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String Field Theory[⋆][†]

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ABSTRACT

We present an account of string field theory which aims to introduce the reader to the important concepts of the subject as well as to aid access to the literature. Light cone string theory is briefly reviewed, but the emphasis is on the covariant approach exploiting world sheet BRST invariance to define string field derivations, integration, and products. Although we try to present the ideas in a general way, we draw on Witten's formulation of open string field theory for most of our examples. We include brief self contained reviews of both the operator and Lagrangian versions of the BRST formalism employed in this article. There is a detailed account of perturbation theory for Witten's theory, including a few explicit sample calculations. Open problems faced by open superstring field theory and by closed string field theory are discussed. We sketch some of the ideas for attacking them but do not give a definitive resolution. Finally, some new formulae relevant to the "bosonization of superghosts are derived in a self contained appendix.

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Table of Contents

1. Introduction

Ever since the discovery of anomaly cancellations in superstring theory[1] in 1984, a tremendous amount of energy has been directed toward the development of a satisfactory formulation of the theory. String theory was discovered[2] and initially developed as a perturbation expansion in a coupling constant[3–9] closely resembling the expansion of quantum field theory as a sum of Feynman diagrams. It was therefore natural to suppose that a generalization of field theory, string field theory, might provide the proper framework for string theory.

The search for a satisfactory string field theory has proved to be surprisingly difficult. Soon after the development of the light cone gauge quantization[8,9] of the Nambu-Goto relativistic string,[10,11] second quantization produced the first version of string field theory.[12,13] This formulation had the unsatisfactory feature of not being manifestly Lorentz invariant. Early attempts at a covariant formulation[14] were stalled by technical complications. The discovery of BRST quantization[15–17] of gauge theories provided a way out of the impasse.[18–23] Although, after more than three years of intensive effort toward a gauge invariant formulation, the ultimate goal has not been reached, nonetheless definite progress has been made. This article is an effort to provide an overview of the major advances in string field theory of the last few years. Although it will be impossible to provide a completely self-contained treatise on the subject, I hope that the reader will find here a thorough introduction to the important ideas underlying string field theory and a useful guide to the vast literature.

Before embarking on this task though, it is important to remember that it is not certain that string field theory is the right goal to pursue. Indeed, one could deservedly claim that every concrete new result in string theory over the last two decades has been obtained from the first-quantized sum over world-sheet approach - in spite of its perturbative flavor. This is true for the discovery of new models (notably the spinning string,[24–28] heterotic strings[29] and strings propagating on orbifolds[30]), for consistency checks (no ghost theorems[31–34] and anomaly cancellations[1]), and even for the crucial new ideas which got the modern covariant approach to string field theory off the ground (world sheet BRST invariance).[20,21,35,36] Even more, the world sheet approach has provided insight into the vacuum structure of string theory (Calabi-Yau compactification).[37] In return, string field theory has brought little new insight into the dynamics of string theory. Moreover, there is the general feeling that the passage to a field theory description sacrifices some of the most basic insights, such as duality, provided by the world sheet approach. From this point of view the two dimensional structure of the world sheet is more fundamental than the notion of loop space, and this $2d$ structure might be central in the final formulation of string theory.

And yet it has been very difficult to envisage, within the world sheet approach, a formulation that doesn't rely in principle on the perturbation expansion. It must be regarded as unsatisfactory to *define* a fundamental theory as a power series, especially since this series is at best an asymptotic one.[38] One way to get around this difficulty and retain the world sheet approach is to invoke a world sheet lattice.[39] More recently Banks and Martinec[40] have proposed using the renormalization group to extract string dynamics from such a cutoff world sheet path integral. Some kind of cutoff is probably a necessary step also for advocates of infinite genus and universal moduli space as the fundamental nonperturbative concepts.[41] Save for these ideas, string field theory has been the only proposal to formulate string theory independently of its perturbation expansion. Thus, it would be premature to discard string field theory as an approach. It could well be that a crucial insight is still missing which would remove some of the objections raised in the previous paragraph.

Recent progress in string field theory has relied heavily on the modern BRST invariant definition of the world sheet path integral. Starting with the next chapter we shall work exclusively with this BRST formalism in developing the subject. However, to set the stage for the modern approach, we shall devote the remainder of this introductory chapter to a sketch of the basic string field concepts without introducing the world sheet ghosts. This will include a brief review of the utility of light cone quantization in motivating string field equations. It will become clear that the approach to covariant string field theory without ghosts faces overwhelming obstacles as soon as nonlinear field equations are contemplated.

1.1. The String Field Concept

A classical field is a collection of real numbers associated with each point x^μ of space-time. ($\mu = 0, 1, 2, \cdots, D-1$, where D is the dimension of space time.) A string field generalizes this concept to the association of a collection of real numbers with each curve $x^\mu(\sigma)$ in space-time, where σ labels each point on the curve

$$A[x^\mu(\sigma)].$$

Thus, a string field is a functional rather than a function. Because of this the mathematics needed for string field dynamics is considerably more complicated than that needed for field dynamics.

To get a grip on the description of a string field, it has been useful to consider it as a Schrodinger wave function for a quantum string moving in D dimensions

$$A[x^\mu(\sigma)] \rightarrow \langle x^\mu(\sigma)|\, A\rangle.$$

In this analogy, the bra $\langle x^\mu(\sigma)|$ is viewed as an eigenstate of a quantum string configuration operator $x^\mu(\sigma)$. We think of $|A\rangle$ as a general state for a D dimensional string and so it corresponds to an essentially arbitrary functional $A[x^\mu]$. But since the state space of a D dimensional string is that of a $1+1$ dimensional quantum field theory where the fields are the components of x, the analogy helps us describe string field configurations.

For example, the fact that there are an infinite number of degrees of freedom in a quantum field theory immediately leads to the existence of several inequivalent representations of the canonical commutation relations

$$[x_\mu(\sigma), \frac{\delta}{\delta x^\nu(\sigma')}] = -g_{\mu\nu}\delta(\sigma - \sigma'). \tag{1.1}$$

The quantum field theory of interest here is defined on a finite interval and imposing different boundary conditions leads to different sectors of the state space. Thus a string field defined on closed curves corresponds to a state of a closed string whereas a string field defined on open curves corresponds to a state of an open string. More complex possibilities arise when one considers spinning strings and strings defined on compact manifolds and orbifolds.

In order to work with an infinite number of degrees of freedom, it is useful to specify a concrete basis for the space of string fields by labelling the basis states by the eigenvalues of a suitable positive linear operator built from x^μ and $\frac{\delta}{\delta x^\mu}$. A particularly convenient choice is just the Hamiltonian of the analogue quantum string system:

$$H = \frac{\pi}{2}\int_0^\pi d\sigma(-\frac{\delta^2}{\delta x^2} + (\frac{x'}{\pi})^2). \tag{1.2}$$

The prime denotes differentiation with respect to σ. In this equation the indices of the x^μ are contracted with the metric used in (1.1). Since space-time has an indefinite Minkowski metric ($g_{00} = -1$), H is not strictly a positive operator. However by choosing an indefinite metric on the state space we can consistently interpret H as an operator bounded below within each subspace of definite space-time momentum $p = \int d\sigma \frac{1}{i}\frac{\delta}{\delta x(\sigma)}$. Clearly, H is just the hamiltonian for an infinite collection of harmonic oscillators, the frequencies of which depend on boundary conditions.

To spell this out in more detail, let us work out the eigenstates of H for closed and open strings. In the first case one defines raising and lowering operators by the expansions

$$\frac{1}{i}\frac{\delta}{\delta x_\mu} \pm \frac{x^{\mu\prime}}{\pi} = \frac{2}{\pi}\sum_m \binom{\tilde{\alpha}_m^\mu}{\alpha_m^\mu} e^{\mp 2im\sigma} \tag{1.3}$$

with $\alpha_0^\mu = \tilde{\alpha}_0^\mu = p^\mu/2$. We need to supplement (1.3) with the definition of the zero mode

$$q^\mu = \frac{1}{\pi}\int_0^\pi d\sigma x^\mu. \tag{1.4}$$

Then the commutation rules (1.1) are equivalent to

$$\begin{aligned}
[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] =& m g^{\mu\nu}\delta_{m,-n} \\
[\alpha_m^\mu, \alpha_n^\nu] =& m g^{\mu\nu}\delta_{m,-n} \\
[q^\mu, p^\nu] =& i g^{\mu\nu}
\end{aligned} \tag{1.5}$$

with all other combinations vanishing. According to the last of (1.5) p^μ generates global translations, and so it is indeed the space-time wave vector of the classical string field which is properly interpreted as the energy momentum of a quantum string field. In terms of the α's, H becomes

$$H_{closed} = \frac{p^2}{2} + 2\sum_{m=1}^{\infty}(\tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m + \alpha_{-m}\cdot\alpha_m) + \text{const.} \tag{1.6}$$

The additive constant is just the infinite zero point energy of the oscillators. The eigenbasis of H can then be described by defining for each value of p the "vacuum" ket $|0,p\rangle$ satisfying

$$\tilde{\alpha}_m\,|0,p\rangle = \alpha_m\,|0,p\rangle = (\alpha_0 - \frac{p}{2})\,|0,p\rangle = 0, \quad m > 0. \tag{1.7}$$

Ignoring the constant in (1.6) we see that the spectrum of H is given by

$$2N + \frac{p^2}{2}, \quad N = 0, 1, 2, \ldots \tag{1.8}$$

and is bounded below for each value of p. (Note, however, that because p is a Minkowski D vector p^2 can be arbitrarily negative.) For each value of N and p

the degeneracy is given by the coefficient of q^{2N} in

$$Z_{closed} = \prod_{n=1}^{\infty} (1 - q^{2n})^{-2D} \tag{1.9}$$

and grows exponentially with $\sqrt{N}$.

The corresponding results for open strings are obtained from the mode expansions

$$\frac{1}{i}\frac{\delta}{\delta x_\mu} \pm \frac{x^{\mu\prime}}{\pi} = \frac{1}{\pi}\sum_m \alpha_m^\mu e^{\mp im\sigma}. \tag{1.10}$$

where there is now only one set of modes and $\alpha_0^\mu = p^\mu$. The algebra of the α's is still that in (1.5), but H now simplifies to

$$H_{open} = \frac{p^2}{2} + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \text{const} \tag{1.11}$$

The eigenstates of H_{open} are again oscillator states generated by the single set of operators α_m, and the spectrum is, for the open string case,

$$N + \frac{p^2}{2}, \quad N = 0, 1, 2, \ldots \tag{1.12}$$

For a fixed N and p, the degeneracy of the level (1.12) is given by the coefficient of w^N in

$$Z_{open} = \prod_{n=1}^{\infty} (1 - w^n)^{-D} \tag{1.13}$$

and it also grows exponentially with $\sqrt{N}$.

We have developed this "oscillator" basis for string fields in such detail because we shall use it repeatedly both in the analysis of the structure of the theory and in explicit calculations. In later chapters we shall develop this formalism much further, but at this stage we have used it to begin to understand the nature of string fields. Thus we have illustrated the important role of boundary conditions in differentiating sectors of string field space, and we have noted the rich (Hagedorn) level densities characteristic of quantized strings.

