indispensable tool for the classification and structure of hadrons and nuclei. We are
all very familiar with the group $SU(2)$ because it is also the group of ordinary rotations. We just have to remember that it refers to an internal symmetry and not rotations.

First we note that the 3 NGB’s from spontaneous symmetry breaking, $\pi^\pm, \pi_0$ fall into the $I = 1$ representation, with $I_3 = (\pm 1, 0)$ respectively. In general, different members of an isospin multiplet will have different electric charges, so QED certainly breaks the isospin symmetry even with $m_u = m_d = 0$. However, charge conjugation is a symmetry of both QED and QCD, so it represents a discrete symmetry. Since $\pi_0$ decays into 2 photons, it follows that the $\pi_0$ has $C = +1$. However, $C$ interchanges the positive and negative pions. We fix phases to define $C|\pi^+\rangle = -|\pi^-\rangle$.

Recall the relation between the $I_3$ basis and Cartesian basis: With the convention that $|\pi^-\rangle \sqrt{2} = |\pi_1\rangle - i |\pi_2\rangle$, it follows from the Lie algebra that $|\pi^+\rangle \sqrt{2} = -|\pi_1\rangle + i |\pi_2\rangle$. In analogy with the rotation group, $C$ is like reflection in the 13 plane, reversing only the 2 component of an isovector. Then by combining $C$ with an isorotation by 180° about the 2-axis,

$$e^{i\theta I_2} = 1 + i I_2 \sin \theta + I_2^2 (\cos \theta - 1)$$

$$\rightarrow 1 - 2I_2^2, \quad \theta \rightarrow \pi,$$

$$\rightarrow 1 - I(I + 1) + I_3^2 + \frac{I_2^2 + I_2^2}{2}$$

$$\rightarrow -1 + I_3^2 + \frac{I_2^2 + I_2^2}{2}, \quad I = 1$$

We can define a more convenient $G$-parity $G = Ce^{i\pi I_2}$, which reverses all three components of an isovector. In particular, The isovector pion field is odd under $G$-parity. We can then list the quantum numbers of the pion as $I^G = 1^-, J^P = 0^-, m_\pi \approx 140\text{MeV}$. And we list several other mesons and their quantum numbers and masses:

- $\pi$ 
  
  $I^G = 1^- \quad J^P = 0^- \quad (135, 140)\text{MeV}$

- $\eta$ 
  
  $I^G = 0^+ \quad J^P = 0^- \quad 549\text{MeV}$

- $\eta'$ 
  
  $I^G = 0^+ \quad J^P = 0^- \quad 958\text{MeV}$

- $K = (K^+, K_0)$ 
  
  $I = \frac{1}{2} \quad J^P = 0^- \quad (494.498)\text{MeV}$

- $\bar{K} = (K^0, K^-)$ 
  
  $I = \frac{1}{2} \quad J^P = 0^- \quad (498.494)\text{MeV}$

- $\rho$ 
  
  $I^G = 1^+ \quad J^P = 1^- \quad 771\text{MeV}$

- $\omega$ 
  
  $I^G = 0^- \quad J^P = 1^- \quad 782\text{MeV}$

- $\phi$ 
  
  $I^G = 0^- \quad J^P = 1^- \quad 1029\text{MeV}$

- $K^* = (K^{*+}, K^{*0})$ 
  
  $I = \frac{1}{2} \quad J^P = 1^- \quad (892, 896)\text{MeV}$

- $\bar{K}^* = (K^{*0}, K^{*-})$ 
  
  $I = \frac{1}{2} \quad J^P = 1^- \quad (896, 892)\text{MeV}$

Note that $C$ changes $K$ isodoublets into $\bar{K}$ isodoublets and so the $K$’s have no definite value of $G$. In terms of quarks, these mesons can be viewed as formed from the “light” quarks,
$u, d, s$ and their antiquarks. Indeed if we imagine the quark/antiquark are in an $s$-wave the spin and parity assignments are exactly those seen in nature.

Mesons made of the heavy quarks, $c, b, t$, also exist, with the caveat that the top quark has an extremely short lifetime, so that if the top is included the corresponding “meson” would be highly unstable. A meson made of heavy quarks can resemble positronium very closely, because the strong force is relatively weak at those energies and the quarks are moving nonrelativistically. In some ways the $s$ quark is “light” and in other ways “heavy”. The NGB’s with strangeness behave more like $u, d$. But not for the spin 1 mesons: For example the $\phi$ behaves as an $s\bar{s}$ system like $c\bar{c}$ or $b\bar{b}$.

Now let’s consider the baryons (3 quark composites) and their isospin assignments.

\[
N = (p,n) \quad I = \frac{1}{2} \quad J^P = \frac{1^+}{2} \quad (938.3, 939.6)\text{MeV}
\]

\[
\Lambda \quad I = 0 \quad J^P = \frac{1^+}{2} \quad 1116\text{MeV}
\]

\[\Sigma = (\Sigma^+, \Sigma^0, \Sigma^-) \quad I = 1 \quad J^P = \frac{1^+}{2} \quad (1189, 1193, 1197)\text{MeV}\]

\[\Xi = (\Xi^0, \Xi^-) \quad I = \frac{1}{2} \quad J^P = \frac{1^+}{2} \quad (1315, 1321)\text{MeV}\]

\[
\Delta \quad I = \frac{3}{2} \quad J^P = \frac{3^+}{2} \quad 1230\text{MeV}
\]

\[\Sigma^* = (\Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}) \quad I = 1 \quad J^P = \frac{3^+}{2} \quad 1385\text{MeV}\]

\[\Xi^* = (\Xi^{*0}, \Xi^{*-}) \quad I = \frac{1}{2} \quad J^P = \frac{3^+}{2} \quad 1520\text{MeV}\]

\[\Omega \quad I = 0 \quad J^P = \frac{3^+}{2} \quad 1672\text{MeV}\]

All these baryons are made of the light quarks. Of course any of the light quarks can be replaced by any of the heavy quarks to form more baryons. The only stable baryon is the proton, and even its stability is called into question by grand unified models. The rest of the spin 1.2 baryons are relatively long lived, long enough to make visible tracks in a bubble chamber. The only similarly long lived spin 3/2 baryon is the $\Omega$. The others are resonances observed in scattering experiments. In addition to these 3 quark states there are also all of the nuclei formed as bound states of protons and neutrons.

29.1.2 $n_f = 3$: Flavor $SU(3)$

In the previous subsection we presented a spectroscopy of mesons and baryons classified according to their spin and isospin properties. They can be further organized according to flavor $SU(3)$ by which we mean the symmetry obtained by setting $m_u = m_d = m_s = 0$.
after the spontaneous breaking of $SU(3) \times SU(3)$. In this case the NGB’s are an octet of pseudoscalar mesons, which includes the 3 pions, the four kaons, and the $\eta$, all of which we have mentioned in the previous subsection.

We briefly summarize some of the smallish representations that arise from fundamental triplets and anti triplets. The $SU(3)$ Lie algebra is specified by

$$[F_a, F_b] = if_{abc}F_c.$$  \hspace{1cm} (29.3)

The triplet representation is the fundamental one given for example by $F^a = \lambda^a/2$ where the $\lambda^a$’s are the Gell-Mann matrices discussed last semester. The anti-triplet representation is the complex conjugate of the triplet representation $F_a = -\lambda^*_a/2$. Unlike the case of $SU(2)$, for which the complex conjugate doublet representation is equivalent to the doublet representation ($-\tau^*_a = \tau_2\tau_a\tau_2$), $-\lambda^*_a$ is not equivalent to $\lambda^a$! If $v, w$ are triplet vectors, transforming as $v \rightarrow Uv, w \rightarrow Uw$. Then $w^iv$ is an invariant under $SU(3)$. Here $w^i$ is thought of as a row vector. But if we want to think of this invariant in terms of a tensor product representation of two triplets we write $w^iU^i = (U^*w^*)^T$ and see that the $w$ factor in the tensor product transforms under the antitriplet representation. Thus we see that the tensor product of a triplet with an antitriplet must contain the singlet representation.

It is convenient to distinguish a representation from its conjugate by using raised indices for the triplet and lowered indices for the antitriplet, writing $U^i_j \equiv (U^i_j)^*$, so that $w^i v^j \rightarrow U^i_j w^i U^j_k v^k = w^i v^j$ is invariant. This shows that the representation $\bar{3} \otimes 3$ contains the singlet representation. The remaining eight states in this tensor product

$$w^i v^j - \frac{1}{3} \delta^i_j w^k v^k$$  \hspace{1cm} (29.4)

should transform in the octet representation. An explicit realization of this can be written $v^i \lambda^*_a v$ which transforms under the adjoint (octet) representation by construction. The decomposition now reads

$$\bar{3} \otimes 3 = 1 \oplus 8$$  \hspace{1cm} (29.5)

Next we consider $3 \otimes 3$. With our index convention, the components of the tensor product are $w^i v^j$. The antisymmetrized product, $\epsilon_{ijk} w^i v^k$ clearly transforms as $\bar{3}$, because the epsilon symbol is invariant under $SU(3)$:

$$\epsilon_{ijk} \rightarrow U^i_j U^m_j U^n_k \epsilon_{lmn} = \epsilon_{ijk} \det U = \epsilon_{ijk}$$  \hspace{1cm} (29.6)

because $U$ is in $SU(3)$. The remaining 6 states transform in a new representation:

$$3 \otimes 3 = 3 \oplus 6, \quad \bar{3} \otimes \bar{3} = \bar{3} \oplus \bar{6}$$  \hspace{1cm} (29.7)

To analyze $3 \otimes 6$, think of the 6 as a symmetric tensor product of two 3’s. Then one irreducible representation is the completely symmetric triple tensor product, with $3 \times 4 \times 5/6 = 10$ states. The remaining $3 \times 6 - 10 = 8$ states can only be in the octet representation.

$$3 \otimes 6 = 10 \oplus 8, \quad 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$  \hspace{1cm} (29.8)
29.1.3 Assignment of Flavor $SU(3)$ irreps to Mesons

We can think of a meson as a quark antiquark composite state. Then we should expect mesons to fall into singlet or octet representations of $SU(3)$. Since quarks carry ordinary spin 1/2, each has two spin states in addition to the 3 flavor states. The possible total spins are 0 and 1, and in the $s$-wave, the parity of quark antiquark system is odd. So we should expect a pseudoscalar $0^-$ singlet and octet and also a $1^-$ singlet and octet. The singlet octet combination is sometimes referred to as a nonet. we find these patterns in the particle spectrum. The pseudoscalar octet can be presented as

\[
\begin{array}{ccc}
K^0 & K^+ \\
\pi^- & \pi^0, \eta & \pi^+ \\
K^- & \bar{K}^0
\end{array}
\]

In this display the horizontal direction represents the $I_3$ isospin of each particle and the vertical direction represents a quantum number dubbed strangeness by Gell-Mann, which is the number of $\bar{s}$ quarks minus the number of $s$ quarks. So the top line has $S = +1$ with $K^0 = d\bar{s}$, $K^+ = u\bar{s}$, the middle line has $S = 0$ and the bottom line has $S = -1$. This octet of pseudoscalar mesons is just the NGB octet we studied at the end of last semester. The nonet is completed by adding the $\eta'$, which is too heavy to be an NGB. We shall see that the QCD anomaly accounts for this difference.

There is also an octet of $1^-$ vector bosons displayed as

\[
\begin{array}{ccc}
K^{*0} & K^{*+} \\
\rho^- & \rho^0, \omega_8 & \rho^+ \\
K^{-} & \bar{K}^{*0}
\end{array}
\]

with an $\omega_1$ rounding out the nonet. However unlike the pseudoscalar octet $\omega_8$ and $\omega_1$ are not particle states, but rather superpositions of particle states. The $\phi$ vector meson resonance decays mostly into kaons instead of pions, which in the quark model suggests that $\phi \approx |s\bar{s}\rangle$ and that $\omega$ be the orthogonal combination $\omega \approx (|u\bar{u}\rangle + |d\bar{d}\rangle)/\sqrt{2}$. In terms of particle states,

\[
\omega_8 \approx \frac{1}{\sqrt{3}}(|\omega\rangle - |\phi\rangle \sqrt{2}), \quad \omega_1 \approx \frac{1}{\sqrt{3}}(|\omega\rangle \sqrt{2} + |\phi\rangle)
\]

This feature of the vector octet is called singlet-octet mixing. But it is the very special mixing that corresponds to the $\phi$ being almost pure $s\bar{s}$. this is sometimes called ideal mixing, but it is more simply described by saying that the strange quark is independent of the up down system for the case of vector mesons.

Finally we should note that these are only the lowest mass meson states. They can be accounted for by assuming the quarks are in $s$-wave states, so angular momentum only arises from the spin of the quarks. At higher masses there are many meson resonances, some of which have very high angular momentum. We shall return later to the story of how the patterns of these high angular momentum states inspired the discovery of string theory. But for now we focus on just these low mass states.
29.1.4 Assignment of Flavor $SU(3)$ irreps to Baryons

Again we concentrate here on the low mass baryons, which are roughly understood as composites of three quarks, each of which are in s-waves, i.e. have no orbital angular momentum. The baryons must also be color singlets, which implies that the 3 quark state is totally antisymmetric in color. Since quarks are fermions, the state must be completely symmetric in the spin and $SU(3)$ flavor degrees of freedom. Since each quark has 6 such degrees of freedom, there are a total of $6 \cdot 7 \cdot 8/3! = 56$ internal degrees of freedom for a given baryon multiplet.