1.2. String Field Dynamics

We have defined string fields as amplitudes associated with curves in space-time, but so far we have been discussing parametrized curves. The parameter σ has no significance beyond labelling a point on the curve: many different functions $x^\mu(\sigma)$ can be used to describe the same curve. We might therefore be tempted to impose a constraint on string fields that they be independent of the chosen parametrization of the curve. Actually, we only need that string fields evaluated on different parametrizations of the same curve be physically equivalent, a constraint, like gauge invariance, that limits the dynamics rather than the kinematics of the field. This more flexible view has been crucial for the recent progress in this subject, but in this chapter we shall pursue the narrower view a bit further to gain some intuition on the significance of the later developments.

We begin by recalling some basic facts about the (first-quantized) Nambu-Goto relativistic string. The fundamental dynamical variables are the functions $x^\mu(\sigma,\tau)$ which describe the world-sheet swept out in space time by an evolving string. The action is proportional to the area of the world sheet. The proportionality costant is T_0, the rest tension of the string. In this article we choose units for which $T_0 = 1/\pi$. This action is invariant under reparametrizations of the world sheet:

$$\begin{aligned} \sigma &\to \tilde{\sigma}(\sigma,\tau) \\ \tau &\to \tilde{\tau}(\sigma,\tau). \end{aligned} \tag{1.14}$$

As a consequence, the phase space of the string is subject to the first-class constraints

$$x' \cdot \mathcal{P} = 0 \tag{1.15}$$

$$\mathcal{P}^2 + (\frac{x'}{\pi})^2 = 0 \tag{1.16}$$

where $\mathcal{P}_\mu$ is the momentum canonically conjugate to x^μ. If we now construct the quantum theory in Schrodinger picture wherein

$$\mathcal{P} \to \frac{1}{i}\frac{\delta}{\delta x}, \tag{1.17}$$

and we choose to identify the Schrodinger wave functional with the string field A, the phase space constraints suggest the candidate set of linear string field equations

$$x'(\sigma) \cdot \frac{\delta}{\delta x(\sigma)} A = \ 0 \tag{1.18}$$

$$\left(-\frac{\delta^2}{\delta x(\sigma)^2} + \frac{x'(\sigma)^2}{\pi^2}\right) A = \ 0. \tag{1.19}$$

The first of these equations simply states that A is invariant under reparametrizations of the curve $x(\sigma)$. As mentioned above this is the strong form of the kinematic requirement that the string field be defined on curves. The second equation, associated with τ reparametrizations on the world-sheet, provides dynamical constraints on the string fields. We shall devote the remainder of this introduction to a heuristic discussion of the physical content of these equations. In subsequent chapters we shall return to the systematic development of the modern gauge invariant formulation of string field theory.

To understand the meaning of the string field equations, let us consider the restriction of A to the particular class of curves satisfying

$$x^+(\sigma) \equiv \frac{1}{\sqrt{2}}(x^0 + x^1) = \tau \tag{1.20}$$

independent of σ. Call this restriction

$$\hat{A}(\tau, x^-, \mathbf{x}),$$

where $x^- = (x^0 - x^1)/\sqrt{2}$ and $\mathbf{x}$ are the "transverse" components of x^μ, i.e. $\mu = 2, \ldots, D-1$. Then the term in (1.18) involving $\frac{\delta}{\delta x^+}$ drops out and that equation reduces to the statement that $\hat{A}$ is invariant under any changes of $(x^-, \mathbf{x})$

produced by reparametrizations. The second field equation (1.19) involves derivatives which take x out of the class (1.20). However, we can use that equation to compute how $\hat{A}$ changes with τ:

$$i\frac{\partial}{\partial\tau}\hat{A} = i\int d\sigma \frac{\delta}{\delta x^+(\sigma)}\hat{A} = \int d\sigma \frac{1}{2}\left(i\frac{\delta}{\delta x^-}\right)^{-1}\left(-\frac{\delta^2}{\delta \mathbf{x}(\sigma)^2} + \frac{\mathbf{x}'(\sigma)^2}{\pi^2}\right)\hat{A}. \quad (1.21)$$

It is clearly useful to Fourier transform the x^- variable

$$\hat{A}(\tau, x^-, \mathbf{x}) = \int \mathcal{D}\mathcal{P}^+(\sigma) \exp\left(-i\int d\sigma x^-(\sigma)\mathcal{P}^+(\sigma)\right)\tilde{A}(\tau, \mathcal{P}^+, \mathbf{x}). \quad (1.22)$$

For the string field $\tilde{A}$, reparametrization invariance says that $\tilde{A}$ is invariant under changes in $(\mathcal{P}^+, \mathbf{x})$ of the form

$$\begin{aligned} \mathcal{P}^+(\sigma) &\to \frac{df}{d\sigma}\mathcal{P}^+(f(\sigma)) \\ \mathbf{x}(\sigma) &\to \mathbf{x}(f(\sigma)). \end{aligned} \quad (1.23)$$

Thus the dependence of $\tilde{A}$ on $\mathcal{P}^+$ can be transferred to a reparametrization of $\mathbf{x}$:

$$\tilde{A}(\tau, \mathcal{P}^+(\sigma), \mathbf{x}(\sigma)) = \tilde{A}(\tau, 1, \mathbf{x}(f(\sigma))) \quad (1.24)$$

where f is defined by

$$\frac{df}{d\sigma}\mathcal{P}^+(f(\sigma)) = 1.$$

Here we must restrict the allowed $\mathcal{P}^+$ to be positive for all σ, since f should be monotonic.

The Fourier transform of the field equation (1.21) can then be written as

$$\begin{aligned} i\frac{\partial}{\partial\tau}\tilde{A} &= \int d\sigma \frac{1}{2\mathcal{P}^+}\left(-\frac{\delta^2}{\delta \mathbf{x}(\sigma)^2} + \frac{\mathbf{x}'(\sigma)^2}{\pi^2}\right)\tilde{A} \\ i\frac{\partial}{\partial\tau}\tilde{A}(\tau, 1, \mathbf{x}_f) &= \int d\sigma_0 \frac{1}{2}\left(-\frac{\delta^2}{\delta \mathbf{x}_f(\sigma_0)^2} + \frac{\mathbf{x}_f'(\sigma_0)^2}{\pi^2}\right)\tilde{A} \end{aligned} \quad (1.25)$$

where in the second equation we have defined $\mathbf{x}_f(\sigma) \equiv \mathbf{x}(f(\sigma))$ and we have changed integration variables to σ_0 given by $\sigma = f(\sigma_0)$. From (1.25) it is clear that the normal mode frequencies (the eigenvalues of $i\frac{\partial}{\partial\tau}$) are independent of $\mathcal{P}^+$, and their values and degeneracies are precisely those of P^-, the "$-$" component of energy momentum for the Nambu-Goto string quantized in light-cone gauge. Notice that had we ignored the requirement of reparametrization invariance, each value of the normal mode frequencies would have an infinite degeneracy labelled by the functions $\mathcal{P}^+$. We have analyzed the *classical* string field equations but we can expect that in the corresponding *quantum* string field theory the normal mode frequencies would be identified with energies (really P^-'s). Thus our heuristic treatment of the string field equations has precisely reproduced the standard quantum relativistic string spectrum[8].

Returning to the field equations (1.18) and (1.19), let us consider how one might generalize them to nonlinear equations which could describe interacting strings. At first sight it would seem reasonable to leave the first of these unmodified.[14] After all, it merely imposes the kinematic constraint that string fields be defined on curves. Thus one might try to add nonlinear terms only to (1.19). At this point the heuristic approach we have been following becomes quite unwieldy. The problem is that the field equations are actually an infinite set of equations - one for each σ - and even at the linearized level the mutual consistency of these equations is a subtle matter. As a matter of fact the first equation together with the integral of the second equation imply the whole set. Also, the cavalier manner in which we transformed them to their light-cone versions was not at all sensitive to the need for the critical dimension. Even apart from this, there are several points of the heuristic discussion that might inspire skepticism, e.g. the procedure of restricting the curves to the class (1.20), and the assumption that $\mathcal{P}^+$ is positive.

In the next chapter we shall remove these objections to the treatment of the linearized field equations by exploiting the elegant BRST method of handling first class constraints. This method involves augmenting the arguments of the string field to include Feynman-Fadeev-Popov ghosts. At the same time one relaxes the requirement of reparametrization invariance, replacing it with the requirement that reparametrizations change the string field by a gauge transformation. The problem of introducing nonlinearities into the field equation is much more tractable in this formulation largely because the infinite set of equations in (1.18) and (1.19) is replaced by a single global equation.

2. Free String Fields

2.1. General Considerations

In the preliminary discussion of the last chapter we motivated the choice of free (i.e. linear) string field equations by taking over the phase space constraints of the Nambu-Goto string. This led to the field equations (1.18) and (1.19). The technical obstacle to finding a suitable nonlinear generalization of these equations is the uncertainty about how to handle the mutual consistency of an infinite number of equations.

The BRST approach to quantizing gauge theories provides an elegant, and probably profound, method of dealing with such equations. In the general formulation[42–44] of this approach one starts with an arbitrary system of first class constraints

$$\phi_a \left|phys\right\rangle = 0. \tag{2.1}$$

The requirement that these be first class is that commuting them among themselves and with all observables Ω does not lead to new constraints:

$$\begin{aligned} [\phi_a, \phi_b] &= f^c_{ab}\phi_c \\ [\phi_a, \Omega] &= \omega^b_a \phi_b. \end{aligned} \tag{2.2}$$

In (2.2) the f and ω might be operators, but in that case it is essential that the ϕ appear on the right. It is not guaranteed that a given system of constraints for a classical system will satisfy this requirement upon quantization. Indeed the constraints of first quantized string theory generically produce a nonvanishing constant term on the right hand side of (2.2).

Putting aside such subtleties for the moment we pursue the BRST method by introducing for each constraint a pair of canonically conjugate fermionic Fadeev-

Popov ghost fields c^a and b_b satisfying

$$\{c^a, b_b\} = \delta^a_b \tag{2.3}$$

with all other anticommutators vanishing. Then one defines the BRST operator

$$Q = c^a \phi_a - \frac{1}{2} f^c_{ab} c^a c^b b_c. \tag{2.4}$$

In this discussion we simplify matters by assuming the f and ω of (2.2) are c-numbers. (The general situation has been thoroughly analyzed[42–44] in the literature but will not be needed in this chapter.) Then it is not difficult to check that the algebra (2.2), including the Jacobi identity on f, implies that Q is nilpotent

$$Q^2 = 0. \tag{2.5}$$

The converse is also true. Thus the mutual consistency of the first class constraints is summarized in the extremely compact (2.5). The subtleties introduced by operator ordering can cause a violation of (2.5) so it is important to check this property for each system.