One option is that the state be separately completely symmetric in spin and flavor. In that case the $J = 3/2$ and the flavor $SU(3)$ representation must be the decuplet $10$. The remaining 16 states must be spin 1/2 (doublets) and in an $SU(3)$ octet. The latter case is the spin 1/2 baryon octet containing the nucleon:

\[
\begin{array}{cccc}
\Sigma^- & n & p \\
\Xi^- & \Xi^0, \Lambda & \Sigma^+ \\end{array}
\]

The top row has strangeness $S = 0$ and the bottom row $S = -2$. To make the display similar to the meson display, it is convenient to replace $S$ by hypercharge $Y = S + B$, where $B$ is baryon number. Then in both meson and baryon octet displays the top row has $Y = +1$, the middle $Y = 0$ and the bottom row $Y = -1$.

The rest of the three quark baryon states are filled out by the spin 3/2 $SU(3)$ decuplet:

\[
\begin{array}{cccc}
\Delta^- & \Delta^0 & \Delta^+ & \Delta^{++} \\
\Sigma^{*-} & \Sigma^{*0} & \Sigma^{*+} \\
\Xi^{*-} & \Xi^{*0} & \Xi^{*+} \\
\Omega^- & & & \\
\end{array}
\]

The top row has $Y = +1$ and the bottom row $Y = -2$. When Gell-Mann was proposing $SU(3)$ as the organizing symmetry principle for strongly interacting particles, there were missing members of some of the multiplets. The most conspicuous one being the $\Omega^-$, with quark content $sss$. Unlike other members of the decuplet, it has a lifetime of order $10^{-10}$s so it shows up as a charged track in a bubble chamber. Its discovery was an important confirmation of the $SU(3)$ proposal.

29.1.5 Gell-Mann Okubo Mass Relations

We are taking the symmetry limit of QCD to be the vanishing of all $n_f$ light quark masses. The symmetry breaking is then via nonzero masses.

\[
\sum_f m_f \bar{q}_f q_f = 0
\]  

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Since isospin is a very good symmetry, we start just giving the $s$ quark a mass. Thus we assume a perturbation that transforms as the 8 component of an octet.

By the Wigner-Eckart theorem the matrix element between members of the same multiplet of the perturbation is proportional to the matrix element of a general octet function of the actual generators. Of course one such function is the generator $F_a$ itself, and that would be the only option for $SU(2)$. However for $SU(3)$ there is another possibility

$$d_{abc} F_b F_c, \quad \{\lambda_a, \lambda_b\} = \frac{4}{3} \delta_{ab} + 2d_{abc} \lambda_c$$

(29.11)

We then can use

$$AF_a + Bd_{abc} F_b F_c$$

(29.12)

as the effective perturbation. Of course $A$ and $B$ will depend on which multiplet is being analyzed. The perturbation is taken to be the 8 component, so the mass splittings for $s$ given multiplet can be read off from

$$\langle R | \Delta | R \rangle = \langle R | AF_8 + Bd_{8bc} F_b F_c | R \rangle$$

(29.13)

We have tabulated the $d_{abc}$ in Section[?], but here we only need the nonzero ones with at least one 8 index

$$d_{888} = -\frac{1}{\sqrt{3}}, \quad d_{811} = d_{822} = d_{833} = \frac{1}{\sqrt{3}}$$

$$d_{844} = d_{855} = d_{866} = d_{877} = -\frac{1}{2\sqrt{3}}$$

$$D_8 = d_{8bc} F_b F_c = \frac{1}{\sqrt{3}} I^2 - \frac{1}{\sqrt{3}} F_8^2 - \frac{1}{2\sqrt{3}} (F^2 - I^2 - F_8^2)$$

$$= -\frac{F^2}{2\sqrt{3}} + \frac{\sqrt{3}}{2} I^2 - \frac{F_8^2}{2\sqrt{3}}$$

(29.14)

In these formulas $I$ is a 3-vector whose components are the isospin generators, The hypercharge and $F_8$ are related by $F_8 = Y \sqrt{3}/2$. Then

$$D_8 = -\frac{F^2}{2\sqrt{3}} + \frac{\sqrt{3}}{2} I^2 - \frac{Y^2 \sqrt{3}}{8} = \frac{\sqrt{3}}{2} \left[ -\frac{F^2}{3} + I(I + 1) - \frac{Y^2}{4} \right]$$

(29.15)

Here $F^2 = \sum_a F_a^2$ has a fixed value over the whole multiplet. So we can write a mass formula within any fixed irrep as

$$M = m_0 + m_1 Y + m_2 \left[ I(I + 1) - \frac{Y^2}{4} \right]$$

(29.16)

A curious feature of the decuplet irrep is that $I = 1 + Y/2$ so the term in square brackets is a linear function of $Y$, implying equal mass splittings within the decuplet. The observed
mass differences are 155, 135, and 152 averaging 147 MeV, which is not too bad! Historically only the first 2 mass differences were known. Gell-Mann predicted the mass of the Ω by the average of the first two (145 GeV) to $m_{\xi}$ getting 166 GeV. The Ω was then discovered at 1672 GeV.

For the baryon octet we have four masses to determine 3 parameters, so there is one relation (Gell-Mann/Okubo) which we get by observing

$$m_N + m_{\Xi} = 2m_0 + m_2, \quad 3m_{\Lambda} + m_{\Sigma} = 2(2m_0 + m_2)$$

(29.17)

The left side is 939 + 1318 = 2257 MeV and the right side is (3348 + 1193)/2 = 2270, which is very good.

For the pseudoscalar mesons, the GMO relation works significantly better for mass squared comparisons. Remembering that $K$ and $\bar{K}$, being antiparticles, have the same mass, GMO for mass squared reads

$$4m^2_K = 3m^2_\eta + m^2_\pi$$

(29.18)

than mass comparisons. Here the left side is 0.98 MeV$^2$ and the right side is 0.92 MeV$^2$, a 6% discrepancy. Using masses in the relation gives instead 1.984 MeV on the left and 1.787 MeV on the right, a 10% discrepancy. In fact we found last semester, when we used chiral perturbation theory to infer the masses of the NGB’s due to small quark masses, we found that the square masses were linear in the quark masses. Neglecting isospin breaking, chiral dynamics gave precisely GMO for mass squared!

For the vector mesons, we can use GMO to determine $m_8$ the mass of the would be octet isosinglet,

$$m^2_8 = \frac{4}{3}m^2_{K^*} - \frac{1}{3}m^2_\rho, \quad m_8 \approx 930\text{MeV}. \quad (29.19)$$

Then we can model octet-singlet mixing by assuming a 2 × 2 matrix of perturbation of the form

$$M = \begin{pmatrix} m_8 & \Delta \\ \Delta^* & m_1 \end{pmatrix} \quad (29.20)$$

so the physical masses are determined by

$$m_\omega + m_\phi = m_8 + m_1, \quad m_\omega m_\phi = m_1 m_8 - |\Delta|^2 \quad (29.21)$$

The first equation then determines $m_1 = 880\text{MeV}$, and then the second determines $|\Delta| = 117\text{MeV}$. Writing the mixing vector $(a, b)$ we then have $b/a = (m_8 - m)/\Delta = 1.30$ in magnitude for the $\omega$ and 0.77 in magnitude for the $\phi$. This is compared to the ideal mixing ratios of $\sqrt{2} \approx 1.4$ and $1/\sqrt{2} \approx 0.70$ respectively.
29.1.6 Rotational Excitations of Mesons

We have been considering only the lowest mass mesons and baryons, corresponding to quarks in s-waves. But the spectral data show many resonances with increasing values of angular momentum. In fact there are such recurrences for each member of each flavor multiplet.

To illustrate this feature, we tabulate and graph the mass squared and angular momentum of nonstrange resonances with $T^G = 1^{(-)^{J+1}}$. The lightest such meson is the $\rho$:

\[
\begin{align*}
\rho, J &= 1, M^2 = 0.59; & A_2, J &= 2, M^2 = 1.7; & \rho_5, J &= 3, M^2 = 2.9; \\
A_4, J &= 4, M^2 = 4.2; & \rho_5, J &= 5, M^2 = 5.5; & A_6, M^2 &= 6.0
\end{align*}
\]

\[\text{rho trajectories: } l=1, P=-G=(-)^{J+1}\]

The function $\alpha(t)$, which interpolates $\alpha(M^2) = J(M^2)$ when evaluated at $t = M^2$, is called a Regge trajectory, and it is related to the high energy behavior of scattering amplitudes. For example a 2 to 2 scattering amplitude has the high energy behavior

\[
\mathcal{M}(s,t) \sim \beta(t)s^{\alpha(t)} \quad s \to \infty. \tag{29.22}
\]

Here $s$ and $t$ are the usual Mandelstam invariants. we shall say more about Regge behavior later in the course. Here we show that the linear behavior $J(M^2)$ is indicative of the dynamics of a relativistic string.

The coordinates and energy-momentum densities of a string are denoted

\[
x^\mu(\sigma,t), \quad P^\mu(\sigma,t)
\]
Here $\mathcal{P}^\mu d\sigma = dp^\mu$ is the energy momentum carried by the element $d\sigma$ of string. In analogy to the particle mass shell constraint $p^2 + m^2 = 0$, which follows from taking its action proportional to the length of the worldline $S = -m \times (\text{Length})$ swept out by the motion of the particle, Nambu proposed that the action for a relativistic string be proportional to the area of the worldsheet swept out in spacetime $S = -T_0 \times (\text{Area})$. This leads to the phase space constraints (Exercise).

$$\mathcal{P}^2 + T_0^2 x'^2 = 0, \quad x' \cdot \mathcal{P} = 0 \quad (29.23)$$

The left side of the second constraint is the generator of reparametrization of $\sigma$. So this constraint says that the string dynamics doesn’t depend on how the points of a string are parameterized. Thus we can solve the constraints by fixing a convenient parameterization such that the first constraint is tractable.

Lightcone parameters are particularly convenient: Single out a spatial direction, say $x^1$, and denote the remaining “transverse” coordinates $x$. Then instead of $t$, $x^1$ and $\mathcal{P}^0$, $\mathcal{P}^1$, use

$$x^\pm = \frac{1}{\sqrt{2}}(t \pm x^1), \quad \mathcal{P}^\pm = \frac{1}{\sqrt{2}}(\mathcal{P}^0 \pm \mathcal{P}^1) \quad (29.24)$$

With these choices, the Minkowski scalar product of two 4 vectors is $v \cdot w = v \cdot w - v^+ w^- - v^- w^+$. Now we choose to specify initial data on equal $x^+$ surfaces, and at the same time fix $\sigma$ so that $\mathcal{P}^+ = 1$, and the constraints simplify to

$$\mathcal{P}^- = \frac{1}{2}(\mathcal{P}^2 + T_0^2 x'^2), \quad x'^- = x' \cdot \mathcal{P} \quad (29.25)$$

and we remember that the length of $\sigma$ space is now $p^+$. The constraints now give $x'^-$ and $\mathcal{P}^-$ in terms of the transverse coordinates and momenta. The variable $p^- = \int d\sigma \mathcal{P}^-$ is the generator which connects two infinitesimal equal $x^+$ surfaces, and there has the interpretation of canonical Hamiltonian. Hamilton’s equations then give

$$\dot{x} = \mathcal{P}, \quad \dot{\mathcal{P}} = T_0^2 x'', \quad x^{|p^+} = 0. \quad (29.26)$$

The boundary conditions can be met either by periodic boundary conditions (closed string) or by independently requiring $x' = 0$ at $\sigma = 0, p^+$ (open string). In the first case periodicity of $x^-$ imposes the side constraint $\int d\sigma x' \cdot \mathcal{P} = 0$. Since the left side of that side constraint commutes with the Hamiltonian, it is simply a constraint on the allowed physical states\(^1\) The

\(^1\)These equations also follow from the action

$$S = \int dt \int_0^{p^+} d\sigma \left( \dot{x} \cdot \mathcal{P} - \frac{1}{2} \mathcal{P}^2 - T_0^2 x'^2 \right)$$

$$S \rightarrow \int dt \int_0^{p^+} d\sigma \frac{1}{2} \left( \dot{x}^2 - T_0^2 x'^2 \right)$$
equations of motion for the open string, which is a model for a $q\bar{q}$ meson with a confining linear potential, are:

$$\frac{\partial^2 x}{\partial x^2} = T_0 \frac{\partial^2 x}{\partial \sigma^2}$$

$$\frac{\partial x}{\partial \sigma} = 0, \quad \sigma = 0, p^+$$

Here we consider a simple pinwheel motion in the 23-plane

$$(x^2, x^3) = A \cos \frac{\pi \sigma}{p} + \left(\cos \omega x^+, \sin \omega x^+\right)$$

(29.27)

For this motion it is easy to check that $x' \cdot P = 0$, so that there is no motion out of the 23-plane. Then $\omega = T_0 \pi/p$. and

$$P^2 = A^2 T_0^2 \pi^2 \cos^2 \frac{\pi \sigma}{p^+} = T_0^2 x'^2$$

$$x'^3 p^3 - x'^3 p^2 = A^2 T_0^2 \cos^2 \frac{\pi \sigma}{p^+}$$

$$2p^+ p^- = A^2 T_0^2 \pi^2$$

$$J = A^2 T_0 \pi^2$$

(29.28)

from which it follows that $M^2 = 2\pi T_0 J$, or $J = \alpha' M^2$, $\alpha' = 1/(2\pi T_0)$. All normal modes can easily be found, but for now we content ourselves with this simple case, which illustrates the association of linear trajectories with the motion of 1 dimensional objects.