Observables must satisfy the second of the equations (2.2) and we can use it to construct a modified observable which commutes with Q

$$\hat{\Omega} = \Omega + \omega^b_a c^a b_b \tag{2.6}$$

so that

$$[Q, \hat{\Omega}] = 0 \tag{2.7}$$

In particular the operators ϕ_a themselves can be so modified

$$\hat{\phi}_a = \phi_a - f^c_{ab} c^b b_c. \tag{2.8}$$

We note that the $\hat{\phi}_a$ belong to the class of operators which are (anti)commutators of Q with another operator:

$$\hat{\phi} = \{b^a, Q\}. \tag{2.9}$$

Such operators are trivial in the sense that any monomial, formed from at least one of them and any number of BRST invariant operators, has vanishing matrix elements between states in the kernel of Q.

We have now encoded all of the consistency conditions in the elegant statements (2.5) and (2.7). The final step of the method is to translate (2.1) into the BRST language. We notice that on states annihilated by all the b's these conditions are equivalent to

$$Q\,|phys\rangle = 0. \tag{2.10}$$

At first glance it seems that this condition would be much weaker than (2.1) since there are a large number of states not annihilated by the b's. It is certainly true that there are many more solutions to (2.10) than to (2.1).

But the nilpotency of Q saves the day. Any state of the form

$$Q\,|\Lambda\rangle \tag{2.11}$$

automatically satisfies (2.10) and, moreover, any such state is null in the subspace of states satisfying (2.10). Thus it is consistent to associate physical states with states in the kernel of Q modulo the range of Q. So in addition to the condition (2.10) we postulate the equivalence

$$|phys\rangle \equiv |phys\rangle + Q\,|\Lambda\rangle\,. \tag{2.12}$$

This equivalence reduces the size of the solution space of (2.10) enormously. The detailed question of whether the physical space so defined is isomorphic to the solution space of (2.1) is difficult to answer in general, considering that the conditions in the form (2.1) are frequently problematic when there are an infinite number of constraints. For now we simply note that applying $\hat{\phi}_a$ to a state annihilated by Q yields a state of the form (2.11):

$$\hat{\phi}_a \left|phys\right\rangle = Qb_a \left|phys\right\rangle . \tag{2.13}$$

It is in this sense that the BRST method relaxes the constraints of the original system; now the constraints no longer annihilate physical states but rather map them into "pure gauge" states which are equivalent to zero.

The BRST formalism has also been generalized to include fermionic constraints in addition to the bosonic constraints discussed above.[44,43] This generalization will be used when we come to the spinning string, but we will not present it in full generality. Finally, there are cases where the operator formalism either is awkward to deal with or (as in the case of interacting string field theory) is as yet undeveloped. For such systems a Lagrangian approach to the BRST method is desirable, and there is an extensive literature on this topic.[45,46] We shall return to it when we treat the problem of gauge-fixing string field theory.

The BRST approach to constrained systems is tailor made for the construction of free string field theory. Once we apply the formalism to the field equations in (1.18) and (1.19), we will be able to replace those infinite sets of equations with the single equation (2.10), where the equivalence (2.12) is understood. The price for this simplification is that the string field will be a function of more variables, e.g. the c_a's. In this form string field theory bears a striking resemblance to gauge field theories as was first appreciated by Witten.[47] The nilpotency of Q is analogous to that of the exterior derivative. The substitution

$$\left|A\right\rangle \to \left|A\right\rangle + Q\left|\Lambda\right\rangle \tag{2.14}$$

can be interpreted as a gauge transformation, and the equivalence (2.12) is just gauge equivalence. The problem of finding the physical states is that of computing the cohomology of Q. Guided by this analogy Witten has constructed a nonlinear open string field theory which will be the subject of Chapter 3.

In this chapter we shall give a systematic development of free string field theory for the various known string theories. For each case we shall describe the BRST construction and sketch the various checks that are needed. In particular we shall discuss the cohomology of Q and show how the spectrum of the light-cone quantized string is reproduced. For later discussions of the interacting theory we shall need the "bosonized" description of the various ghost fields that arise. The development of this alternative formalism is presented in Appendix A.

2.2. Free Bosonic Open String Field Theory: A Useful Paradigm

Although bosonic strings are probably unphysical, it is useful to consider them first, both because of their simplicity and because the same variables and analysis reappear in the more physically realistic models. The basic concepts of string field theory will be developed in great detail for the open bosonic string. We will then find that most of the analysis for the other string models is so similar that a sketch of the new features will suffice.

The bosonic string variables are simply the coordinates of space-time $x^\mu(\sigma)$. In accordance with the BRST method, we should add a ghost field c and its conjugate b for each of the differential operators appearing in the field equations (1.18) and (1.19). It is convenient to describe these operators by their normal modes as determined by the boundary conditions appropriate to the sector of interest. For the bosonic string there are two such sectors corresponding to open and closed strings. In this section we concentrate on the open string sector.

Mode Expansions, Ghost Number and the BRST Operator.

The mode expansions of x and $\delta/\delta x$ are given in (1.10) which determines the expansions

$$\left(\frac{1}{i}\frac{\delta}{\delta x} \pm \frac{x'}{\pi}\right)^2 = \frac{2}{\pi^2}\sum_m L_m e^{\mp im\sigma} + const \tag{2.15}$$

where the Virasoro generators[48] are defined by

$$L_m = \frac{1}{2}\sum_k : \alpha_{-k}\cdot\alpha_{k+m} : \tag{2.16}$$

In (2.16) the colons indicate normal ordering with positive-moded operators standing to the right. The algebra of the L's is given by[49,50]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m,-n}. \tag{2.17}$$

Thus we should introduce the ghosts $c^m = c_{-m}$ for each integer m and the conjugate antighosts b_m satisfying

$$\{c_m, b_n\} = \delta_{m,-n}. \tag{2.18}$$

It will be important in the development of string field theory to define the concept of ghost number. In general, each class of ghost fields will have its own ghost number but only the total will be conserved. For the open bosonic string the only ghost number operator is

$$G^b = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{k=1}^{\infty}(c_{-k}b_k - b_{-k}c_k). \tag{2.19}$$

Next we construct the BRST operator

$$Q^{open} = \sum_m c_{-m}L_m - \frac{1}{2}\sum_{m,n}(m-n) : c_{-m}c_{-n}b_{m+n} : -c_0 \tag{2.20}$$

where we have again normal ordered. The last term in (2.20) is needed for nilpotency. As shown by Kato and Ogawa[35] Q^{open} is nilpotent if and only if the coefficient of this term is as written and D = 26. To show this one must be careful with the ordering of operators. The safest procedure is to apply Q

twice to oscillator states with finite occupation number. For example, it is easy to evaluate

$$\begin{aligned}
b_n Q^2 b_{-n} \left|0,p\right\rangle = b_n Q(\hat{L}_{-n} - b_{-n}c_0(\frac{p^2}{2}-1)) \left|0,p\right\rangle \\
=&\hat{L}_n(\hat{L}_{-n} - b_{-n}c_0(\frac{p^2}{2}-1)) \left|0,p\right\rangle \\
=&(L_nL_{-n} + L_n^{gh}L_{-n}^{gh}) \left|0,p\right\rangle - 2n(\frac{p^2}{2}-1) \left|0,p\right\rangle \\
=&(np^2 + \frac{D}{12}(n^3-n) - 2n - \frac{26}{12}(n^3-n)) \left|0,p\right\rangle - (np^2-2n) \left|0,p\right\rangle \\
=&\frac{D-26}{12}(n^3-n) \left|0,p\right\rangle
\end{aligned}$$

where the ket $|0,p\rangle$ is annihilated by b_0 and $\alpha_0 - p$ and by α_n, b_n, and c_n for $n > 0$. We have also anticipated the definition of $\hat{L}_n$ and L_n^{gh} below. Actually, this simple calculation establishes the result. This is because Q^2, being an anticommutator of two cubic fermionic operators, is at most quartic in the oscillator variables. The general discussion guarantees the vanishing of the quartic terms, so any potential anomaly resides in the quadratic terms. But since Q^2 has ghost number 2 and commutes with $\hat{L}_0$, it must then be of the form

$$Q^2 = \sum_{k=1}^{\infty} \xi_k c_{-k} c_k$$

and the above calculation determines the ξ_k.

In our general discussion the nilpotency condition was equivalent to the consistency condition for the constraints. The constraint algebra written in (2.17) appears inconsistent for any D due to the constant term. The BRST method has therefore given a consistent interpretation to the problematic naive constraint algebra. Indeed when we construct $\hat{L}_m$ according to

$$\hat{L}_m = \{b_m, Q^{open}\}, \tag{2.21}$$

we find that the corresponding Virasoro algebra has a vanishing constant term provided $D = 26$. The anomaly in (2.17) is cancelled by a similar anomaly from the ghosts. For later convenience we quote here the ghost part of $\hat{L}$:

$$L_m^{gh} = -\sum_k (m-k) : c_{-k} b_{m+k} : -\delta_{m,0} \tag{2.22}$$

<u>The Free Open String Field Equation and its Solution.</u>

The open bosonic string field is a general state in the Fock space of all the oscillators (α, b, c). In Witten's scheme the classical string field is further restricted to be a state with ghost number $G = -1/2$. However in the process of quantizing this theory the other ghost number sectors are reintroduced playing the role of string field FFP ghosts, so it is important to keep all ghost number sectors in the formalism. A basis for this Fock space is conveniently labelled by eigenvalues of $\hat{L}_0$

$$\hat{L}_0 = \frac{p^2}{2} - 1 + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{m=1}^{\infty} m(c_{-m} b_m + b_{-m} c_m). \tag{2.23}$$

The ground state ket $|0,p\rangle$ is annihilated by the positive moded operators together with b_0 and is an eigenstate of α_0 with eigenvalue p. It clearly has the minimum eigenvalue of $\hat{L}_0$ for fixed p and has ghost number $-\frac{1}{2}$. It is degenerate with the state $c_0 |0,p\rangle$ which has ghost number $+\frac{1}{2}$. A general string field is obtained by applying an arbitrary function of the negative moded operators and c_0 to $|0,p\rangle$.

The free open bosonic string field equation is simply

$$Q^{open} |A\rangle = 0. \tag{2.24}$$

We assume that $D = 26$ so that (2.24) possesses the gauge invariance

$$\Delta |A\rangle = Q^{open} |\Lambda\rangle . \tag{2.25}$$

Our next task is to solve these equations and show that the most general solution is gauge equivalent to a physical state of the dual string model. This will confirm that the string field theory developed in this chapter reproduces the results of the heuristic discussion of the last chapter. But now it is clear why $D = 26$ is crucial. Furthermore, the other objections raised in that discussion do not apply to the present treatment.

A number of authors[35,51–53] have shown that the general solution to (2.24) with $p \neq 0$ is gauge equivalent to the DDF states[54] with no ghost excitation. These states are annihilated by $L_0 - 1$, c_n, b_n, and L_n for $n > 0$, and, in addition, are annihilated by $k \cdot \alpha_n$ for $n > 0$ where k is a fixed light-like 26-vector such that $k \cdot p \neq 0$. They are in one-to-one correspondence with the states of the first quantized relativistic string in light-cone gauge.[55,8] We shall outline the argument below.

We first deal with the special case $p = 0$ not covered by the general analysis. Because $\hat{L}_0$ commutes with Q we can work in eigenspaces of definite $\hat{L}_0$. Applying b_0 to (2.24) leads to

$$\hat{L}_0 \left|A\right\rangle = Q b_0 \left|A\right\rangle \tag{2.26}$$

from which it is clear that solutions of the field equation are pure gauge if they lie in a non-vanishing eigenspace of $\hat{L}_0$. Since this means that the only non-trivial solutions of (2.24) have $\hat{L}_0 = 0$, we shall refer to states annihilated by $\hat{L}_0$ as being on-shell. If $p = 0$, the vanishing eigenspace of $\hat{L}_0$ is spanned by the $2D + 4$ states of mode number 1. Of these the $2D + 2$ states

$$c_{-1} c_0 \left|0,0\right\rangle \quad \alpha^{\mu}_{-1} c_0 \left|0,0\right\rangle \quad \alpha^{\mu}_{-1} \left|0,0\right\rangle \quad b_{-1} \left|0,0\right\rangle \tag{2.27}$$

satisfy (2.24). Of these, the first two may be regarded as the $p \to 0$ limit of trivial states, and as we shall see, with a suitable interpretation are trivial. We shall return to the interpretation of the remaining states which are truly nontrivial after discussing the $p \neq 0$ solutions. The two other states of unit mode number are $c_0 b_{-1} \left|0,0\right\rangle$ which is not a solution and $c_{-1} \left|0,0\right\rangle = \frac{1}{2} Q c_0 b_{-1} \left|0,0\right\rangle$ which is trivial.

Now we sketch the analysis of the open string field equation for $p \neq 0$. We first note that any state annihilated by $\hat{L}_0$, L_n, c_n, and b_n for all $n > 0$ satisfies (2.24). Since these states contain no ghost excitations, they are isomorphic to the "physical states" of the dual resonance model. As is well-known from the analysis of the no-ghost theorem,[31,32] for $D = 26$ this space of physical states contains a subspace of null states spanned[53] by the states

$$\begin{aligned} L_{-1}\,|phys,0\rangle &= Qb_{-1}\,|phys,0\rangle \\ (L_{-2}+\frac{3}{2}L_{-1}^2)\,|phys,-1\rangle &= Q(b_{-2}+\frac{3}{2}L_{-1}b_{-1})\,|phys,-1\rangle\,. \end{aligned} \tag{2.28}$$

The states on the left hand side of these equations contain no ghost excitations. The kets in (2.28) are annihilated by L_n, c_n, and b_n for all $n > 0$ but are eigenstates of L_0 with values 0 and -1 respectively. The equalities show that these null states are also trivial in the BRST language. In the proof of the No-ghost theorem in Ref.[32] (see also Ref.[53]), one shows that any physical state with $p \neq 0$ (for $D = 26$) is a DDF state plus one of these null states. Here we would like to extend this to the statement that any solution of (2.24) with $p \neq 0$ is gauge equivalent to a DDF state.

For fixed $p \neq 0$ select a light-like vector k such that $k \cdot p = 1$ define $K_n = k \cdot \alpha_n$ and define DDF states by

$$(b_n, c_n, K_n, L_n)\,|DDF\rangle = 0, \quad n > 0. \tag{2.29}$$

Notice that we do not require that they be on-shell and that due to the doubly degenerate ghost vacuum there are two copies of the dual model DDF states with ghost numbers $\pm\frac{1}{2}$. The ones with ghost number $+\frac{1}{2}$ are solutions of (2.24) for all p^2 and can be regarded as the limit $\hat{L}_0 \to 0$ of trivial states. Those with ghost number $-\frac{1}{2}$ are only solutions on-shell and are genuinely nontrivial.

To study the solutions of (2.24) we introduce the following linearly independent basis[53] of the full state space

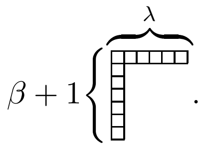
$$\hat{L}_{-\{\ell_1} \cdots \hat{L}_{-\ell_\lambda\}} b_{-[b_1} \cdots b_{-b_\beta]} K_{-\{k_1} \cdots K_{-k_\kappa\}} c_{-[c_1} \cdots c_{-c_\gamma]} |DDF\rangle \tag{2.30}$$

where { }, [ ] denote respectively complete symmetrization, complete antisymmetrization of the enclosed indices. The first stage of the argument is to reduce the size of this basis by a gauge transformation. For this purpose consider the (anti) commutator bracket

$$\begin{aligned}
\Big[Q, \hat{L}_{-\{\ell_1} \hat{L}_{-\ell_2} & \cdots \hat{L}_{-\ell_{\lambda-1}\}} b_{-[b_1} b_{-b_2} \cdots b_{-b_{\beta+1}]}\Big]_{\pm} \\
&= \sum_k (-)^{k-1} \hat{L}_{-\{\ell_1} \hat{L}_{-\ell_2} \cdots \hat{L}_{-\ell_{\lambda-1}\}} b_{-[b_1} \cdots \hat{L}_{-b_k} \cdots b_{-b_{\beta+1}]} \\
&= \sum_k (-)^{k-1} \hat{L}_{-\{\ell_1} \cdots \hat{L}_{-\ell_{\lambda-1}} \ \hat{L}_{-b_k\}} b_{-[b_1} \cdots \hat{b}_{-b_k} \cdots b_{-b_{\beta+1}]} \\
&\quad + \text{other terms with } \beta \ b'\text{s and less than } \lambda \ \hat{L}'\text{s.}
\end{aligned} \tag{2.31}$$

The bracket is an (anti)commutator if the number of b's is (odd) even. The hat over the b in (2.31) signifies that b_{-b_k} is absent.

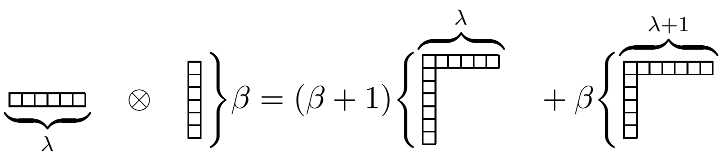
The sum on the RHS of (2.31) can be recognized as the Young (anti) symmetrizer for the tableaux



(2.32)

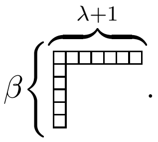
If we now apply (2.31) to a state in the subspace $\mathcal{K}$ spanned by the states in (2.30) with no $\hat{L}$'s or b's, we learn that the linear combinations of states in (2.30) with λ $\hat{L}$'s and with the symmetry (2.32) can be expressed as a pure gauge plus states with less than λ $\hat{L}$'s. Since we are working in an eigenspace of $\hat{L}_0$ with fixed p there is a maximum possible value of λ. Working recursively downward in λ we can systematically remove all linear combinations with symmetry (2.32).

The tableaux identity (see *e.g.* Ref.[56], Section 7-12)



(2.33)

then enables us to conclude that what remains in (2.30) are the linear combinations with the symmetry



(2.34)

At this point we consider (2.24) with $|A\rangle$ a linear combination of the states (2.30) where the ℓ's and b's are symmetrized according to (2.34) and use (2.31) to conclude that

$$\begin{aligned} Q \sum_{\{\ell,b\}} \left(\hat{L}^{\lambda}_{-\{\ell\}} b^{\beta}_{-\{b\}}\right)_{\{\lambda+1,1^{\beta-1}\}} &\left|\phi, \{\lambda+1, 1^{\beta-1}\}\right\rangle \\ = \sum \beta \left(\hat{L}^{\lambda+1}_{-\{\ell\}} b^{\beta-1}_{-\{b\}}\right)_{\{\lambda+1,1^{\beta-1}\}} &\left|\phi, \{\lambda+1, 1^{\beta-1}\}\right\rangle \\ + \sum \left(\hat{L}^{\lambda}_{-\{\ell\}} b^{\beta}_{-\{b\}}\right)_{\{\lambda+1,1^{\beta-1}\}} &\left|\phi', \{\lambda+1, 1^{\beta-1}\}\right\rangle \\ + \sum_{\lambda'<\lambda} \left(\hat{L}^{\lambda'+1}_{-\{\ell\}} b^{\beta-1}_{-\{b\}}\right) &\left|\phi, \{\lambda+1, 1^{\beta-1}\}\right\rangle = 0 \end{aligned} \tag{2.35}$$

where the superscripts indicate the number of $\hat{L}$'s and b's in the polynomials in parentheses. Also $|\phi, \{\lambda+1, 1^{\beta-1}\}\rangle$, $|\phi', \{\lambda+1, 1^{\beta-1}\}\rangle$ belong to $\mathcal{K}$. In (2.35), we have denoted a tableau with λ boxes in the first row and one box in each of the next β rows by the symbol $\{\lambda, 1^{\beta}\}$. Now consider how the RHS of (2.35) can vanish. For each β the terms in the first sum with maximal λ must cancel among themselves. This implies the $|\phi, \{\lambda+1, 1^{\beta-1}\}\rangle$ with λ maximal all vanish and can be dropped from the first and third sums, leaving a smaller maximal

λ. Then these $|\phi, \{\lambda+1, 1^{\beta-1}\}\rangle$ must all vanish and so on. The terms in the middle sum can never cancel any of these terms because they have the wrong symmetry. Thus the RHS of (2.35) can only vanish if all the $|\phi, \{\lambda+1, 1^{\beta-1}\}\rangle$ with $(\beta, \lambda) \neq (0,0)$ vanish. Thus we conclude that

$$Q|A\rangle = 0 \Rightarrow |A\rangle = |\phi\rangle + Q|\Lambda\rangle \tag{2.36}$$

where $|\phi\rangle$ is in $\mathcal{K}$.

The argument must now be completed[53] by following a completely parallel procedure for the states in $\mathcal{K}$. In this phase the K's play the role of the b's and the c's that of the $\hat{L}$'s. One reduces the basis of $\mathcal{K}$ to polynomials with κ K's and γ c's symmetrized according to

$$\gamma+1\left\{\begin{array}{l}\overbrace{\square\square\square\square\square\square}^{\kappa}\\ \square\\ \square\\ \square\\ \square\\ \square\\ \square\end{array}\right. \tag{2.37}$$

applied to DDF states. On this reduced basis it then follows as before that

$$Q\,|A\rangle = 0 \Rightarrow |A\rangle = |DDF\rangle + Q\,|\Lambda\rangle\,. \tag{2.38}$$

Interpretation of Solutions to the Field Equation.

Having completed the mathematical analysis of the field equations, we next turn to the interpretation of their solutions. For $p \neq 0$, we have reduced the solution space to two classes of DDF states,

$$|DDF, \downarrow\rangle\,, \qquad G = -\frac{1}{2} \tag{2.39}$$

and

$$|DDF, \uparrow\rangle\,, \qquad G = +\frac{1}{2}. \tag{2.40}$$

Those with $G = +\frac{1}{2}$ are annihilated by c_0 and for this reason satisfy (2.24) for all values of p. Because of (2.26) these states are pure gauges off-shell and by continuity their on-shell limits should also be regarded as pure gauges.