**Normal modes and Quantization**

For the open string the boundary conditions are met by the normal mode functions $\cos n\pi \sigma/p^+$, so we can expand

$$x^k(\sigma, t) = x^k_0 + \frac{p^k}{p^+} t + \sqrt{\frac{2}{p^+}} \sum_{n=1}^{\infty} a_n^k e^{-i \omega_n t} + a_n^{k\dagger} e^{i \omega_n t} \cos \frac{n \pi \sigma}{p^+}$$

(29.29)

The equations of motion determine the eigenfrequencies $\omega_n = T_0 n \pi/p^+$, Since the Hamiltonian is quadratic in $x^k, p^k$, quantization with lightcone parameterization is immediate:

$$[a_n^k, a_m^{k\dagger}] = \delta_{mn} \delta_{kl}, \quad [x_0^k, p^l] = i \delta_{kl}$$

(29.30)

We can write the Hamiltonian operator

$$H = p^- = \frac{p^2 + M^2}{2p^+}$$

$$\alpha' M^2 = \sum_{n=1}^{\infty} n a_n^k a_n^{k\dagger} - D - 2 \frac{24}{24}$$

$$\alpha' = \frac{1}{2\pi T_0}$$

(29.31)

(29.32)

The mass eigenstates are of form

$$a_n^{k_1\dagger} \cdots a_n^{k_l\dagger} |0\rangle$$
with maximal $J = l$ and

$$\text{mass}^2 = \frac{n_1 + \cdots + n_l}{\alpha'}$$

Leading trajectory has minimal mass for fixed $J = l$, so $n_1 = n_2 = \cdots = n_l = 1$, and

$$\alpha' M^2 = l - \frac{(D - 2)}{24},$$

predicting a linear trajectory

$$\alpha (M^2) = l = \alpha' M^2 + \frac{(D - 2)}{24}.$$ 

Nonminimal masses lie on parallel trajectories below the leading one.
29.2 Vector and Axial Symmetries and Electroweak Processes

The key to extracting precise predictions for the weak interactions of hadrons at low energies relies on symmetry principles. Carrying out perturbation theory in $g_2, g_1$ requires the input of matrix elements of purely hadronic operators for $g_1 = g_2 = 0$. To identify these operators we recall the interaction part of the low energy effective Lagrangian for the electroweak theory,

$$L'_{	ext{eff}} = \frac{G_F}{\sqrt{2}} \left[ J^\mu_C(x)J^\dagger_{C\mu}(x) + \rho J^\mu_0(x)J_0\mu(x) \right]$$ \hspace{1cm} (29.33)

$$J^\mu_C(x) = \sum_f \bar{f}t_\gamma(1 - \gamma_5)f(x)$$ \hspace{1cm} (29.34)

$$J^\mu_0(x) = \sum_f \bar{f}t_3\gamma(I - \gamma_5)f - J_{em}\sin^2\theta_W$$ \hspace{1cm} (29.35)

where $G_F/\sqrt{2} = g_2^2/(8M_W^2)$. After the fermion mass terms are diagonalized by redefinition of the Fermi fields, the charged current acquires flavor mixing matrices as follows

$$J^\mu_C(x) \rightarrow \sum_{i,j} \bar{U}_i \gamma^\mu(1 - \gamma_5)D^iV^Q_{ij} + \bar{\nu}_i \gamma^\mu(1 - \gamma_5)L^iV^L_{ij})$$ \hspace{1cm} (29.36)

where $V^Q$ is the CKM mixing matrix for quarks and $V^L$ is the analogous matrix for leptons. In the approximation of massless neutrinos $V^L \rightarrow \delta_{ij}$. In our notation $U^i$ are the charge $2/3$ quark fields $u, c, t$ and $D^i$ are the charge $-1/3$ quark fields $d, s, b$. Also $L^i$ are the charged lepton fields $e, \mu, \tau$ and $\nu^i$ are the neutrino fields $\nu^1, \nu^2, \nu^3$. Note that the neutrino mass eigenstates are not paired with the leptons, so we use neutral labels for them. It is noteworthy that the neutral currents remain diagonal in flavor in the standard model. This is arranged by the doublet assignments in $SU(2)$. The extreme suppression of flavor changing neutral decay processes was the reason Glashow, Iliopolous, and Maiani (GIM) predicted the existence of the charmed quark several years before its discovery.

The hadronic parts of the weak currents can be built up from linear combinations of the symmetry currents

$$V^\mu_a = \bar{q}t_a \gamma^\mu q, \quad A^\mu_a = \bar{q}t_a \gamma_5 \gamma^\mu q$$ \hspace{1cm} (29.37)

If all 6 quarks were massless, the spatial integrals of the time components of these currents would be the generators of the chiral $SU(6) \times SU(6)$ symmetry. But in practice, only the $u, d, s$ quarks are light enough to reflect an approximate symmetry, the chiral $SU(3) \times SU(3)$ we have been discussing.

We can divide hadron decays into semi-leptonic and nonleptonic. In the former the hadronic part of the Feynman amplitude is just the single particle matrix element of a current,
and symmetry considerations give powerful constraints on their structure. Nonleptonic weak decays are less tractable because the amplitude involves the matrix element of the product of two currents, for which symmetry constraints are less effective.

29.2.1 Vector Symmetries

Focussing first on the vector currents involving only $n_f$ light quarks ($n_f = 2, 3$), we see that they are built from various components of the currents associated with flavor $SU(n_f)$, \[ F_a = \int d^3x V_a^0(x). \] (29.38)

To see the implications of symmetry on decay processes, consider the matrix element of $V_a$ between one particle states

\[ \langle p' | V_a^\mu(x) | p \rangle = e^{i(p' - p) \cdot x} \langle p' | V_a^\mu(0) | p \rangle \] (29.39)

Integrating the time component over all space, we learn that

\[ \langle p' | F_a | p \rangle = (2\pi)^3 \delta(p' - p) \langle p | V^0_a(0) | p \rangle \] (29.40)

The matrix element on the right [times $2E(2\pi)^3$] is the time component of the hadronic vertex of some semi-leptonic process in the forward direction. If the initial and final particles are members of the same $SU(n_f)$ multiplet, the matrix element on the left is determined by group theory, i.e. symmetry principles alone.

As a simple example, consider the beta decay process

\[ \pi^- \to \pi^0 + e^- + \bar{\nu}_e \] (29.41)

The Feynman amplitude is determined by

\[ \mathcal{M} = \frac{iG_F}{\sqrt{2}} \bar{u}_e \gamma^\mu (1 - \gamma_5) u_e V^{Q}_{ud} \langle \pi_0, p', \bar{u} \gamma^\mu (1 - \gamma_5) d | \pi^-, p \rangle \] (29.42)

We estimate the unknown pion matrix element as follows. First we work to zeroth order in the electroweak perturbation theory. This means we calculate the matrix element in pure QCD, which conserves parity. The axial vector part of the matrix element must vanish because you can’t make an axial vector out of only two four momenta.

For the vector part, Lorentz invariance implies that

\[ (2\pi)^3 \sqrt{4EE'} \langle \pi_0, p', \bar{u} \gamma^\mu d | \pi^-, p \rangle = F_1(q^2)(p + p')^\mu + F_2(q^2)q^\mu \] (29.43)

where $q = p' - p$. In the limit of exact isospin symmetry (CVC) $F_2 = 0$ because the current is conserved in that limit. In this limit $F_1(q^2)$ determines the decay rate. We get one more
Of course $q^2 \neq 0$, but it is very small: $q^2 = (p' - p)^2 = -m_{\pi^0}^2 - m_{\pi^-}^2 + 2E' m_{\pi^-}$ and $m_{\pi^0} < E_{\pi^0} < m_{\pi^-}$. Since $q$ is the energy momentum of the $e\bar{\nu}$ subsystem in the final state $q^2 < -m_e^2$, so it is constrained by $-(m_{\pi^-} - m_{\pi^0})^2 < q^2 < -m_e^2$. It is reasonable to take the scale over which $F_1(q^2)$ varies to be $m_{\pi^-}^2$, so approximating $F_1(q^2) \approx F_1(0) \approx \sqrt{2}$ should be an excellent approximation.

$$F_1(0) = \langle 1, 0|t_+|1, -1 \rangle = \sqrt{1(1+1) - 0 \cdot (-1)} = \sqrt{2} \tag{29.44}$$

$$\mathcal{M} \approx G_F \bar{u}_e(p + p') \cdot \gamma(1 - \gamma_5) v_e V_{ud}^Q$$

$$\sum |\mathcal{M}|^2 = G_F^2 |V_{ud}^Q|^2 \text{Tr}(m_e - \gamma \cdot p_e)(p + p') \cdot \gamma(1 - \gamma_5)$$

$$(-q \cdot \gamma)(p + p') \cdot \gamma(1 - \gamma_5)$$

$$= 2G_F^2 |V_{ud}^Q|^2 \text{Tr} \gamma \cdot p_e(p + p') \cdot \gamma q \cdot \gamma(p + p') \gamma(1 - \gamma_5)$$

$$= 8G_F^2 |V_{ud}^Q|^2 [2p_e \cdot (p + p') q \cdot (p + p') - q \cdot p_e(p + p')^2] \tag{29.46}$$

It is a lengthy but straightforward calculation to plug this result into the formula for the decay rate.

The noteworthy feature of this process is that it gives a clean measurement of the vector part of the electroweak couplings. The reason the axial current decoupled was that the pion has zero spin. The same simplification will happen for the beta decay of a spin 0 nucleus into a spin 0 nucleus, and there are several examples:

$$O^{14} \rightarrow N^{14} + e^+ + \nu_e; \quad Cl^{34} \rightarrow Se^{34} + e^+ + \nu_e; \quad Al^{26} \rightarrow Mg^{26} + e^+ + \nu_e$$

Careful measurement of all these $0 \rightarrow 0$ decays gives an accurate determination of the CKM element $V_{ud} \approx 0.974$ and the departure from unity is significant: it exactly correlates with the measured value of $V_{us}$ from strangeness changing weak decays.

### 29.2.2 Conserved Vector Currents (CVC) in Baryon Decay

In contrast to the $0 \rightarrow 0$ decays just discussed, the beta decay amplitudes for baryons are a linear combination of vector and axial currents. We can still consider the vector currents and axial currents separately, but the calculation of rates involves the square of the sum, and hence interference cross terms. In addition to this complication there are three form factors for the matrix element of the vector current. On grounds of Lorentz invariance and parity invariance, we can write

$$\left\langle B', \mathbf{p}', s' | \mathbf{q} \gamma^\mu \frac{\lambda_2}{2} q \right| B, \mathbf{p}, s \right\rangle$$

$$= \bar{u}(p', s') \left[ f_1^\alpha(q^2) \gamma^\mu - if_2^\alpha(q^2) \sigma^{\mu\nu} q_\nu + f_3^\alpha(q^2) q^\mu \right] u(p, s) \tag{29.47}$$
where \( q = p' - p \). With these definitions time reversal invariance implies that the \( f_i^a \) are all real. In the flavor symmetry limit the vector current operator is conserved, which implies that the scalar product of the left side by \( q_a \) is zero, and hence that \( f_3(q^2) = 0 \).

As we have already mentioned, at \( q = 0 \) the flavor symmetry limit implies that

\[
 f_1^a(0) = (F_a)_{BB} 
\]

where \( F_a \) is the generator of the flavor symmetry \( SU(n_f) \) in the representation that \( B, B' \) belong to. Since the energy release in nucleon beta decay is relatively tiny, it is a good approximation to drop the \( f_2, 3 \) terms and approximate \( f_1^a(q^2) \approx f_1^a(0) \). In the particular case of nucleon beta decay in the isospin symmetry limit, the \( pn \) matrix element of the vector current can be related to electromagnetic form factors via isospin rotations:

\[
\langle p|\bar{q}\gamma\mu t_+q|n\rangle = 2\langle p|\bar{q}\gamma\mu t_3q|p\rangle 
\]

and writing \( \langle p| = \langle n|T_- \) and moving \( t_- \) to the right shows that this same matrix element equals \( -2\langle n|\bar{q}\gamma\mu t_3q|n\rangle \), so

\[
\langle p|\bar{q}\gamma\mu t_+q|n\rangle = \langle p|\bar{q}\gamma\mu t_3q|p\rangle - \langle n|\bar{q}\gamma\mu t_3q|n\rangle = \langle p|j_{EM}\mu|p\rangle - \langle n|j_{EM}\mu|n\rangle 
\]

and the matrix elements on the last line determine the electromagnetic form factors. This then gives a prediction of the \( q \) dependence of the weak decay form factors.

### 29.2.3 Axial Symmetries

As we know the limit of \( n_f \) massless quarks expands the flavor symmetry to chiral \( SU(n_f) \times SU(n_f) \), with the implied conservation of axial currents as well as vector currents. But these axial symmetries must be spontaneously broken, because otherwise the conserved axial charges would imply that every hadron has a nearby opposite parity partner, for which there is no evidence.