On the other hand, the DDF states with $G = -\frac{1}{2}$, annihilated by b_0, are solutions of the field equations only on-shell and represent the true nontrivial cohomology of Q for $p \neq 0$. It is these states that we identify with the physical field configurations, so there is a one-to-one correspondence between $p \neq 0$ physical field configurations and the P^- eigenstates of the light-cone first-quantized string. Here we presume the usual identification of wave-number p of classical string field theory with space-time momentum P in quantum string field theory.

The DDF states with $G = +\frac{1}{2}$ are very important for the construction of gauge invariant vertex functions. This is because with the conventional hermitian inner product on the BRST state space,

$$\begin{aligned}(\alpha_n)^\dagger &= \alpha_{-n} \\ (c_n)^\dagger &= c_{-n} \\ (b_n)^\dagger &= b_{-n}\end{aligned} \tag{2.41}$$

a state of definite ghost number has zero norm. The states $\langle DDF, \uparrow|$ have non-zero inner product with the states $|DDF, \downarrow\rangle$. To construct gauge invariant vertex functions we must find states $\langle V|$ in the dual to the string field space with the property

$$\langle V|\, Q\, |\Lambda\rangle = 0 \tag{2.42}$$

for arbitrary gauge functions Λ. Thus we require

$$\langle V|\, Q = 0 \tag{2.43}$$

and this for all values of p. It had better also be true that $\langle V|DDF, \downarrow\rangle \neq 0$, and we see that the states $\langle DDF, \uparrow|$ are precisely singled out by these two requirements. In fact, the vertex bra must not contain the DDF states $\langle DDF, \downarrow|$ because of the requirement (2.43) which must be true off shell.

As we have seen the states of the BRST first quantized formalism play two distinct roles in the application to string field theory. These two roles are neatly separated by distinguishing the state space (kets) from its dual space (bras). In this article string fields will always be associated with kets, and vertex functions with bras. Then (2.24) is a string field equation. In spite of its similarity to (2.24), (2.43) is not a field equation but, rather, the requirement that the vertex be gauge invariant. This asymmetric distinction between kets and bras will be maintained throughout this article.

From this discussion we see that regarded as string fields, the kets $|DDF, \uparrow\rangle$ are indeed trivial, since even on-shell they decouple from gauge invariant vertex functions. We stress that this conclusion depends on the continuity and off-shell gauge invariance of vertex functions in a neighborhood of $\hat{L}_0 = 0$. Continuity is meaningful only if p^2 is continuous so that $\hat{L}_0$ has continuous eigenvalues. If space-time were compact, p would be discrete and the $\uparrow$ states might no longer be trivial[57,53].

Returning to the solutions of the field equations for $p = 0$, (2.27), we can now interpret them. The second and third entries have ghost number $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. The transverse components of these can be regarded as $p \to 0$ limits of up and down DDF states respectively. The $\pm$ components are extra solutions for $p = 0$. The presence of these extra solutions is completely understandable. The states at this level correspond to a massless vector particle. The $p = 0$ component of the vector potential is just a constant which can be gauged away with a gauge function linear in x. Such a gauge function would be a derivative of a delta function in momentum space, and we have not included such gauge functions in our analysis. The first and last entries of (2.27) do not correspond in this way to particle states. The ghost number $-\frac{3}{2}$ state $b_{-1}|0,0\rangle$ is just the $SO(2,1)$ invariant vacuum of the BRST first quantized string theory

$$\hat{L}_{\pm 1} b_{-1}|0,0\rangle = \hat{L}_0 b_{-1}|0,0\rangle = 0. \tag{2.44}$$

As already mentioned the ghost number $+\frac{3}{2}$ state $c_{-1}c_0|0,0\rangle$ is the limit of a trivial state viewed as a string field, but, of course the bra $\langle 0,0| c_0 c_1$ is a nontrivial element of the dual space.

The vertex function we discussed above involved only one off shell string. It could, for example, describe an amplitude involving several on-shell physical strings and a single off shell string. The concept is easily generalized to the case of several off-shell strings. A multi-string state is a ket in the tensor product of the ket spaces of the individual strings. The BRST operator for this multi-string space is simply the sum of those of the component single string states and it will be nilpotent with the convention that Grassmann odd operators acting in different factor spaces anticommute with one another. A sufficient requirement to insure that only the nontrivial cohomology of Q, *e.g.* the DDF states, couples to the multi-string vertex functions is simply

$$\langle V_N| \sum_{i=1}^{N} Q_i = 0. \tag{2.45}$$

This is because the many string physical state condition

$$\sum_{i=1}^{N} Q_i \, |N \ strings\rangle = 0 \tag{2.46}$$

implies (by applying the above argument to each string in turn) that the N string ket is a multi-DDF state plus $\sum Q_i \,|\Lambda\rangle$, so (2.45) is sufficient to guarantee that only the DDF components of physical string states couple. In the nonlinear string field theory discussed in later chapters the identities (2.45) should be a consequence of the Ward-Takahashi-Slavnov-Taylor identities for N string vertex functions. These in turn are a consequence of string field theoretic BRST invariance.

The form of our postulated Ward-like identities (2.45) is somewhat simpler than the form of those for ordinary nonabelian gauge theories. All terms involve the derivation operator, and there are no terms involving vertex functions for a smaller number of strings, as in the QED identity which relates the longitudinal part of the vertex to the self energy. This is possible because the single condition $Q\,|A\rangle = 0$ embodies both the physical state condition *and* the equations of motion. This in turn can be traced to the fact that the action is a Chern-Simons three form. Although the string field theoretic BRST invariance to be discussed in Chapter 4 implies the decoupling of nonlinear functions of the fields, there are indications, at least from low orders of perturbation theory, that the nonlinear terms (which would lead to lower string functions in (2.45)) are precisely cancelled by the anticommutator of Q with the gauge-fixed propagator when the Q is moved from its position acting on the field past the external propagator to act on the (amputated) vertex. If this feature can be shown to hold generally, an explicit understanding of the relation between string field theoretic gauge invariance and world sheet BRST invariance will be gained.

The String Field Action Principle.

We now turn to the problem of finding a gauge invariant action principle which implies the free string field equations. It is tempting to try to take the action as simply $\langle A| Q |A\rangle$[58,59]. This is close to what we want, except it violates the notational conventions established in the previous paragraphs, and it confuses the generalization to the nonlinear case. We seek to express the action in the form

$$S = \frac{1}{2} \langle \mathcal{V}_2| A\rangle Q |A\rangle \tag{2.47}$$

where $\langle \mathcal{V}_2|$ is a suitable vertex for the two string state space. We would like to choose this vertex so that, for certain "real" B, $\langle \mathcal{V}_2| B\rangle |A\rangle = \langle B| A\rangle$. Then for A "real" S would be guaranteed to be real and of the physical sign for physical DDF fields. In fact it would reduce precisely to the light-cone string field theory action of Ref.[12,13].

In addition a reality condition is necessary so that, without Chan-Paton factors, the tachyon corresponds to a real scalar field. Because complex conjugating a field generally corresponds to charge conjugation, it is reasonable to impose

$$A(x(\sigma))^* = A(x(\pi - \sigma)) \tag{2.48}$$

since charge conjugation should reverse the orientation of a string. This condition translates the statement $\langle \mathcal{V}_2 | \, B \rangle \, |A\rangle = \langle B | \, A \rangle$ to the condition

$$\langle \mathcal{V}_2 | \, (\alpha^1_{-n} + (-)^n \alpha^2_n) = 0 \tag{2.49}$$

where the superscripts label the oscillators for the two strings. Since we also want

$$\langle \mathcal{V}_2 | \, (Q^1 + Q^2) = 0, \tag{2.50}$$

the ghost dependence of $\langle \mathcal{V}_2 |$ should satisfy

$$\begin{aligned} \langle \mathcal{V}_2 | \, (c^1_{-n} + (-)^n c^2_n) = 0 \\ \langle \mathcal{V}_2 | \, (b^1_{-n} - (-)^n b^2_n) = 0 \end{aligned} . \tag{2.51}$$

In the first of these equations the + sign is dictated by the + sign in (2.50). Then the − sign in the overlap condition on the b's is dictated by consistency of the pair of equations with the b, c anticommutation relations. Also, the signs in (2.49) and (2.51) are correlated with the world sheet CP properties of $\mathcal{P}^\mu$, c, and b. All these conditions uniquely determine the vertex

$$\langle \mathcal{V}_2 | = \int dp \, \langle 0, p |_1 \, \langle 0, -p |_2 \, (c^1_0 + c^2_0) \exp(-\sum_{n=1}^{\infty} (-)^n (\alpha^1_n \cdot \alpha^2_n + c^2_n b^1_n + c^1_n b^2_n)). \tag{2.52}$$

We adopt the normalization convention for vacuum states implied by

$$\langle \mathcal{V}_2 | 0, p \rangle_1 \, c_0 \, |0, p' \rangle_2 \equiv \delta(p + p'). \tag{2.53}$$

In the expression for the free action (2.47) we have two string field concepts rolled into one. First of all there is the operation of "multiplying" two string fields, i.e. a rule for turning two string fields into one

$$A, B \rightarrow A * B. \tag{2.54}$$

In this case the two string fields are A and QA. Then there is the operation of "integration," i.e. a rule for turning a string field into a number

$$A \to \int A. \tag{2.55}$$

Hence the free action can be written symbolically as

$$S = \frac{1}{2} \int A * QA. \tag{2.56}$$

One cannot infer from the final expression for the free action how to define these individual operations, but as Witten has emphasized,[47] their definition is the key to generalizing the linear gauge principle to a nonlinear one. The detailed elaboration of Witten's construction will be deferred until Chapter 3. Here we only mention some simple consequences of ghost number counting that guide the construction.

Inspection of (2.47) reveals that the ghost number $-\frac{1}{2}$ component of a string field is the only one that has a diagonal contribution to S. This is simply because twice the ghost number of such a component must cancel the unit ghost number of Q. Thus one can consistently restrict the classical field to carry ghost number $-\frac{1}{2}$, and this choice turns out to be correct. It relies on ghost number conservation. Then the nonlinear generalization of the linear gauge transformation

$$\Delta A = Q\Lambda + A * \Lambda - \Lambda * A, \tag{2.57}$$

which Witten postulates in analogy with Yang-Mills gauge invariance, requires that $*$ carries ghost number $+\frac{3}{2}$. It then follows that $\int$ carries ghost number $-\frac{3}{2}$. These properties of $*$ and $\int$ will be crucial for the equivalence of this version of string field theory to the dual resonance model.