Instead the spontaneously broken symmetry implies massless Nambu-Goldstone bosons, whose low energy couplings are determined by that symmetry. We have already discussed this phenomenon in detail at the end of last semester. Here we focus on its implications for baryon decay. The axial vector contribution to beta decay of baryons involves the matrix element

\[
\langle B', p', s' | \bar{q}\gamma\mu \gamma_5 \frac{\lambda_a}{2} q | B, p, s \rangle 
\]

\[
= \bar{u}(p', s') [g_1^a(q^2)\gamma^\mu \gamma_5 + \gamma_5 g_2^a(q^2)q^\mu - ig_3^a(q^2)\sigma^{\mu\nu}q_\nu] u(p, s) \tag{29.51}
\]

The weaker symmetry requirements of time reversal and G-parity invariance also imply \( f_3 = 0 \).
In the chiral limit, the conservation of the axial current means that the scalar product of the left side with $q\mu$ gives zero. The last $g_3$ term is automatically conserved, but the first two terms are not. Using the Dirac equation on the first term, we see that conservation requires

$$-2m_N g_1(q^2) + q^2 g_2(q^2) = 0, \quad g_3(q^2) = \frac{2m_N g_1(q^2)}{q^2}$$

(29.52)

since $g_1(0) = g_A \neq 0$, we see that CAC requires a pole in $g_2(q^2)$ at $q^2 = 0$ with residue $2m_N g_A$. This pole is supplied by the NGB pions, which must couple to the axial current:

$$\langle 0 | \bar{q} \gamma\mu \gamma_5 \gamma t_a q | \pi_b \rangle \equiv \delta_{ab} \frac{iq^\mu F}{(2\pi)^{3/2} \sqrt{2E_\pi}}$$

writing the pion nucleon vertex as $\bar{u} G_{\pi N} \gamma_5 \gamma t a u$ the contribution of the pion pole to the nucleon matrix element is $(-i/q^2)2G_{\pi N} i q^\mu F$ giving $g_3 \sim 2FG_{\pi N}/q^2$. Comparison then gives the Goldberger-Treiman relation $FG_{\pi N} = m_N g_A$, which relates $g_A$ to $F$ and $g_{\pi N}$. When combined with the Adler-Weisberger sum rule, the electroweak parameters $F, g_A$ can be successfully calculated in terms of purely hadronic dynamics.

A confusing point about neutron beta decay is that, although the $g_2$ term is essential for the argument relating $g_A$ to strong interaction dynamics, it actually makes a negligible contribution to the beta decay process. The point is that the momentum $q^\mu$ is tiny, of order $m_e$. In the theory, this smallness is compensated by the pion pole. But after the effects of finite quark masses have given mass to the pions, the pole location is moved to $q^2 = -m_\pi^2$.

$$\frac{q^\mu}{q^2 + m_\pi^2} = \frac{q^\mu}{m_\pi^2} [1 + O(q^2/m_\pi^2)]$$

(29.54)

which is tiny when $q \sim m_e$. All together the $g_2$ term is smaller than the $g_1$ term by a factor of order $m_e m_N/m_\pi^2 \approx 0.025$.

### 29.3 More Semi-Leptonic Hadronic Decay Processes

Probably the simplest example of a semi-leptonic decay process is pion decay $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. The hadronic part of the process is the disappearance of the pion which is described by the matrix element of the axial current between the vacuum and the one pion state.

$$\langle 0 | \bar{q} \gamma\mu (1 - \gamma_5) d | \pi_-, p \rangle = - \langle 0 | \bar{u} \gamma\mu \gamma_5 \gamma t a q | \pi_-, p \rangle = - \langle 0 | \bar{q} \gamma\mu \gamma_5 t a q | \pi_1 - i\pi_2, p \rangle \frac{1}{\sqrt{2}}$$

$$= - \frac{ip^\mu \sqrt{2} F}{(2\pi)^{3/2} \sqrt{2E_\pi}}$$

(29.55)

The vector part of the current gives zero contribution due to parity conservation (the pion is a pseudoscalar). The Feynman amplitude for the decay is then

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{ud} F \sqrt{2} \bar{u}_\mu (1 - \gamma_5) \nu_\mu$$

$$|\mathcal{M}|^2 = 2G_F^2 |V_{ud}|^2 F^2 m_\mu^2 \bar{u}(-q \cdot \gamma(1 + \gamma_5) u$$

(29.56)
where we used the Dirac equation to write
\[ \bar{u}_\mu \gamma \cdot p (1 - \gamma_5) v_\nu = \bar{u}_\mu \gamma \cdot (p_\mu + q) (1 - \gamma_5) v_\nu = -m_\mu \bar{u}_\mu (1 - \gamma_5) v_\nu \]

In these formulas \( q \) is the neutrino energy momentum. Specializing to the pion rest frame \( (q = -p_\mu, |q| + \omega_\mu = m_\pi) \), the spinor matrix element in the chiral representation is
\[ \bar{u}(-q \cdot \gamma)(1 + \gamma_5)u = 2\phi^\dagger(|p_\mu| + p_\mu \cdot \sigma) (\omega + p_\mu \cdot \sigma) \phi \]
\[ = 2|m_\mu|\phi^\dagger(1 + \hat{p}_\mu \cdot \sigma) \phi \] (29.57)

Energy conservation \( m_\pi = p_\mu + \sqrt{m_\mu^2 + p_\mu^2} \) determines
\[ p_\mu = \frac{m_\mu^2 - m_\pi^2}{2m_\pi} \] (29.58)

so the squared amplitude becomes
\[ |M|^2 = 2G_F^2 |V_{ud}|^2 F_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \phi^\dagger(1 + \hat{p}_\mu \cdot \sigma) \phi \] (29.59)

The differential rate in the pion rest frame is
\[ \frac{d\Gamma}{d\Omega} = \frac{p_\mu}{32\pi^2 m_\pi^2} |M|^2 \]
\[ = G_F^2 |V_{ud}|^2 F_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2 \frac{\phi^\dagger(1 + \hat{p}_\mu \cdot \sigma) \phi}{32\pi^2 m_\pi^2} \] (29.60)

The total rate includes the sum over muon spin and integrating over \( d\Omega \) for a net factor of \( 8\pi \)
\[ \Gamma = G_F^2 |V_{ud}|^2 F_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2 \frac{\phi^\dagger(1 + \hat{p}_\mu \cdot \sigma) \phi}{4\pi m_\pi^3} \] (29.61)

The Fermi coupling \( G_F \) is known from muon decay, and \( |V_{ud}| \) from pion beta decay, and the masses are known from direct measurement. Thus Comparison of \( \Gamma \) to the reciprocal of the pion lifetime times the branching ratio for this process (essentially unity) determines \( F_\pi \approx 93 \text{MeV} \).

The formula for the differential rate shows that the muon in the final state has only right handed helicity. This is a consequence of angular momentum conservation: the massless antineutrino has right handed helicity and the pion has spin 0. On the other hand, if the muon were massless, the \( (1 - \gamma_5) \) factor would force the muon to be left handed, so the rate must vanish if \( m_\mu = 0 \). This explains the overall factor of \( m_\mu^2 \) in the formula for the decay rate.

We next consider other decays of this type. First the decay of the pion into electron and antineutrino is described by the same physics and its rate is obtained by substituting \( m_\mu \to m_e \). This predicts
\[ \frac{\Gamma_{\pi \to e\bar{\nu}_e}}{\Gamma_{\pi \to \mu\bar{\nu}_\mu}} = \frac{m_e^2 (m_e^2 - m_\mu^2)^2}{m_\mu^2 (m_\pi^2 - m_\mu^2)^2} \approx 1.3 \times 10^{-4} \] (29.62)
which is dramatically confirmed by inspection of the particle book. Of course the charge conjugate processes will have the same rates, but the antilepton and neutrino will come out left handed.

There are also analogous decays of the kaons $K^- \rightarrow l^- + \bar{\nu}_l$. This differs from the pion decay not only because the masses are different, but also because $V_{ud}$ is replaced by $V_{us}$: one is measuring a different element of the CKM matrix. The coupling of the kaon to the axial current $F_K$ might not be the same as $F_\pi$, although the approximate validity of chiral flavor $SU(3)$ indicates they should not be too different. Making all these substitutions we can write the kaon decay rate as

$$\Gamma_{K \rightarrow l\bar{\nu}_l} = G_F^2 |V_{us}|^2 F_K^2 \frac{m_l^2 (m_K^2 - m_l^2)^2}{4\pi m_K^3}$$

$$\frac{\Gamma_{K \mu \bar{\nu}_\mu}}{\Gamma_{\pi \mu \bar{\nu}_\mu}} = \frac{F_K^2 |V_{us}|^2}{F_\pi^2 |V_{ud}|^2} \left[ \frac{m_K (1 - m_\mu^2/m_K^2)^2}{m_\pi (1 - m_\mu^2/m_\pi^2)^2} \right]$$

(29.63)

The ratio of masses inside the square brackets evaluates to 17.6. The experimental rate relative to the pion rate is

$$\frac{\Gamma_{K \mu \bar{\nu}_\mu}}{\Gamma_{\pi \mu \bar{\nu}_\mu}} \approx \frac{2.60 \times 0.636}{1.23} \approx 1.34$$

(29.64)

Comparing we conclude that

$$\frac{F_K^2 |V_{us}|^2}{F_\pi^2 |V_{ud}|^2} \approx \frac{1.34}{17.6} = 0.076$$

(29.65)

With this one measurement we cannot separate the ratio of mixing matrix elements from the ratio of decay constants. However in the flavor $SU(3)$ symmetry limit the decay constants would be the same and $\sqrt{0.076} \approx 0.276$ is an estimate for $|V_{us}/V_{ud}| = \tan \theta_C$, where $\theta_C$ is the Cabibbo angle. But the same factor $V_{us}$ appears in all strangeness changing amplitudes, so combining measurements of many different processes including baryon decays combined with symmetry arguments gives a good value for this ratio, namely $|V_{us}/V_{ud}| \approx 0.231$.

It is important to appreciate that the lowest order electroweak theory provides an interpretation of decay rates of hadrons as a measurement of hadronic matrix elements of certain operators (the various currents) belonging to the purely hadronic theory. Even though we can’t calculate these matrix elements directly, by analyzing symmetries of the strong interactions, we can predict many relations among different matrix elements and check whether the std model electroweak theory agrees. In this way rather spectacular supporting evidence for the low energy standard model is accumulated.

One example of another process which involves $V_{us}$ is the kaon analogue of pion beta decay: $K^- \rightarrow \pi^0 + e + \bar{\nu}_e$. For the Kaon there is also enough phase space to allow substituting $\mu\bar{\nu}_\mu$ for the electron. These processes have branching fractions of about 5.1% and 3.3% respectively. The CVC hypothesis for these modes involves flavor $SU(3)$ rather than isospin, but it is good enough to support our estimate of $|V_{us}|$ that comes from $K \rightarrow \mu + \bar{\nu}_\mu$. 

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29.3.1 The Neutral Kaon System

The physics of neutral kaons is very interesting and subtle. One striking difference from the charged kaons is that the mass eigenstates of neutral kaons are linear combinations of $K^0$ and its charge conjugate $\bar{K}_0$. This is known as $K^0 - \bar{K}^0$ mixing and plays an essential role in the understanding of their electroweak decays. Because of the CKM matrix strangeness is, unlike electric charge, only approximately conserved: electroweak interactions violate strangeness and indeed other flavor conservation. We shall see in the next section that strangeness changing processes without a corresponding change in charge are nonzero but very highly suppressed. But for now we give a slightly oversimplified discussion of how this mixing is described.

The electroweak theory explicitly breaks charge conjugation and parity symmetries, but their combination $CP$ is a very weakly broken symmetry. This means that to a very good approximation we can take the mass eigenstates in the kaon rest frame to be eigenstates of $CP$.

\[ |K, \pm\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle \mp CP|K^0\rangle) \]  

(29.66)

(Here we note that $CP|K^0\rangle = -|\bar{K}\rangle^0$ in the kaon rest frame because the kaon is a pseudoscalar.) Because the amplitude for the transition is so suppressed, these two states will have slightly different masses. We defer the details to the next section, where we will also incorporate $CP$ violation. For now we assume $CP$ is an exact symmetry to argue that these two states have different decay patterns.

The dominant decay modes are nonleptonic into two or three pions. We can easily see that the two pion system in the center of mass has a definite eigenvalue of $CP$. Its parity is of course $(-)^l$, but what about charge conjugation? For two $\pi^0$s $C = +$. But because they are identical bosons $l$ must be even, and hence $CP = +$. For the $\pi^+\pi^-$ final state $l$ can be any integer. However $C$ just switches the roles of $\pi^+$, $\pi^-$ which is the same action as parity. So $C$ on this final state is $(-)^l$, so the action of $CP^+$ for any two pion system. Thus we have the selection rule that the eigenstate odd under $CP$ cannot decay into two pions. The 3 pion final state is possible for the odd $CP$ eigenstate, but its limited phase space suppresses the decay relative to the 2 pion final state of the even $CP$ eigenstate. The $CP$ even state is named $K_s$ because it has a shorter lifetime and the $CP$ odd eigenstate is named $K_L$. In the particle book the decay processes are labelled separately for $K_s$ and $K_L$.

29.4 Flavor Changing Neutral Current Interactions

From experimental measurements it is found that strangeness changing neutral processes are very highly suppressed. If we ask for a neutral kaon analogue of $K^- \rightarrow \mu\bar{\nu}_\mu$, we would pick $K \rightarrow \mu^+\mu^-$ or $\nu\bar{\nu}$. The latter would be very difficult to detect, but the former very easy. Its branching fraction is tiny

\[ \frac{\Gamma_{K_L \rightarrow \mu^+\mu^-}}{\Gamma_{K_L}} \approx (0.684 \pm 0.011) \times 10^{-8} \]  

(29.67)
but it definitely occurs.