2.3. Other Free String Field Theories

Since we have developed in great detail the general principles for constructing linear string field equations in our application of the BRST formalism to the bosonic open string field, the extension of the formalism to other string theories is quite straightforward.[60–62] The discovery of an action principle leading to these field equations is quite another matter, as we shall see. Indeed for closed strings a completely satisfactory free action has not yet been found. Here we shall present the BRST formalism needed for open spinning strings (including the superstring) as well as for closed strings. This then immediately yields the string field equations. We shall discuss the solutions of these equations and their interpretation. Then we shall close with a discussion of the problems associated with constructing an appropriate action.

BRST Formalism for the Open Spinning String

The spinning string has spin degrees of freedom in addition to the space-time coordinates $x^\mu(\sigma)$. There are two approaches to the description of these extra degrees of freedom. In the formulation developed by Ramond, Neveu and Schwarz, and Thorn[24–28] (NSR), they are represented by a Grassmann odd world-sheet spinor field $\psi^\mu_a(\sigma)$, carrying a space-time vector index μ and a world sheet spinor index $a = 1, 2$. In this formulation the Virasoro Algebra is enlarged to a graded extension[24]

$$
\begin{aligned}
[L^R_m, L^R_n] &= (m-n)L^R_{m+n} + \frac{D}{8}(m^3 - m)\delta_{m,-n} \\
[L^R_m, F_n] &= (\frac{m}{2} - n)F_{m+n} \\
\{F_m, F_n\} &= 2L^R_{m+n} + \frac{D}{2}(m^2 - \frac{1}{4})\delta_{m,-n}.
\end{aligned}
\tag{2.58}
$$

Here we have displayed the algebra for the fermion (Ramond) sector for which the mode numbers of the F's are integers. For the boson sector (Neveu-Schwarz) the F's are replaced by G_r where r is a half odd integer, but the algebra is identical[25]:

$$\begin{aligned}
[L_m^{NS}, L_n^{NS}] &= (m-n)L_{m+n}^{NS} + \frac{D}{8}(m^3 - m)\delta_{m,-n} \\
[L_m^{NS}, G_r] &= (\frac{m}{2} - r)G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s}^{NS} + \frac{D}{2}(r^2 - \frac{1}{4})\delta_{r,-s}.
\end{aligned} \tag{2.59}$$

The L's in (2.58) and (2.59) include, of course, a contribution from the spin degrees of freedom. The explicit formulas for the super Virasoro generators are

$$\begin{aligned}
F_n &= \sum_k \alpha_{-k} \cdot \psi_{k+n} \\
G_r &= \sum_k \alpha_{-k} \cdot \psi_{k+r} \\
L_n^R &= \frac{1}{2}\sum_k : \alpha_{-k} \cdot \alpha_{k+n} : + \frac{1}{2}\sum_k (k + \frac{n}{2}) : \psi_{-k} \cdot \psi_{k+n} : + \frac{D}{16}\delta_{n,0} \\
L_n^{NS} &= \frac{1}{2}\sum_k : \alpha_{-k} \cdot \alpha_{k+n} : + \frac{1}{2}\sum_s (s + \frac{n}{2}) : \psi_{-s} \cdot \psi_{s+n} :
\end{aligned} \tag{2.60}$$

and the ψ_n's satisfy the anticommutation relations

$$\{\psi_m^\mu, \psi_n^\nu\} = g^{\mu\nu}\delta_{m,-n}. \tag{2.61}$$

The other description, pioneered by Bardakci and Halpern for dual models,[63] and developed for the superstring by Green and Schwarz,[64] employs a doublet of Grassmann odd space-time *spinor* valued fields on the world sheet. Although initially[63] it was thought that these should be world sheet spinor fields, as developed by Green and Schwarz, they are world sheet scalars. So far, progress with this formulation has relied on a Lagrangian approach,[65–67] and we do not yet have the analog of (2.58). Thus we are not yet in a position to apply the BRST formalism. In this article we shall, for this reason, largely concentrate on the NSR formulation. As we shall see this approach is also problematic for interacting string field theory and it might well be that the Green-Schwarz formalism is a more appropriate vehicle for superstring field theory.

So returning to (2.58) we see that we need to introduce ghost variables for the fermionic generators in addition to the usual reparametrization ghosts b, c. These are called superghosts β, γ and, because they are associated with fermionic generators, they are bosons. They have the same moding as the odd generators: integer for the Ramond sector and half integer for the Neveu-Schwarz sector. Their commutation relations are

$$[\gamma_m, \beta_n] = \delta_{m,-n} \tag{2.62}$$

and their hermiticity properties are

$$\begin{aligned} \gamma_m^\dagger &= \gamma_{-m} \\ \beta_m^\dagger &= -\beta_{-m} \end{aligned} \tag{2.63}$$

Of course the superghosts carry their own ghost number

$$\begin{aligned} G^s &= -\sum_m :\gamma_{-m}\beta_m: +\frac{1}{2} \qquad && R \text{ sector} \\ G^s &= -\sum_r :\gamma_{-r}\beta_r: && NS \text{ sector}, \end{aligned} \tag{2.64}$$

but we shall only insist that the string field theory respect total ghost number $G = G^b + G^s$. In (2.64) the colons signify normal ordering for nonzero modes with the ordering of zero modes as written. The BRST operator corresponding to (2.58) is

$$\begin{aligned} Q_R^{open} = \sum_m c_{-m}L_m + \sum_m \gamma_{-m}F_m - \frac{1}{2}\sum_{m,n}(m-n) :c_{-m}c_{-n}b_{m+n}: \\ -\sum_{m,n}\gamma_{-m}\gamma_{-n}b_{m+n} + \sum_{m,n}(\frac{3m}{2}-n) :c_{-m}\gamma_{m-n}\beta_n: -\frac{5}{8}c_0 \end{aligned} \tag{2.65}$$

and that corresponding to (2.59) is the same except for the modes of the oscillators and a different reordering term

$$
\begin{aligned}
Q_{NS}^{open} = \sum_m c_{-m} L_m + \sum_r \gamma_{-r} G_r - \frac{1}{2} \sum_{m,n} (m-n) : c_{-m} c_{-n} b_{m+n} : \\
- \sum_{r,s} \gamma_{-r} \gamma_{-s} b_{r+s} + \sum_{m,r} (\frac{3m}{2} - r) : c_{-m} \gamma_{m-r} \beta_r : -\frac{1}{2} c_0 .
\end{aligned}
\tag{2.66}
$$

The m, n in (2.65) and (2.66) range over all integers whereas the r, s in (2.66) range over all half odd integers.

Just as in the bosonic case, one can establish the nilpotency of Q_R and Q_{NS} for $D = 10$ by the relatively simple calculation of $b_n Q^2 b_{-n} |0\rangle$ and $\beta_n Q^2 \beta_{-n} |0\rangle$, where $|0\rangle$ is annihilated by all positive moded oscillators. It is also easy to check that $\hat{L}_m = \{Q, b_m\}$ and $\hat{F}_m = [Q, \beta_m]$ obey the extended algebra (2.58) with vanishing c number term for $D = 10$, and similarly for $\hat{L}, \hat{G}$. We also quote the ghost contribution to these generators:

$$
\begin{aligned}
L_n^{gh,R} &= \sum_m (m-n) : c_{-m} b_{m+n} : + \sum_m (\frac{3n}{2} - m) : \gamma_{n-m} \beta_m : -\frac{5}{8} \delta_{n,0} \\
L_n^{gh,NS} &= \sum_m (m-n) : c_{-m} b_{m+n} : + \sum_r (\frac{3n}{2} - r) : \gamma_{n-r} \beta_r : -\frac{1}{2} \delta_{n,0} \\
F_n^{gh} &= -2 \sum_m \gamma_{-m} b_{m+n} + \sum_m (\frac{m}{2} - n) c_{-m} \beta_{m+n} \\
G_r^{gh} &= -2 \sum_s \gamma_{-s} b_{r+s} + \sum_m (\frac{m}{2} - r) c_{-m} \beta_{m+r} .
\end{aligned}
\tag{2.67}
$$

BRST Formalism for Closed Strings

Any closed string state space is isomorphic to the tensor product of two open string state spaces, one of which is identified with right moving modes and the other with left moving modes. Generally, we shall attach a bar to all operators acting in the left space. As is clear from the discovery of the heterotic string and the subsequent study of asymmetric compactifications, there is no *a priori* requirement that right and left spaces be isomorphic. For example, the right space of the heterotic string[29] is a ten dimensional superstring and its left space is a twenty six dimensional bosonic string with sixteen compactified dimensions. There is, however, the important constraint on the tensor product space that there is only one space-time momentum associated with uncompactified coordinates. This is achieved by identifying the left and right zero mode momenta for these coordinates

$$\alpha_0 = \bar{\alpha}_0 = \frac{p}{2} \tag{2.68}$$

To construct a BRST formalism for closed strings, all one has to do is form the tensor product of two BRST open string state spaces and define the BRST charge as the sum of the charges for the factor spaces:

$$Q^{closed} = Q + \bar{Q} \tag{2.69}$$

where all Grassmann odd operators acting in one factor space anticommute with the odd operators acting in the other space. Then the nilpotency of Q^{closed} follows from that of Q and $\bar{Q}$.

There is an important consequence of the constraint on the right and left momenta for the cohomology of Q^{closed}. Because they are equal, $\hat{L}_0 - \hat{\bar{L}}_0$ has a discrete spectrum. Thus the continuity argument we used to determine the cohomology of Q for the open string does not have the same force here. For the closed string there are four classes of on shell DDF states for $p \neq 0$. Continuity of $\hat{L}_0 + \hat{\bar{L}}_0$ shows that those annihilated by $c_0 + \bar{c}_0$ are trivial, but those annihilated by $c_0 - \bar{c}_0$ are not. Thus the cohomology of Q^{closed} is not the tensor product of the cohomologies of Q and $\bar{Q}$, but rather a doubled version of that space.

Free String Field Equations and their Solution

We are now in a position to write down the free field equations for any string theory. One simply identifies the string field with a state in the corresponding

BRST state space and declares that the field equation is just

$$Q\,|A\rangle = 0. \tag{2.70}$$

It is evident that this formulation of string field theory has considerable flexibility. As far as BRST invariance is concerned, all one needs is a representation of the Virasoro algebra or its graded extension with the right value of the central charge to guarantee the nilpotency of Q. To construct a first quantized interacting string theory one actually needs the representations to be further constrained by the requirement that they yield a local two dimensional (world-sheet) quantum field theory. The string field theoretic version of these further constraints has not yet been sharply formulated, although it is fairly obvious that without them it would be difficult to discover consistent nonlinear string field equations. There is always, of course, a part of the "matter" sector that is simply some number (less than the critical dimension) of uncompactified coordinates, but the remaining degrees of freedom needed to provide the correct central charge can apparently be any conformally invariant two dimensional quantum field theory which possesses an $SO(2,1)\times SO(2,1)$ invariant vacuum.

As we have seen, the basic building blocks of any string field theory are the bosonic and spinning open string state spaces. Before turning to the analysis of the general field equations, which will closely parallel that for the open bosonic string, we have to discuss some special features of the superghosts. We recall that the main virtue of the BRST formalism is that it provides a consistent projection of an infinite set of first class constraints onto finite dimensional subspaces labelled by the eigenvalues of $\hat{L}_0$. This is achieved by defining the ghost or superghost Fock space so that the eigenspectrum of $\hat{L}_0$ is bounded below.

For the ordinary reparametrization ghosts the choice of a suitable "vacuum" for the Fock space is more or less unique. To avoid a spectrum unbounded below, one must certainly interpret all but a finite number of positive moded oscillators as annihilation operators. But because the b's and c's are fermionic any such vacuum can be obtained by applying a finite number of negative moded oscillators to the vacuum annihilated by *all* the positive moded oscillators. So all suitable choices lead to the same Fock space.

The bosonic nature of the superghosts introduces an important new feature. Just as before, we must interpret all but a finite number of positive moded ocscillators as annihilation operators. But now each such vacuum leads to an inequivalent Fock space. In the Neveu-Schwarz sector there is a unique choice which leads to a spectrum of $\hat{L}_0$ bounded below. This is the Fock space built on the "vacuum" annihilated by *all* positive moded oscillators. This is the simplest choice. If one were only interested in free strings one could simply make this choice and be done with it. However, at least in the first quantized approach, it is very useful to incorporate other choices into the formalism. The different pictures for superghosts, as introduced by Friedan, Martinec, and Shenker,[68] are defined by selecting a vacuum $|0, P\rangle$ by the conditions

$$\begin{aligned} \beta_n \, |0, P\rangle = 0 \qquad & \text{for } n > -P - \frac{3}{2} \\ \gamma_n \, |0, P\rangle = 0 \qquad & \text{for } n > +P + \frac{1}{2} \end{aligned} \tag{2.71}$$

The case $P = -1$ is the simplest one mentioned above. This definition of pictures applies also to the Ramond sector for which P takes on half odd integer values.

In the Ramond sector, one gets an $\hat{L}_0$ bounded below if all positive moded oscillators annihilate the vacuum. Inspection of (2.71) reveals that two values of P, $-1/2$ and $-3/2$, satisfy this criterion. In the first case β_0 is an annihilation operator, and in the second case γ_0 is. An unavoidable feature of the Ramond sector is the infinite degeneracy caused by the zero mode creation operators γ_0, respectively β_0. Of course, in sectors of fixed ghost number, the degeneracy is finite.

It is because of the conserved ghost number that one can make sense out of the more general pictures in both Neveu-Schwarz and Ramond sectors. Consider, for example, the $P = 0$ picture for the NS sector. Then $\gamma_{+1/2}$ is a creation operator that adds $-1/2$ to $\hat{L}_0$. But $\beta_{-3/2}$ is the lowest mode creation operator carrying negative ghost number. Thus all monomials of creation operators with zero ghost number increase the value of $\hat{L}_0$, so that $\hat{L}_0$ is bounded below in each sector with fixed ghost number. This property is shared by all of the pictures.

We now see that there is some flexibility in setting up a field theory for the spinning string. We have to decide what picture to use. If we wish the theory to include fermions, our string field should have both a bosonic component associated with a state in the NS sector and a fermionic component associated with a state in the R sector. Within each sector we must further specify the Fock space by choosing the value of P. In the analysis of the field equations we shall choose $P = -1$ in the NS sector and $P = -1/2$ in the R sector. It should be borne in mind that, since different pictures are inequivalent, the equations must be reanalysed if different picture choices are made.

The analysis of the free string field equations of motion in the general situation is closely parallel to the analysis for the case of the open bosonic string. We first discuss the NS sector of the open spinning string. Again, for fixed $p \neq 0$, one picks a light-like vector k with $k \cdot p = 1$ and defines $H_r = k \cdot \psi_r$ and DDF states by

$$(b_n, c_n, \beta_r, \gamma_r, K_n, L_n, G_r, H_r)\,|DDF\rangle = 0, \quad n, r > 0. \tag{2.72}$$

Because there are no zero mode spin oscillators in the NS sector there are precisely two copies of dual model DDF states, just as for the bosonic string, and those with ghost number $+\frac{1}{2}$ can again be regarded as limits of trivial states.

The analogue of the basis (2.30) is obtained by applying all possible symmetrized monomials of the various negative moded operators to the DDF states. The key to the analysis of the equations of motion is the tableau identity (2.32). The product of a symmetric tensor with an antisymmetric tensor produces precisely two symmetry patterns. The operators appearing in the basis can be paired off according to how they behave under bracketing with Q. Each pair has one bosonic and one fermionic member. For the bosonic string the pairings were

$$\begin{aligned} \hat{L}_n &\leftrightarrow b_n \\ K_n &\leftrightarrow c_n. \end{aligned} \tag{2.73}$$

The bracket of Q with a product of two monomials, one containing the bosonic members and the other containing the fermionic members of a pairing, then shows that one of the two symmetry patterns can be gauged away, and that the equations of motion imply the other pattern is absent. All one has to do for the spinning string is add the pairings

$$\begin{aligned} \hat{G}_r &\leftrightarrow \beta_r \\ H_r &\leftrightarrow \gamma_r \end{aligned} \tag{2.74}$$

to the list. Then repeated use of the basic tableau identity and brackets with Q as in (2.31) allows one to conclude that any solution of the equations of motion is gauge equivalent to a DDF state. As we have mentioned, for the NS sector the doubling of the on-shell DDF states associated with the zero modes of c and b has the same interpretation as for the bosonic string.

For the R sector the new pairings are

$$\begin{aligned} \hat{F}_n &\leftrightarrow \beta_n \\ H_n &\leftrightarrow \gamma_n. \end{aligned} \tag{2.75}$$

Application of the above described analysis in this sector leaves an infinite ultiplicity of DDF states caused by the zero modes of the superghosts. In the $P = -\frac{1}{2}$ picture we have chosen, this multiplicity can be enumerated by applying a multiple of γ_0 to DDF states annihilated by β_0. We can analyse this degeneracy within subspaces of fixed ghost number. The lowest ghost number sector, annihilated by b_0 and β_0, has only one set of DDF states, and in this sector, the field equation reduces to

$$Q\left|DDF\right\rangle_0 = (c_0 L_0 + \gamma_0 F_0)\left|DDF\right\rangle_0 = 0 \tag{2.76}$$

which implies

$$L_0\left|DDF\right\rangle_0 = F_0\left|DDF\right\rangle_0 = 0. \tag{2.77}$$

In the sector with ghost number N the general DDF state may be written

$$\left|DDF\right\rangle_N = \gamma_0^N\left|T_1\right\rangle + c_0\gamma_0^{N-1}\left|T_2\right\rangle \tag{2.78}$$

where $\left|T_i\right\rangle$ is a DDF state annihilated by b_0 and β_0. Noting that

$$Qc_0\gamma_0^{N-2}\left|T_1\right\rangle = -\gamma_0^N\left|T_1\right\rangle + F_0 c_0\gamma_0^{N-1}\left|T_1\right\rangle, \tag{2.79}$$

we see that one can remove the T_1 term from (2.78) by a gauge transformation for $N \geq 2$. Then applying Q to the remaining term in (2.78) we find that the field equation implies that $\left|DDF\right\rangle_N = 0$ for all $N \geq 2$. This takes care of the infinite multiplicity. Turning to the sector with $N = 1$, we write the field equation as

$$Q\left|DDF\right\rangle_1 = c_0(L_0 - \frac{D}{16})\gamma_0\left|T_1\right\rangle + \gamma_0^2 F_0\left|T_1\right\rangle + \gamma_0 F_0 c_0\left|T_2\right\rangle - \gamma_0^2\left|T_2\right\rangle = 0 \tag{2.80}$$

which implies that $\left|T_2\right\rangle = F_0\left|T_1\right\rangle$ so

$$\left|DDF\right\rangle_1 = (\gamma_0 + c_0 F_0)\left|T_1\right\rangle. \tag{2.81}$$

The DDF state (2.81) is annihilated by Q both on and off shell. Off shell it is, of course, a pure gauge so it plays the role of the ghost number $+\frac{1}{2}$ states of the bosonic open string or NS sector open spinning string.

So just as with the bosonic string we have found that the linear spinning string field equations for $p \neq 0$ possess solutions in one-to-one correspondence with the eigenstates of the light cone first quantized spinning string. For $p = 0$ there are also the analogues of the special solutions (2.27). In the NS sector in the $P = -1$ picture they are, respectively,

$$\gamma_{-1/2} c_0 \left|0,0\right\rangle \qquad \psi^{\mu}_{-1/2} c_0 \left|0,0\right\rangle \qquad \psi^{\mu}_{-1/2} \left|0,0\right\rangle \qquad \beta_{-1/2} \left|0,0\right\rangle , \tag{2.82}$$

and their interpretation is exactly the same as with the bosonic case.

The solution of the linear closed string field equations, for the most part, follows by doubling all of the bose fermi pairings and simply applying the analysis using tableau and bracket identities to each pairing in turn. This allows one to conclude that the $p \neq 0$ solutions are DDF states with some multiplicity. But as we have already mentioned above, the fact that $L_0 - \bar{L}_0$ has a discrete spectrum means that one cannot conclude that all of the multiple copies are trivial. For example, the closed bosonic string possesses four copies of on shell DDF states which solve the field equations, with total ghost numbers $-1, 0, 0, +1$. One can argue that those annihilated by $c_0 + \bar{c}_0$ are limits of trivial states, but not those annihilated by $c_0 - \bar{c}_0$. Thus, there are two nontrivial cohomology sectors with total ghost numbers -1 and 0. For closed string field theory, this means that one has a choice for the ghost number of the classical closed string field.

Action Principles

We have seen that the extension of the BRST style linear field equations to the general situation presents no difficulties of principle. Likewise their solution and interpretation is, for the most part, completely parallel to that of the open bosonic string. As we shall see, however, the discovery of suitable action principles for these more general string theories is not so straightforward.

Let us first consider finding an action for the free open spinning string. In attempting to follow the example of the bosonic string we encounter the difficulty that the dual to a ket $|P\rangle$ in the picture P state space is a bra $\langle -P-2|\, c_0$ in the picture $-P-2$ state space. This is simply (in the notation of Appendix A) ϕ_0, χ_0, and g_0 conservation. Thus we can only construct non vanishing matrix elements

$$\langle -P-2|Q|P\rangle\,.$$

Only for the $P=-1$ bosonic component of the string field does this matrix element involve states in the same picture. No matter what picture we choose for the fermionic component of the string field the nonzero matrix elements of Q connect states in different pictures. To handle this problem, we need an "inverse picture changing operator" Y_{-2P-2} that changes picture P states into picture $-P-2$ states. This operator is required to be BRST invariant and of zero conformal weight. In addition, since the cohomology of Q in picture P has ghost number $P+1/2$, we require that Y_{-2P-2} also carry $-2P-2$ units of total ghost number. Then the matrix elements

$$\langle \lambda', P|\xi_0 Q Y_{-2P-2}|\lambda, P\rangle$$

can be nonzero with $G_\lambda = G_{\lambda'} = P+1/2$.