Since the neutral current is diagonal in flavor in the standard model, there are no tree level contributions to this process. Looking back to the construction of the model, the reason this happened was that every quark flavor comes as a doublet: for each charge $-1/3$ quark there is a charge $+2/3$ quark. Before the discovery of the charm quark there was a mismatch between the two charge sectors. Only one doublet coupling to the $SU(2)$ part of the gauge group was present, say $(u, d)$, so only one of the charge $-1/3$ quarks could couple to $SU(2)$. The other one $s$ could only couple to the $U(1)$ factor. Thus when a general mass matrix

$$G \bar{u}_L u_R + G_{dd} \bar{d}_L d_R + G_{ss} \bar{s}_L s_R + G_{sd} \bar{s}_L d_R + G_{ds} \bar{d}_L s_R + h.c. \quad (29.68)$$

is diagonalized by a pair of unitary transformations, the neutral part of the $SU(2)$ gauge couplings involve the linear combination $(U^t_L)_{kj} (U_L)_{dj} \bar{d}_k W_3 \cdot \gamma d_j$. That is there is no sum on the $d$ index. With the charm quark added this index would be summed, and the flavor changing part of the neutral current would be erased by the unitarity of $U_L$.

This argument explains why tree amplitudes contain no strangeness changing neutral current processes. But this only suppresses the effect by a single power of $\alpha$, which is still about 5 orders of magnitude bigger than observed. Also the process is indeed observed so we don’t want to banish it forever. So let’s look at the sort of loop diagram which could mediate $K_L \rightarrow \mu \bar{\mu}$. $K_L$ is a linear combination of an $s \bar{d}$ and a $d \bar{s}$ system, so consider a 1 loop vertex diagram with a fermion line entering the diagram as an $s$ quark and exiting as a $d$ quark. The loop is formed by the exchange of a $W$ boson between the $s$ and $d$ lines. At its $s$ vertex the $s$ quark is converted to a $u$, $c$, or $t$ quark and at the $d$ vertex, the $u$, $c$, or $t$ quark is converted to a $d$ quark. Finally a $Z$ boson is emitted from the $u$, $c$, or $t$ quark.

At the $s$ vertex we have a factor of the mixing matrix $V_{is}$ and at the $d$ vertex a factor $V_{id}^*$, where $i$ is $u$, $c$, or $t$. Thus the diagram would evaluate to

$$\sum_{i=u,c,t} V_{is} V_{id}^* X_i \quad (29.69)$$

where $X_i$ is the result of carrying out the loop integral. If $X_i$ were independent of $i$ the sum would be $\delta_{sd} = 0$. But of course this is not the case because the quark masses are not the same!

This diagram is log divergent but that divergence will be removed after renormalization. However for loop momenta much larger than $M_W$ and the quark masses the cancellation among the different masses actually makes the integral convergent. A low energy estimate for $X_i$ would be $X_i \sim G_F m_i^2$. Assuming that $V_{is}$ and $V_{id}$ are small enough to neglect the contribution of the top quark, the diagram would give a value of order

$$m_c^2 G_F V_{es} V_{cd} \sim G_F m_c^2 \cos \theta_C \sin \theta_C \sim \frac{m_c^2}{M_W^2} \alpha \sim 4 \times 10^{-4} \alpha$$

which is in the right ballpark to explain the $10^{-8}$ suppression. This suppression mechanism is called the GIM mechanism for Glashow, Iliopoulis and Maiani who proposed the idea to
explain the suppression several years before charm was discovered. It is interesting that the needed suppression gave an impressive estimate of what the charm quark mass should be!

In order to exploit this mechanism without repeating the same calculation multiple times, we can use the diagrams to infer a term in the effective Lagrangian that describes processes not present at tree level. For a given process find diagrams at the level of quarks and leptons which mediate the process. Then translate the diagram to an effective tree vertex. As an example consider the process $K^+ \rightarrow \pi^+ + \nu + \bar{\nu}$. At the level of quarks and leptons we replace the $K^+$ with a $u\bar{s}$. Then the $u$ quark is a spectator, and the $\bar{s}$ quark transforms to a $\bar{d}$ quark through the diagram described above where the $Z$ boson is hooked onto an $\nu\bar{\nu}$ pair.

There is another diagram where two $W$ bosons are emitted from the $\bar{s}$ line converting the $\bar{s}$ via $u,c,t$ to $\bar{d}$ The two $W$ bosons are then absorbed by the neutrino line with leptons $e,\mu,\tau$ propagating between the two vertices.

Calculating these two diagrams then gets interpreted as a term

$$
\frac{G_F g_3^2}{\sqrt{2}} F\left(\frac{m_t^2}{M_W^2}, \frac{m_c^2}{M_W^2}\right) \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{\nu} \gamma_\mu (1 - \gamma_5) \nu 
$$

(29.70)

in the effective Lagrangian. Once $F$ is known one can then calculate semileptonic processes by gleaning information about hadron matrix elements of the given process. For this example that means

$$
\langle \pi^+ | \bar{s} \gamma^\mu (1 - \gamma_5) d | K^+ \rangle = \sqrt{2} \langle \pi^0 | \bar{s} \gamma^\mu u | K^+ \rangle 
$$

(29.71)

where we have dropped the $\gamma_5$ term by parity and used the Wigner-Eckart theorem for isospin to relate the process to kaon beta decay.
However, on its face, tree level suppression is not enough to account for $10^{-8}$ had. counting $= 0$. $\alpha (\alpha) \times 10^{-2} - 10^{-3}$

\[
\text{Lagrangian:} \quad \mathcal{L} = \left( \mathcal{L}_0 + \mathcal{L}_1 \right)
\]

\[
\mathcal{L}_0 = \frac{1}{2} (D \bar{\nu} D \nu) - \frac{1}{2} m^2 \nu \nu
\]

\[
\mathcal{L}_1 = \frac{\alpha}{4 \pi} \frac{G_F}{\sqrt{2}} \left( \bar{\nu} \gamma^\mu (1 - \gamma^5) \nu \right) \mathcal{A}_\mu
\]

This is GIM Mechanism (Glashow-Iliopoulos-Maiani).

**Example:** $K = \pi + \nu \bar{\nu}$

\[
\text{Explicit loop calculation gives an estimate for an effective term in the Lagrangian:}
\]

\[
\left( \frac{m_\nu^2}{m_W^2} \right)^2 \frac{G_F}{\sqrt{2}} = \frac{\alpha}{4 \pi} \frac{G_F}{\sqrt{2}} \left( \frac{m_\nu^2}{m_W^2} \right) \approx 10^{-4}
\]

\[
\text{SI coupling:} \quad \langle n^+ 1 S \tau \left( \frac{1 + r}{2} \right) d \mid k^+ \rangle \approx \frac{G_F}{\sqrt{2}} \left( \frac{m_\nu^2}{m_W^2} \right) \langle n^+ 1 S \tau \mid k^+ \rangle
\]

\[
= \langle n^+ 1 S \tau \mid d \mid k^+ \rangle = \sqrt{2} \langle n^+ 1 S \tau \mid u \mid k^+ \rangle
\]

\[
\approx \alpha m W \langle n^+ 1 S \tau \rangle
\]
29.5 Non leptonic Weak Decays

For kaons and strange baryons (hyperons), the dominant decay modes are non-leptonic, meaning that the final particles are all hadrons. As we have already mentioned in these cases symmetry arguments have considerably less power than with semi-leptonic processes. We list some examples:

**Hyperons**

\[
\begin{align*}
\Lambda(I = 0) & \rightarrow \begin{cases} p + \pi^- & 64\% \quad I_f = \frac{1}{2}, \frac{3}{2} \\
n + \pi^0 & 36\% \end{cases} \\
\Sigma^+(I = 1) & \rightarrow \begin{cases} p + \pi^0 & 52\% \quad I_f = \frac{1}{2}, \frac{3}{2} \\
n + \pi^+ & 48\% \end{cases} \\
\Sigma^-(I = 1) & \rightarrow n + \pi^- \quad 100\%, \quad I_f = \frac{3}{2} \\
\Xi & \rightarrow \Lambda + \pi \quad 100\%, \quad I_f = 1 \\
\end{align*}
\]

**Kaons**

\[
\begin{align*}
K^0(I = 1/2) & \rightarrow \begin{cases} \pi^+ + \pi^- & 69\% \quad I_f = 0, 2 \\
\pi^0 + \pi^0 & 31\% \end{cases} \\
K^+(I = 1/2) & \rightarrow \pi^0 \pi^+, \quad 20.67\%, \quad I_f = 2 \\
\end{align*}
\]

Note that for the 2 pion final states, \( I_f = 1 \) is excluded because that case is odd under interchange of isospin labels, and Bose statistics would require that the spatial exchange also be odd, which for a two particle state means \( l = 1 \), but \( l = 0 \) in these decays because \( k \) is pseudoscalar.

If we ignore the contribution of \( b, t \) quarks because of their weight, the interaction terms relevant to these strangeness changing processes are

\[
\begin{align*}
G_F \sqrt{2} \left[ V_{ud} V_{us}^* \gamma^\mu (1 - \gamma_5) u \bar{u} \gamma_\mu (1 - \gamma_5) s + V_{cd} V_{cs}^* \gamma^\mu (1 - \gamma_5) c \bar{c} \gamma_\mu (1 - \gamma_5) s \right] \\
= G_F \sqrt{2} \sin \theta_C \cos \theta_C \left[ \gamma^\mu (1 - \gamma_5) u \bar{u} \gamma_\mu (1 - \gamma_5) s - \gamma^\mu (1 - \gamma_5) c \bar{c} \gamma_\mu (1 - \gamma_5) s \right]
\end{align*}
\]

The first of these terms can change isospin by both 1/2 and 3/2, whereas the second term is pure \( \Delta I = 1/2 \). It is an experimental fact that nonleptonic decays show a strong enhancement of those processes for which \( \Delta I = 1/2 \) over those with \( \Delta I = 3/2 \).

Consider the kaon decays for example. The process \( K^+ \to \pi^+ \pi^0 \) is only \( \Delta I = 3/2 \). This is because the pion system is in an s-wave and hence must be in an isospin state symmetric under interchange of the two pions. But it can’t be \( I = 0 \) because \( I_3 = +1 \). So it must be \( I = 2 \), and \( \Delta I = 2 - 1/2 = 3/2 \). In contrast the two pions emerging from \( K^0 \) decay can have either \( I = 0 \) or \( I = 2 \). The experimental rates are

\[
\begin{align*}
\Gamma_{K^+ \to \pi^+ \pi^0} &= 0.21 \times 10^8 / 1.24 = 0.169 \times 10^8 \text{s}^{-1} \\
\Gamma_{K^{0\,\text{S}} \to \pi \pi} &= 1.12 \times 10^{10} \text{s}^{-1} = 664 \times \Gamma_{K^+ \to \pi^0 \pi^+} \\
\end{align*}
\]
A factor of $\sqrt{663} \approx 26$ in amplitude.

We can also analyze the isospin changes in the two possible final states in neutral $K$ decay.

$$
|2, 2\rangle = |\pi^+\pi^+, \pi^0\pi^0\rangle, \quad |2, 1\rangle \frac{\sqrt{6} - 2}{2} = (|\pi^0\pi^+\rangle + |\pi^+\pi^0\rangle) \frac{\sqrt{2}}{2},
$$

$$
|2, 0\rangle \frac{\sqrt{6}}{2} = |\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle + 2|\pi^0\pi^0\rangle
$$

$$
|0, 0\rangle \frac{\sqrt{3}}{2} = |\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle - |\pi^0\pi^0\rangle
$$

The last two equations can be solved:

$$
|\pi^0\pi^0\rangle = \frac{2}{\sqrt{3}}|2, 0\rangle - \frac{1}{\sqrt{3}}|0, 0\rangle
$$

$$
\frac{1}{\sqrt{2}}(|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle) = \frac{1}{\sqrt{3}}|2, 0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|0, 0\rangle
$$

This shows that we can write

$$
\mathcal{M}_{\pi^+\pi^-} = \frac{A_2 + \sqrt{2}A_0}{\sqrt{3}}
$$

$$
\mathcal{M}_{\pi^0\pi^0} = \frac{A_2\sqrt{2} - A_0}{\sqrt{3}}
$$

Then we have

$$
\frac{\Gamma_{\pi^+\pi^-}}{\Gamma_{\pi^0\pi^0}} = \frac{\int d\Omega |A_2 + A_0\sqrt{2}|^2}{\int d\Omega |A_2\sqrt{2} - A_0|^2}
$$

---

3 The normalization of scattering states is a bit subtle. To clarify we represent the final states in Fock space, with the isospin decompositions

$$
a^\dagger (p_1) a^\dagger (p_2) |0\rangle = \frac{|2, 0\rangle \sqrt{2} - |0, 0\rangle}{\sqrt{3}}
$$

$$
a^\dagger (p_1) a^\dagger (p_2) |0\rangle = \frac{|2, 0\rangle + |0, 0\rangle \sqrt{2} + |1, 0\rangle \sqrt{3}}{\sqrt{6}}
$$

We had to include the isospin 1 states to isolate the appropriate scattering state. Then using proper scattering states

$$
\mathcal{M}_{\pi^0\pi^0} = \frac{A_2\sqrt{2} - A_0}{\sqrt{3}}
$$

$$
\mathcal{M}'_{\pi^+\pi^-} = \frac{A_2 + \sqrt{2}A_0 + \sqrt{3}A_1}{\sqrt{6}}
$$

$$
\rightarrow \frac{A_2 + \sqrt{2}A_0}{\sqrt{6}}
$$

where the last line follows from $A_1 = 0$. The formula for $\mathcal{M}'_{\pi^+\pi^-}$ is a factor of $1/\sqrt{2}$ smaller than $\mathcal{M}_{\pi^+\pi^-}$! But this discrepancy disappears in the ratio of rates, because the total rate for the $\pi^0\pi^0$ final state has an extra factor of $1/2$ because the particles are identical.

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According to the $\Delta I = 1/2$ enhancement, we expect that $A_2 \ll A_0$, so that the ratio should be close to 2. The experimental ratio is $69/31 \approx 2.2$ supporting $\Delta I = 1/2$ enhancement.

Symmetry alone can’t explain this enhancement. It has to come from the dynamics of QCD. One can identify classes of Feynman diagram which contribute only $\Delta I = 1/2$, and hope that for some reason they dominate over diagrams which allow both $\Delta I = 3/2, 1/2$. To change flavor there must be at least one W exchange. Focus on neutral $K \rightarrow \pi \pi$ decay. The kaon minimal quark content is $s \bar{d}$. The pions contain no s quark. The diagrams in which the s quark changes to a u quark by emitting a W which then converts to a $d \bar{u}$ pair will contain both $\Delta I = 1/2, 3/2$. Two pions can form either by the pair from the $W$ converting to a $\pi^-$ and the initial $\bar{d}$ combining with the the $u$ created from the $s$ combining into a $\pi^+$; or the $u\bar{u}$ and $d\bar{d}$ convert to two $\pi^0$’s.

But there is another class of diagrams in which the s quark emits a $W$, converting to u, c, or t quarks which then reabsorb the $W$ converting to a d quark. The strangeness changing process here is pure $\Delta I = 1/2$ because an isoscalar $s$ quark has changed to an iso-doublet $d$ quark. In both classes of diagrams there have to be enough gluon exchanges so that the whole diagram is connected. In perturbative QCD one can see that this second class of “penguin” diagrams is mildly enhanced. But the observed enhancement is much more dramatic than this, and can only be understood by nonperturbative dynamics.
\[ \Gamma_{K^+ \pi^0 \pi^0} = 0.21 \left[ \frac{10^{-5}}{1.2 \text{ fm/au}} \right] \]
\[ \Gamma_{K^0 \pi^+ \pi^-} = \frac{10^{-10}}{1.2 \text{ fm/au}} \approx 1.2 \times 10^{-11} \approx 6 \text{ MeV} \pi^+ \pi^- \pi^0 \text{ Both} \]

**Iso-spin Addition:**

\[ M_{K^0 \to \pi^+ \pi^-} = A_0 + \frac{A_2}{\sqrt{2}} \]
\[ M_{K^0 \to \pi^+ \pi^-} = A_0 - \frac{A_2}{\sqrt{2}} \]

\[ \Rightarrow \frac{\Gamma_{\pi^+ \pi^-}}{\Gamma_{\pi^+ \pi^-}} = 2 \text{ approximate} \]

Similar conclusion from hyperon decay: enhancement of \[ \Delta I = \frac{1}{2} \] deca is dramatic.

We can calculate Matrix element \( \langle \pi \pi | H_{\text{ew}} | K \rangle \)

Flavor symmetry alone allows both \( \Delta I = \frac{3}{2}, \frac{1}{2} \) with

more reason and gives a justification for \( \Delta I = \frac{1}{2} \) enhancement.

[Symmetry here is much less powerful than in leptonic or semi-leptonic decays!]

From perturbative QCD: \( n_p(M_K) \approx 0.1 \), pretty small.

but \( n_p(5 \text{ GeV}) \approx 1 \). Raritan coupling effects provide \( n_p \) (factor of 2) type enhancement of \( \Delta I = \frac{1}{2} \)

Typical diagram:

```
  K_0  \rightarrow  d \pi^0 \pi^0

\Delta I = \frac{3}{2}, \frac{1}{2}
```

Maybe for some reason this class of diagram has enhanced matrix elements.
29.5.1 Perturbative Enhancement in Charm Decays
Example of Perturbative Enhancement in Charm Decays

\[ \Delta I = 0 \quad \Delta I = 1 \]

Effective Term:

\[ \frac{t^u}{c} \frac{t^u}{c} \frac{c}{s} (1 + \gamma_5) s \ ar{d} \gamma_\mu (1 + \gamma_5) u \]

\( \text{Note: } c \neq u \)

Large loop corresp. for \( t^u \)

\[ \begin{array}{c|c|c}
\hline
u & c & \bar{d} \\
\hline
\end{array} \]

\[ \text{not for } \begin{array}{c|c|c}
\hline
\bar{c} & \bar{s} & \bar{s} \\
\hline
\end{array} \]

\[ \text{Recall } \beta \text{ decay: ineffective correction} \]

\[ \text{Tree level: } \begin{array}{c|c|c}
\hline
\bar{a} & \bar{a} & \bar{a} \\
\hline
\end{array} \]

\[ \text{Abstract: } s \text{ vs } s \text{ suppress.} \]

\[ \text{Color: } \frac{3}{3} \times \frac{3}{3} = \frac{3}{6} + \frac{3}{6} \]

\[ \frac{A}{6} \text{ Repulsive, } \frac{A}{6} \text{ Attractive} \]

Conidrum:

\[ \frac{c}{s} (1 + \gamma_5) s \ ar{d} \gamma_\mu (1 + \gamma_5) u + \frac{c}{s} (1 + \gamma_5) u \ ar{d} \gamma_\mu (1 + \gamma_5) s \]

\[ + \text{ Color interaction, repulsive (suppress) } \]

\[ - \text{ Color interaction, attractive (enhanced) } \]

\[ \frac{t^u}{t^u} = \frac{1}{2} \left[ \frac{2}{3} - \frac{1}{3} - \frac{1}{3} \right] = \left\{ \begin{array}{l}
\frac{-2}{3} \quad \text{Attractive} \\
\frac{1}{3} \quad \text{Repulsive} \\
\end{array} \right. \]

\[ \text{In } 3 \text{, or } \bar{3} \]

\[ T_{\alpha} \frac{t^a}{2} \frac{t^a}{2} = \frac{1}{2} \delta_{\alpha \mu} \delta_{\alpha \mu} = \frac{1}{2} \left( N_c^2 - 1 \right) \]

\[ \frac{1}{2} \left[ \gamma_5 - \gamma_5 \right] = \frac{1}{2} \]

\[ \text{Leading log, } \text{running energy factors of } 2. \]
Flavor Accoum. Number of $O^+$

$SU(3)_L$: 
$(u \bar{u}s)_S = \bar{6}$ 
$(u \bar{u}s)_S \otimes \bar{T} = 6 \times 3 = \bar{15} + 3$

$(u \bar{u}s)_A = \bar{3}$ 
$(u \bar{u}s)_S \otimes \bar{T} = \bar{3} \times \bar{3} = \bar{3} + \bar{\bar{3}}$

$O^+_+$ has $I_J = +1$ not in 3, \( \bar{15}, \bar{\bar{3}} \) respect.

\[ Q = \begin{pmatrix} \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \]

\[ Y_3 = (\frac{2}{3} - \frac{1}{2}) = \frac{1}{3} \]

\[ Y_2 = (-\frac{1}{2} + \frac{1}{2}) = 0 \]

\[ Y_3 = (\frac{2}{3} - \frac{1}{2}) = -\frac{2}{3} \]

\[ (u \bar{u}s, T) = \frac{1}{3} \]

\[ \begin{pmatrix} u_1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \]

\[ 0^+ \text{ is } \bar{15} \] "Smaller flavoring is enhanced"

\[ 0^- \bar{6} \]

Analogous operator for strong decay:

\[ \bar{d} \sigma (1 + \gamma_5) \bar{u} \gamma_\mu (1 + \gamma_5) \bar{s} = \frac{1}{2} \sigma \gamma_\mu (1 + \gamma_5) \bar{u} \gamma_\mu (1 + \gamma_5) \bar{s} \]

\[ 0^+ \] $(u \bar{u}s)_S = 6$ $(T \bar{0} \bar{u})_S = 6$ $6 \times 6 = 27 + 8 + 1$

\[ 0^- \] $(u \bar{u}s)_A = \bar{3}$ $(T \bar{0} \bar{u})_A = 3$ $\bar{3} \times 3 = 8 + 1$

$0^-$ mildly enhanced, $O^+^+$ mildly suppressed.
29.5.2 $K^0$-$\bar{K}^0$ Mixing from Feynman Diagrams

The transitions between $K^0$ and $\bar{K}^0$ are non zero, but they are highly suppressed as shown by experiment. This is a neutral ($\Delta Q = 0$) strangeness changing ($\Delta S = -2$) transition, and the way this suppression comes about in the standard model is via the GIM mechanism. There are two electroweak quark diagrams that mediate these transitions. They both have an $\bar{s}$ and a $d$ quark incoming and a $\bar{d}$ and an $s$ quark outgoing. In the first diagram the incoming lines exchange two $W$ bosons, the top line changing from an $\bar{s}$ to $\bar{u}, \bar{c}, \bar{t}$ to $\bar{d}$, while the bottom line changes from $d$ to $u, c, t$ to $s$. In the second diagram the incoming $\bar{s}, d$ lines annihilate, emitting 2 $W$’s, which then pair create the final $\bar{d}, s$.

Factors of the CKM mixing matrix $V_{i\bar{s}}V_{i\bar{d}}$ and $V_{j\bar{u}}V_{j\bar{s}}$, where $i, j = u, c, t$, enter the calculation, multiplied by the value of the diagrams $X_{ij}$ and then summed over $i, j$. If the $u, c, t$ all had the same mass, $X_{ij}$ would be independent of $i, j$ so the sums would give zero, by the unitarity of the CKM matrix. The transitions are nonzero because these masses are not the same. In the following, we show a partial calculation, omitting the contribution of the NGB part of Higgs field exchange.

To simplify writing, define $\xi_i \equiv V_{i\bar{d}}V_{i\bar{s}}^*$. We work out the Feynman amplitude for the sum of the two diagrams putting the external momenta equal to zero. Remember that the strategy is to use this calculation to obtain a term in the effective Lagrangian which will be used for processes of low energy compared to $M_W$. We get

$$\mathcal{M} = \frac{g_2^4}{64} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + M_W^2)^2} \sum_{ij} \xi_i \xi_j \frac{k^2/D}{(k^2 + m_i^2)(k^2 + m_j^2)} \left[4 [\bar{u}_s \gamma^\mu \gamma^\rho \gamma^\nu(1 - \gamma_5)vd \bar{u}_s \gamma^\rho \gamma^\mu(1 - \gamma_5)ud - \bar{v}_s \gamma^\mu \gamma^\rho \gamma^\nu(1 - \gamma_5)ud \bar{u}_s \gamma^\rho \gamma^\mu(1 - \gamma_5)vd] \right]$$

(29.80)

We can identify the operator

$$\frac{1}{2} \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{s} \gamma_\mu (1 - \gamma_5) d$$

(29.81)
as generating the spinor structure of $\mathcal{M}$. So we write the term in the effective Lagrangian

$$\mathcal{L}_{\Delta S = \pm 2}^{eff} = \frac{G_F}{\sqrt{2}} \frac{g_2^2}{64\pi^2} \sum_{i,j} \xi_i \xi_j A(x_i, x_j) \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{s} \gamma^\mu (1 - \gamma_5) d + \text{h.c.}$$

(29.82)

$$A(x_i, x_j) = \frac{M_W^2}{\pi^2} \int \frac{d^4 k_E}{(k_E^2 + M_W^2)^2} \frac{k_E^2/D}{(k_E^2 + m_i^2)(k_E^2 + m_j^2)}$$

$$= \frac{1}{x_j - x_i} \left[ \frac{x_j \ln x_j}{(x_j - 1)^2} - \frac{x_i \ln x_i}{(x_i - 1)^2} + \frac{x_j - x_i}{(1 - x_i)(1 - x_j)} \right]$$

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where \( x_i = m_i^2/M_W^2 \), and recall \( \xi_i = V_{id}V_{is}^* \).

Write out the CKM matrix:

\[
V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}
\] (29.83)

and we choose the underlined elements to be real, so that as many elements involving light quarks are real as possible, Unitarity of \( V \) implies \( \xi_u + \xi_c + \xi_t = 0 \), so we can eliminate \( \xi_c = -\xi_u - \xi_t \). Since \( \xi_u \) is real and \( \xi_t \) is extremely small, \( \xi_c \) is nearly real. The coefficient of the \( \Delta S = -2 \) term in the effective Lagrangian now reads

\[
\sum_{i,j} \xi_i \xi_j A_{i,j} = \xi_u^2 (A_{uu} + A_{cc} - 2A_{uc}) + \xi_t^2 (A_{tt} + A_{cc} - 2A_{ct}) + 2\xi_u \xi_t (A_{cc} - A_{ct} + A_{ut} - A_{uc})
\] (29.84)

This quantity is real unless \( \xi_t \) is complex, and even then if \( x_c = x_t \) it would also real. Since CP violation depends on complex parameters in the Hamiltonian, its only source for this process in the standard model is the complexity of \( \xi_t \), which is itself very tiny.

Although our expression for the coefficient of the \( \Delta S = \pm 2 \) terms is explicit, it can be greatly simplified given the actual values of the parameters. It is clearly an excellent approximation to set \( x_u = 0 \) and to drop terms of order \( x_c^2 \). After some tedious analysis one arrives at the approximate formula

\[
\sum_{i,j} \xi_i \xi_j A_{i,j} \approx \xi_u^2 x_c + \xi_t^2 \left( \frac{x_t + x_c^2}{(1 - x_t)^2} + \frac{2x_t^2 \ln x_t}{(1 - x_t)^3} \right) + x_c \left[ 3 + 2 \ln x_t - \frac{2}{1 - x_t} - \frac{2 \ln x_t}{(1 - x_t^2)} \right] + 2\xi_u \xi_t x_c \left( 2 + \ln x_c - \frac{1}{(1 - x_t)} - \frac{\ln x_t}{(1 - x_t)^2} \right) \approx \xi_u^2 x_c + \xi_t^2 \left( \frac{x_t + x_c^2}{(1 - x_t)^2} + \frac{2x_t^2 \ln x_t}{(1 - x_t)^3} \right) - 2\xi u \xi_t x_c \left( 1 + \ln x_c - \frac{1}{(1 - x_t)} - \frac{\ln x_t}{(1 - x_t)^2} \right)
\] (29.85)

where we could also drop terms of order \( \xi_t^2 x_c \), since \( \xi_t \) is so tiny. For a rough approximation to the size of the effect, we set \( \xi_t = 0 \) and write the effective term as

\[
\xi_u^2 \frac{G_F m_c^2}{16\pi^2} \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{s} \gamma_\mu (1 - \gamma_5) d + \text{h.c.}
\] (29.86)

To estimate the transition matrix element in the Hamiltonian, we write

\[
\langle \bar{K}_0, \mathbf{p}' | H' | K^0, \mathbf{p} \rangle = \left\langle \bar{K}_0, \mathbf{p}' \right| \int d^3 x \mathcal{L}' | K^0, \mathbf{p} \rangle = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \left\langle \bar{K}_0, \mathbf{p} | \mathcal{L}' | K^0, \mathbf{p} \right\rangle
\] (29.87)
We can therefore describe mixing in the $K$ rest frame by the two state system governed by the effective Hamiltonian

$$h = \begin{pmatrix} m_0 & h' \\ h'\dagger & m_0 \end{pmatrix}$$  \hspace{1cm} (29.88)$$

where $m_0 = m_{K^0} = m_{\bar{K}^0}$ is the neutral kaon mass with the mixing turned off and $h' = (2\pi)^3 \langle \bar{K}_0, 0 | \mathcal{L}' | K^0, 0 \rangle$ is the mixing term in the $K$ rest system.

To estimate $h'$ we first recall the definition of $f_K$

$$\langle 0 | \bar{s} \gamma^\mu (1 - \gamma_5) d | K^0 \rangle (2\pi)^{3/2} \sqrt{2m_K} = ip^\mu \sqrt{2} F_K$$  \hspace{1cm} (29.89)$$

Now consider the matrix element in $h'$

$$(2\pi)^3 \langle \bar{K}_0, 0 | \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{s} \gamma^\mu (1 - \gamma_5) d | K^0, 0 \rangle$$  \hspace{1cm} (29.90)$$

There are several ways to think of this matrix element in terms of $F_K$. First one can insert a complete set of states between the two currents but keep only the vacuum contribution $|0\rangle \langle 0|$. You could also swap the two currents before repeating this procedure. These two contributions would then total $2p^2 F_K^2 / (2m_K) = -4m_K^2 F_K^2 / 2m_K$. But instead of swapping the two currents one could swap the two $d$ fields using the Fierz rearrangement system. and then insert $|0\rangle \langle 0|$. But then the $\bar{s}$ and $d$ in the new current would not be in a color singlet. But the vacuum is a singlet, so

$$\langle 0 | \bar{s} \gamma^\alpha (1 - \gamma_5) d_\beta | K^0 \rangle = \frac{1}{N_c} \delta_\beta^\alpha \langle 0 | \bar{s} \gamma (1 - \gamma_5) d | K^0 \rangle$$

$$\langle \bar{K}^0 | \bar{s} \gamma^\alpha (1 - \gamma_5) d_\alpha | 0 \rangle = \frac{1}{N_c} \delta_\alpha^\beta \langle \bar{K}^0 | \bar{s} \gamma (1 - \gamma_5) d | 0 \rangle$$  \hspace{1cm} (29.91)$$

So the result for this swap is $1/N_c = 1/3$ times the first case. Finally the two $\bar{s}$ fields can be swapped to get a similar contribution. Combining with the first two contributions gives the approximation

$$(2\pi)^3 \langle \bar{K}_0, 0 | \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{s} \gamma^\mu (1 - \gamma_5) d | K^0, 0 \rangle \approx -\frac{8}{3} m_K F_K^2$$  \hspace{1cm} (29.92)$$

Inserting this into the expression for $h'$ gives

$$h' \approx -\xi_u G_F^2 m_c^2 \frac{8}{3} m_K F_K^2 = -\xi_u G_F^2 m_c^2 \frac{6}{\pi^2} m_K F_K^2 \approx 1.8 \times 10^{-6} \text{eV}.$$  \hspace{1cm} (29.93)$$
**K_{0} - \bar{K}_{0} Mixing \& CP Violation**

K_{0} \to \bar{K}_{0} amplitude suppressed: \Delta S = -2, \Delta Q = 0.

The Standard Model achieves this through GIM Mechanism:

\[
\sum_{i} \nu_{i*} V_{id} = (\nu^{+} W)_{sd} = 0 \quad \text{(Unitarity of Mixing Matrix)}.
\]

Inequality of \( m_{u}, m_{c}, m_{t} \) allows mixing \( \neq 0 \).

Define \( \xi_{i} = V_{id} \nu_{i*} \). We work out explicitly, now exchange.

Estimate Mixing completely with external momenta \( = 0 \). For real vector exchange, we have:

\[
M_{K_{0} = \bar{K}_{0}} = \frac{(3g^{2})^{2}}{G_{F}} \frac{1}{8\pi^{2}} \sum_{i} \xi_{i} \xi_{j} \frac{k^{2}}{D_{i} m_{i}^{2}} \frac{1}{D_{j} m_{j}^{2}}
\]

\[
\left[ \bar{u}_{3} \gamma^{\mu}(1+i\gamma_{5}) \frac{\partial^{2}}{\partial \tau_{1}^2} \gamma_{\nu}(1+i\gamma_{5}) \bar{d}_{3} \bar{u}_{3} \gamma^{\nu}(1+i\gamma_{5}) \bar{d}_{3} \bar{u}_{3} \gamma^{\mu}(1+i\gamma_{5}) \frac{\partial^{2}}{\partial \tau_{1}^2} \gamma_{\nu}(1+i\gamma_{5}) \bar{d}_{3} \right]
\]

\[
\sum_{\nu, \mu} \left[ \gamma^{\nu} \gamma^{\mu} \gamma^{\nu}(1+i\gamma_{5}) \right]_{ab} \left[ \gamma_{\nu} \gamma_{\mu} \gamma_{\nu}(1+i\gamma_{5}) \right]_{cd} = 4 \left[ \delta_{\nu}(1+i\gamma_{5}) \right]_{ab} \left[ \nu_{\nu}(1+i\gamma_{5}) \right]_{cd}
\]

\[
M_{K_{0}}^{\nu} = \frac{k^{2}}{4} \left( \sum_{i} \xi_{i} \xi_{j} \sum_{i} \frac{k^{2}}{D_{i} m_{i}^{2}} \frac{1}{D_{j} m_{j}^{2}} \right) \left[ \bar{u}_{3} \gamma^{\mu}(1+i\gamma_{5}) \bar{d}_{3} \bar{u}_{3} \gamma^{\nu}(1+i\gamma_{5}) \bar{d}_{3} \bar{u}_{3} \gamma^{\mu}(1+i\gamma_{5}) \frac{\partial^{2}}{\partial \tau_{1}^2} \gamma_{\nu}(1+i\gamma_{5}) \bar{d}_{3} \right]
\]
29.5.3 CP Violation and Oscillations

We now turn to a more detailed discussion of the consequences of $K^0, \bar{K}^0$ mixing, including the effects of CP violation. As already mentioned if CP were exact we could define $|\bar{K}^0\rangle = -CP|K^0\rangle$. In the $K$ rest frame we discuss mixing in a two state quantum mechanical system. The basis state vector can be taken

$$ |K^0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\bar{K}^0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -CP|K^0\rangle $$

(29.94)

The existence of mixing means that the time evolution is determined by a nondiagonal matrix Hamiltonian. To allow for decay we write

$$ H = M - i\frac{\Gamma}{2} $$

(29.95)

where $M$ is hermitian and $\Gamma$ is hermitian and positive definite.

It is helpful to consider how $M$ and $\Gamma$ could be determined from first principles, even though we won’t be able to go beyond very rough estimates. Writing out perturbation theory to second order would give

$$ M_{jk} - i\frac{\Gamma_{jk}}{2} = \langle j|H_W|k\rangle + \sum_{\lambda, E_\lambda \neq E_k} \frac{\langle j|H_W|\lambda\rangle \langle \lambda|H_W|k\rangle}{E_k - E_\lambda + i\epsilon} $$

(29.96)

where $j, k$ label the two degenerate ($E_j = E_k$) zeroth order eigenstates. Then to see how the imaginary part arises recall the relation

$$ \frac{1}{E_k - E_\lambda + i\epsilon} \sim P \frac{1}{E_k - E_\lambda} - i\pi \delta(E_\lambda - E_k) $$

(29.97)

The first term contributes to $M$ and the second to $-i\Gamma/2$. The challenge is that zeroth order in $H_W$ still means all orders in QCD! The consequences of CP violation depend on the relative phases of the $M_{ji}$ and the $\Gamma_{ji}$.

Before discussing CP violation we first note the consequences of CPT invariance which is a symmetry of any QFT. We restrict attention to this 2 state system. We first argue that $M_{11} = M_{22}$ and $\Gamma_{11} = \Gamma_{22}$. At least in perturbation theory in $H_W$, we have been able to separate the contributions to $M$ from those to $\Gamma$. So for $M$ we write

$$ M_{11} = \langle K^0|M|K^0\rangle = \langle CPTMK^0|CPTK^0\rangle = \langle CPTK^0|M|CPTK^0\rangle = \langle \bar{K}^0|M|\bar{K}^0\rangle = M_{22} $$

(29.98)

Since all that was used about $M$ was that it was hermitian, the same argument shows that $\Gamma_{11} = \Gamma_{22}$. But that is all one gets! applying the argument to $M_{12}$, you just learn that $M_{12} = M_{12}$ and $\Gamma_{12} = \Gamma_{12}$, a tautology.

For CP invariance the symmetry is unitary and one can fix phase conventions by defining $|\bar{K}^0\rangle = -CP|K^0\rangle$ and then, from the symmetry, one learns that $M_{12} = M_{21}$ and $\Gamma_{12} = \Gamma_{21}$. 397 ©1992, 2017 by Charles Thorn
Now we turn to the eigenstates. From CPT we can write

\[ M - i \frac{\Gamma}{2} = \begin{pmatrix} A & p^2 \\ q^2 & A \end{pmatrix} \]  

with \( A, p, q \) complex. Then the eigenvalues are \( \lambda_\pm = A \pm pq \), and the eigenvectors are

\[ |K\pm\rangle = \frac{1}{\sqrt{|p|^2 + |q|^2}} \begin{pmatrix} p \\ \pm q \end{pmatrix} \]  

(29.100)

The states \( |K\pm\rangle \) have been normalized to unity, but they are not orthogonal:

\[ \langle K - |K+\rangle = \frac{|p|^2 - |q|^2}{|p|^2 + |q|^2} \neq 0 \]  

(29.101)

in general. With CP invariance \( p = q \), and the states are orthogonal.

Since CP violation is very small, it is common to introduce a small complex parameter \( \epsilon \) and write \( p = 1 + \epsilon \) and \( q = 1 - \epsilon \). When \( \epsilon = 0 \) \( |K+\rangle \) goes to the CP odd \( K_L \) and \( |K-\rangle \) goes to the CP even \( K_S \). In a strong interaction process the neutral kaons are produced as eigenstates of strangeness not as mass eigenstates. They are linear combinations of \( K_L \) and \( K_S \). But as they travel they decay, with decaying more rapidly. So a kaon beam will after a while contain only \( K_L \)'s. As the \( K_S \)'s decay (\( \tau_S \approx 0.9 \times 10^{-10}s \)) the beam at first produces pions in pairs. The \( K_L \)'s decay more slowly (\( \tau_L \approx 5.2 \times 10^{-8}s \)) mostly into 3 pions.

We have mentioned that because strangeness changing neutral weak processes are highly suppressed (by the GIM Mechanism in the standard model) the consequent mass difference \( m_L - m_S \) is extremely tiny of order \( 10^{-12}\text{MeV}! \) compared to the kaon mass of 500MeV, this tiny difference would seem virtually impossible to detect. Nonetheless it has been measured using a quantum interference experiment called \( K^0 - \bar{K}^0 \) oscillations.

One starts by producing a neutral kaon beam, which quickly becomes a pure \( K_L \) beam. Next send this \( K_L \) beam through a slab of material, and the strong interactions of the \( K_L \) with nuclei causes the \( K_L \) to change to a superposition \( C_1 K_L + C_2 K_S \). Another slab of material is placed a precise distance downstream. Let \( \tau \) be the time of flight between the two slabs. Then at the second slab the superposition becomes

\[ C_1 K_L e^{-i(m_L - i\Gamma_L)/2} + C_2 K_S e^{-i(m_S - i\Gamma_S)/2} \]  

(29.102)

One then measures the \( K_S \)'s (via detection of 2 pion final state) that pass through the second slab. Those coming from the \( K_L \) component have \( \tau \) dependence \( e^{-i(m_L - i\Gamma_L)/2} \) whereas those coming from the \( K_S \) have \( \tau \) dependence \( e^{-i(m_S - i\Gamma_S)/2} \). The flux of \( K_S \) emerging will therefore be proportional to

\[ \left| e^{-i(m_L - i\Gamma_L)/2} + r e^{-i(m_S - i\Gamma_S)/2} \right|^2 = e^{-\Gamma_L \tau} + |r|^2 e^{-\Gamma_S \tau} + 2\text{Re}(re^{-i(m_S - m_L)\tau})e^{-(\Gamma_S + \Gamma_L)\tau/2} \]  

(29.103)

where \( r \) is a complex number depending on the slab material but not on \( \tau \). The period in \( \tau \) of the coefficient of the last term then gives a measurement of \( \Delta m = m_L - m_S \approx 3.490 \times 10^{-12}\text{MeV}. \)
There are 2 important neutral current processes that reveal CP violation. The first is the comparison between

\[
K_L \rightarrow \begin{cases} 
\pi^- e^+ \nu_e & \text{From } K^0 \text{ component} \\
\pi^+ e^- \bar{\nu}_e & \text{From } \bar{K}^0 \text{ component} 
\end{cases}
\]  

(29.104)

One then compares the relative rates in the form

\[
\frac{\Gamma_{e^+} - \Gamma_{e^-}}{\Gamma_{e^+} + \Gamma_{e^-}} = \frac{|1 + i| - |1 - i|}{|1 + i| + |1 - i|} \approx 2 \text{Re } \epsilon
\]

(29.105)

From the data this ratio is measured to be \((3.32 \pm 0.06) \times 10^{-3}\) or \(\text{Re } \epsilon \approx 1.6 \times 10^{-3}\).

The other CP violation measurement is the detection of the process \(K_L \rightarrow \pi \pi\) which implies that \(K_L\) has a CP even admixture. The mere detection of this process establishes CP violation, but its detailed interpretation requires an understanding of the final state interactions in the 2 pion system. Since the pions are in an s-wave, the appropriate interaction factor is the \(\epsilon^{i\delta}\) where \(\delta\) is the \(l = 0\) phase shift in \(\pi \pi\) scattering. Assuming isospin invariance there are two channels \(I = 0\) and \(I = 2\), with a different phase shift in each. Recalling our previous isospin analysis of \(K^-\) decays, we can write

\[
A_{K^0 \rightarrow \pi^+ \pi^-} = A_0 e^{i\delta_0} + \frac{A_2}{\sqrt{2}} e^{i\delta_2}
\]

(29.106)

\[
A_{\bar{K}^0 \rightarrow \pi^- \pi^0} = A_0 e^{i\delta_0} - \frac{\sqrt{2} A_2}{\sqrt{2}} e^{i\delta_2}
\]

(29.107)

and assuming CPT, the corresponding amplitudes for \(\bar{K}^0 \rightarrow \pi \pi\) are obtained from these by the substitution \(A_l \rightarrow -A_l^*\). We can choose phases such that \(A_0 = A^*_0\) is real. Then, using the relation between \(K_L, K_S\) and \(\bar{K}, \bar{K}\) we can write

\[
\eta_{++} = \frac{\langle \pi^+ \pi^- | H_W | K_L \rangle}{\langle \pi^+ \pi^- | H_W | K_S \rangle} = \frac{e^{i\delta_0} A_0 (p - q) + e^{i\delta_2} (pA_2 - qA^*_2)}{\sqrt{2}}
\]

(29.109)

\[
\eta_{00} = \frac{\langle \pi^0 \pi^0 | H_W | K_L \rangle}{\langle \pi^0 \pi^0 | H_W | K_S \rangle} = \frac{e^{i\delta_0} A_0 (p - q) - \sqrt{2} e^{i\delta_2} (pA_2 - qA^*_2)}{\sqrt{2} e^{i\delta_2 - \delta_0} (\text{Re } A_2 + i \text{Im } A_2)}
\]

(29.110)

Remembering the \(\Delta I = 1/2\) rule we expect \(|A_2/A_0| \ll 1\), so define \(\epsilon' \ll 1\) by

\[
\epsilon' = \frac{i e^{i(\delta_2 - \delta_0)} \text{Im } A_2}{A_0 \sqrt{2}}
\]

(29.114)
So we can write
\[
\begin{align*}
\eta_{+-} &= \epsilon + \epsilon' + O(\epsilon^2, \epsilon(A_2/A_0)) \\
\eta_{00} &= \epsilon - 2\epsilon' + O(\epsilon^2, \epsilon(A_2/A_0))
\end{align*}
\] (29.115)

The data give
\[
\begin{align*}
|\eta_{00}| &= (2.220 \pm 0.011) \times 10^{-3}, & |\eta_{+-}| &= (2.232 \pm 0.011) \times 10^{-3} \\
\text{Re}\frac{\epsilon'}{\epsilon} &= (1.66 \pm 0.023) \times 10^{-3} & \text{(29.116)}
\end{align*}
\]
$K_0 - \bar{K}_0$ Mixing

$\text{CP}$ is an approximate symmetry, except for the $SI$ so in absence of weak int can define

$\ket{K_0} = -\text{CP}\ket{\bar{K}_0}$ in $(K_0)$ rest frame.

Describe mixing in the $2d$ subspace $\psi = \begin{pmatrix} \psi_{K_0} \\ \psi_{\bar{K}_0} \end{pmatrix}$

$i \frac{d \psi}{dt} = \left( H - i \frac{P}{2} \right) \psi \quad H = H^\dagger, \quad P = P^\dagger$

$\Gamma \neq 0$ because Kaons decay, let $i = K_0, \bar{K}_0$

Then

$\langle j | U(\theta, \phi) | i \rangle = \sum_{ij} \frac{\delta_{ij}}{\delta_{ij}} \sum \psi_{K_0} \psi_{\bar{K}_0} \psi_{j} \psi_{\bar{j}} \psi_{i} \psi_{\bar{i}}$

$= e^{-i \frac{\Gamma}{2}}$

$i \equiv j = \langle j | H_{\psi | i} \rangle + \sum_{j' \neq j} \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$= \langle j | H_{\psi | i} \rangle + \frac{\Gamma}{2} \sum_{j' \neq j} \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$= \langle j | H_{\psi | i} \rangle + \frac{\Gamma}{2} \sum_{j' \neq j} \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$= \langle j | H_{\psi | i} \rangle + \frac{\Gamma}{2} \sum_{j' \neq j} \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$= \langle j | H_{\psi | i} \rangle + \frac{\Gamma}{2} \sum_{j' \neq j} \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$= \langle j | H_{\psi | i} \rangle + \frac{\Gamma}{2} \sum_{j' \neq j} \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$= \langle j | H_{\psi | i} \rangle + \frac{\Gamma}{2} \sum_{j' \neq j} \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$\Gamma_{ji} \text{ includes } \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$\Gamma_{ii} \text{ includes } \frac{1}{\Delta_{ij}^2} \langle j | H_{\psi | j'} \rangle \langle j' | H_{\psi | i} \rangle$

$\Delta \text{ is } \frac{1}{2} \text{ enhanced, appears in both}$

$\text{CP} \text{ violation depends on relative phase of } \Gamma_{ji}$
Oscillation

\[
\frac{K_L}{\sqrt{2}} \left( \begin{array}{c} 1 + e \\ 1 - e \end{array} \right) \quad \text{SI} \quad C_1 K_L + C_2 K_S \quad C_1 e^{-i\frac{\nu_L}{2} \tau} + C_2 K_S e^{-i\frac{\nu_S}{2} \tau} \quad \text{SI} \quad C_1 \left( C_1 K_L + C_2 K_S \right) e^{-i\frac{\nu_S}{2} \tau} \quad \text{unstable}
\]

\[
1 e^{-i\frac{\nu_L}{2} \tau} + e^{-i\frac{\nu_S}{2} \tau} \left( 1 + |r|^2 e^{-i\frac{\nu_S}{2} \tau} + 2 \text{Re} \left( r e^{-i(H_S - H_L)\tau} \right) e^{-i\frac{\nu_S}{2} \tau} \right)
\]

\[
M_L - M_S = (3.490 \pm 0.006) \times 10^{-12} \text{ MeV} \quad \text{and} \quad \delta_{\text{CP}} \text{ for oscillation by varying } \tau
\]

\[
\begin{align*}
\text{CP violation} & \quad (1) \quad K_L \rightarrow \left\{ \begin{array}{l}
\pi^+ e^- \nu_e \text{ from } K_0 \text{ decay} \\
\pi^- e^+ \bar{\nu}_e \text{ from } \bar{K}_0 \text{ decay}
\end{array} \right. \\
K_0 \rightarrow \bar{u}d \rightarrow e^+ \nu_e + \bar{u}d \\
K_0 \rightarrow \bar{u}d \rightarrow e^- \bar{\nu}_e + u \bar{d}
\end{align*}
\]

\[
E_{\text{exp}} \left( \pm 0.01 \right) \% \quad \frac{F(e^+) - F(e^-)}{F(e^+) + F(e^-)} = \frac{[1 + e^2] - [1 - e^2]}{1 + [1 - e^2][1 - e^2]} \propto 2 \text{Re} \epsilon
\]

\[
e = \frac{|M_{uu} - M_{ee}|}{M_{uu} + M_{ee}} - 1 = \frac{\pi^+ M_{\pi^+ \pi^0}}{\pi^+ M_{\pi^+ \pi^0}} - 1 = \frac{M_{uu} + M_{ee} - 2 M_{\pi^+ \pi^0}}{2 \left( M_{uu} + M_{ee} \right)} - 1
\]
29.5.4 Neutrino Oscillations

Experiments to detect neutrino oscillations work by creating a neutrino beam, tagged by the lepton associated with it, in one location and then detecting it at a location far away, and identifying it by the lepton that is produced with it. The produced and detected neutrino are not mass eigenstates: the amplitude for the process includes lepton mixing matrix factors $V_{ik}^L V_{il}^{L*}$ where $k, l$ are the neutrino types produced and detected, and $i$ labels one of the neutrino mass eigenstates making up the produced neutrino. These factors are multiplied by a propagator factor $\Delta_i$ and the index $i$ is then summed over the contributing neutrino mass eigenstates.

The propagator is in a mixed representation to reflect the distance between production and detection. We use coordinate space in the direction from one location to the other, and we use momentum space for energy and the other spatial directions. It is proportional to

$$\Delta_i = -i \int \frac{dp}{2\pi} e^{iLp} \frac{\left(\gamma^0 E - \gamma^1 p\right)}{p^2 - E^2 + m_i^2 - i\epsilon} \frac{\left(\gamma^0 E - \gamma^1 \sqrt{E^2 - m_i^2}\right)}{2\sqrt{E^2 - m_i^2}} e^{iL\sqrt{E^2 - m_i^2}}$$

(29.117)

In practice $m_i \ll E$ so we can approximate $\sqrt{E^2 - m_i^2} \approx E - m_i^2/(2E)$ in the phase and $\approx E$ elsewhere

$$\Delta_i \approx \frac{\gamma^0 - \gamma^1}{2} e^{iL/E - iLm_i^2/(2E)}$$

(29.118)

Putting in the lepton mixing matrices, we get the oscillatory factor

$$|\sum_i V^L_{ik} V^{L*}_{il} e^{-i\Delta m^2 L/(2E)}|^2$$

(29.119)

in the rate. The oscillations are in the variable $l/E$ and can be revealed either by varying $L$ or by varying $E$. In practice $L$ is fixed by the location of laboratories, so the variation in $E$ is more manageable. Clearly if the neutrino masses were all the same, no oscillations would show up, even if they all had the same nonzero mass.

If oscillations are seen, information on $\Delta m^2 = O(E/L)$ would be determined. Putting $h$ and $c$ back in shows that in units of km and eV, the order of magnitudes would be

$$\Delta m^2 (\text{eV}^2) \sim \frac{E(\text{GeV})}{L(\text{km})}$$

(29.120)

The higher the energy, the longer the baseline to detect a given neutrino mass squared difference.
Diagonalising Majorana Mass Matrices

Let \( M \) be a complex, symmetric matrix.

Then \( \exists U \) \( U^\dagger M U = \text{Diag} > 0 \). \( = U^\dagger M_0 U \)

Call \( M' = \tilde{U} M U \) which is complex symmetric also.

Then \( M' = \tilde{U} M U \).

\[ M' = (R e M' - i d e m M') (R e M' + i d e m M') = \text{Diag} > 0 \]

\[ = (R e M')^2 + (d e m M')^2 + i [R e M', d e m M'] = \lambda > 0 \]

\[ \Rightarrow \ [R e M', d e m M'] = 0 \Rightarrow R e M' = O^T m_0^2 O \quad \text{bilinear} \]

\[ d e m M' = O^T m_0^2 O \]

\[ \Rightarrow m' = O^T (m_0^2 + m_0^2) O = \tilde{U} M U \]

\[ \Rightarrow m = \tilde{U}^T \tilde{O} (m_0^2 + m_0^2) \tilde{O} U \]

\[ m_0^2, m_0^2 = \tilde{U}^T \text{Im}_0 \text{Im}_0 U \quad U' = e^{i \theta} m \]

Note that this is not a similarity transform

\[ \Rightarrow \frac{1}{2} \tilde{O}^T \tilde{O} m_0^2 \phi \psi = \frac{1}{2} \tilde{O}^* \tilde{O} \text{Im}_0 \text{Im}_0 U \psi_\lambda \]

Diagonalise by change of variables \( \psi_\lambda = U^\dagger \psi \).

N.B. to see majorisation because of the fact that produced matrix \( U \) is not a direct sum. \( (\text{Majorisation:\text{bar}}) \).
29.6 Deep Inelastic Lepto-Production
Chapter 30

Quark Confinement

30.1 String Model of Hadrons

30.2 Lattice Gauge Theory
Chapter 31

Physics at High energy and Low Momentum transfer
Chapter 32
Beyond the Standard Model