Clearly for $P=-1$, we can take $Y_0 = 1$. For the fermionic sector in picture $P=-1/2$, we can choose for $Y_{-1}(z)$ the Y constructed in Appendix A. For $P=0$ we would need a Y_{-2} for which there are several possibilities. We shall return to more general pictures in Chapter 6, but for now we restrict our choices to $P=-1$ for the bosonic sector and $P=-1/2$ for the fermionic sector.

For string field theory we wish to replace the sesquilinear matrix elements by bilinear ones $\langle B \mid A\rangle = \langle \mathcal{V}_2 \mid B\rangle\,|A\rangle$ by defining a vertex satisfying

$$\begin{aligned} \left\langle \mathcal{V}_2^{spin}\right|(\alpha^1_{-n} + (-)^n \alpha^2_n) &= 0 && \\ \left\langle \mathcal{V}_2^{spin}\right|(\psi^1_{-r} + i e^{i\pi r}\psi^2_r) &= 0 && \text{NS sector} \\ \left\langle \mathcal{V}_2^{spin}\right|(\psi^1_{-n} + i(-)^n \psi^2_n) &= 0 && \text{R sector.} \end{aligned} \tag{2.83}$$

The coefficient in front of ψ^2 has r dependence fixed by the identification of the point $\pi - \sigma$ on one string with σ on the other. Consistency of these equations with the anticommutation relations of the ψ's then implies the written phase or its negative. The choice we made is by convention. The other choice could be realized, for example, by multiplying the vertex by $(-)^{N_\psi}$.The super Virasoro generators satisfy

$$
\begin{aligned}
\left\langle \mathcal{V}_2^{spin} \right| (L^1_{-n} - (-)^n L^2_n) &= 0 & & \\
\left\langle \mathcal{V}_2^{spin} \right| (G^1_{-r} - ie^{i\pi r} G^2_r) &= 0 & & \text{NS sector} \\
\left\langle \mathcal{V}_2^{spin} \right| (F^1_{-n} - i(-)^n F^2_n) &= 0 & & \text{R sector}
\end{aligned}
\tag{2.84}
$$

and the requirement $\left\langle \mathcal{V}_2^{spin} \right| (Q_1 + Q_2) = 0$ then shows that the ghost oscillators satisfy

$$
\begin{aligned}
\left\langle \mathcal{V}_2^{spin} \right| (c^1_{-n} + (-)^n c^2_n) &= 0 \\
\left\langle \mathcal{V}_2^{spin} \right| (b^1_{-n} - (-)^n b^2_n) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \mathcal{V}_2^{spin} \right| (\gamma^1_{-r} - ie^{i\pi r} \gamma^2_r) &= 0 & & \text{NS sector} \\
\left\langle \mathcal{V}_2^{spin} \right| (\gamma^1_{-n} - i(-)^n \gamma^2_n) &= 0 & & \text{R sector} \\
\left\langle \mathcal{V}_2^{spin} \right| (\beta^1_{-r} + ie^{i\pi r} \beta^2_r) &= 0 & & \text{NS sector} \\
\left\langle \mathcal{V}_2^{spin} \right| (\beta^1_{-n} + i(-)^n \beta^2_n) &= 0 & & \text{R sector.}
\end{aligned}
\tag{2.85}
$$

Let $|0, p^\mu, P\rangle_{NS}$ be the vacuum in the NS sector with space-time momentum p^μ, in picture P, and annihilated by b_0 and the positive moded $c_n, b_n, \alpha_n, \psi_r$. Then $\left\langle \mathcal{V}_2^{spin} \right|$ has the explicit solution

$$
\begin{aligned}
\left\langle \mathcal{V}_2^{spin}, P \right|_{NS} = \int dp \, \langle 0, p, -P-2|_{NS,1} \, \langle 0, -p, +P|_{NS,2} \, (c^1_0 + c^2_0) \\
\exp\Big\{ -\sum_{n=1}^{\infty} (-)^n (\alpha^1_n \cdot \alpha^2_n + c^2_n b^1_n + c^1_n b^2_n) \\
- i \sum_{r=1/2}^{\infty} e^{i\pi r} \psi^2_r \cdot \psi^1_r \\
- i \sum_{r=-P-1/2}^{\infty} e^{i\pi r} \gamma^2_r \beta^1_r + i \sum_{r=P+3/2}^{\infty} e^{i\pi r} \gamma^1_r \beta^2_r \Big\}
\end{aligned}
\tag{2.86}
$$

where $\left\langle \mathcal{V}_2^{spin}, P \mid \psi_1 \right\rangle |\psi_2\rangle \neq 0$ only if $|\psi_1\rangle$ is in picture P and $|\psi_2\rangle$ is in picture $-P-2$.

For the R sector, define the bra $\langle 0, p^\mu, -P-2; 0, -p^\mu, P|$ in picture P to be annihilated by $\psi_0^{1\mu} + i\psi_0^{2\mu}$ for all μ in addition to the negative moded $c_n^{(1,2)}$, $b_n^{(1,2)}$, $\alpha_n^{(1,2)}$, $\psi_n^{(1,2)}$. Then the solution is

$$
\begin{aligned}
\left\langle \mathcal{V}_2^{spin}, P \right|_R = \int dp \, & \langle 0, p^\mu, -P-2; 0, -p^\mu, P| \, (c_0^1 + c_0^2) \\
& \exp\Bigg\{ -\sum_{n=1}^{\infty} (-)^n (\alpha_n^1 \cdot \alpha_n^2 + c_n^2 b_n^1 + c_n^1 b_n^2) \\
& \quad - i \sum_{n=1}^{\infty} (-)^n \psi_n^2 \cdot \psi_n^1 \\
& \quad - i \sum_{n=-P-1/2}^{\infty} (-)^n \gamma_r^2 \beta_n^1 + i \sum_{n=P+3/2}^{\infty} (-)^n \gamma_r^1 \beta_n^2 \Bigg\}
\end{aligned}
$$

In either case a suitable action principle for the string field A in picture P is

$$
\frac{1}{2} \langle \mathcal{V}_2^{spin}, P | A_P \rangle Q Y_{-2P-2} \, |A_P\rangle
$$

So for the spinning string field with the bosonic component A in picture -1 and fermionic component Ψ in picture $-1/2$ the action can be taken to be

$$
S^{spin} = \frac{1}{2} \langle \mathcal{V}_2^{spin}, -1 | A \rangle Q \, |A\rangle + \frac{1}{2} \langle \mathcal{V}_2^{spin}, -1/2 | \Psi \rangle Q Y_{-1} \, |\Psi\rangle \, .
$$

What we have said so far has not specified the linear functinoal of $Y_{-1}(z)$ to insert in the fermionic action. In the next chapter, we shall discuss Witten's formulation of open string field theory in which the midpoint of the open string plays a special role. In this case it is natural and in fact necessary to take Y_{-1} evaluated at this midpoint as the insertion factor.

We have succeeded in constructing an action principle for free open superstring field theory. We close this chapter with a brief comment on the difficulties in doing the same for closed string field theory. The fundamental problem is that the bosonic sector of closed string fields has integer total ghost number. Thus any matrix element of Q connects sectors of different ghost number. This feature is similar to the problem encountered with the fermionic sector of the spinning string. However, now there is no natural candidate for an insertion to balance ghost number. Although various possibilities have been tried,[69,70] none have been completely successful.

3. Interacting String Fields

3.1. Second Quantization in Light-cone Gauge.

It is very instructive to begin our discussion of interacting string fields with the light cone gauge formulation. This is because, in spite of the lack of manifest Poincare covariance, this formulation manages to code the complete nonlinear quantum dynamics by a hamiltonian which is a polynomial in string fields. More importantly, the approach applies to *all* string theories, open and closed, bosonic and fermionic. One therefore has by explicit construction an existence proof that string field theory is possible. In view of the problems of principle encountered in trying to construct a covariant field theory for closed strings, it behooves us to at least understand the one existing closed string field theory. Of course the limitation is that since the gauge has been completely fixed, the underlying nonlinear gauge invariance is hidden, and that is something we would like to avoid. We shall discuss later on the progress toward a gauge invariant nonlinear formulation that has been achieved for the open string theory.

We recall from Chapter 1, that the Schrodinger wave function $\Psi_{P^+}[\mathbf{x}(\sigma),\tau]$ for the first-quantized string in light cone gauge satisfies the time dependent Schrodinger equation

$$\begin{aligned} i\frac{\partial\Psi_{P^+}}{\partial\tau}[\mathbf{x},\tau] &= \frac{\pi}{2}\int_0^{\pi P^+} d\sigma\left[-\frac{\delta^2}{\delta\mathbf{x}(\sigma)^2}+\left(\frac{\mathbf{x}'(\sigma)}{\pi}\right)^2\right]\Psi_{P^+}[\mathbf{x},\tau] \\ &\equiv h\Psi_{P^+}[\mathbf{x},\tau]. \end{aligned} \tag{3.1}$$

In general one can always use the machinery of second quantization to convert a single body theory to a many body theory. Thus we can convert the above theory of a single quantum string to a theory of many free quantum strings by promoting Ψ to an operator with the equal time commutation relations

$$\Big[\Psi_{P^+}[\mathbf{x}(\sigma),\tau],\Psi^\dagger_{Q^+}[\mathbf{y}(\sigma),\tau]\Big]=\delta_{P^+,Q^+}\delta\big[\mathbf{x}(\sigma)-\mathbf{y}(\sigma)\big]. \tag{3.2}$$

The free P_0^- which gives (3.1) as Heisenberg equations is clearly

$$P_0^- \equiv \int \mathcal{D}\mathbf{x}(\sigma) \sum_{P^+} \Psi_{P^+}^\dagger\big[\mathbf{x}(\sigma)\big] h \Psi_{P^+}\big[\mathbf{x}(\sigma)\big]. \tag{3.3}$$

Now we can contemplate adding terms to P_0^- which contain three or more powers of the string field

$$P^- = P_0^- + V. \tag{3.4}$$

In order to understand what kind of nonlinear terms lead to the usual dual amplitudes, we apply time dependent perturbation theory to (3.4). To this end we write

$$e^{-iTP^-} = e^{-iTP_0^-} U_I(T) \tag{3.5}$$

where U_I satisfies the equation

$$\begin{aligned} i\frac{d}{dT}U_I(T) &= e^{iTP_0^-} V e^{-iTP_0^-} U_I(T) \\ &\equiv V_I(T) U_I(T). \end{aligned} \tag{3.6}$$

Iterating (3.6) using the initial condition $U_I(0) = 1$ then gives the perturbation expansion

$$\begin{aligned} e^{-iTP^-} &= e^{-iTP_0^-} - i \int_0^T d\tau_1 e^{-i(T-\tau_1)P_0^-} V e^{-i\tau_1 P_0^-} \\ &+ (-i)^2 \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-i(T-\tau_1)P_0^-} V e^{-i(\tau_1-\tau_2)P_0^-} V e^{-i\tau_2 P_0^-} \\ &+ \quad \cdots \end{aligned} \tag{3.7}$$

To see what V must be we shall interpret (3.7) in terms of the interacting string diagrams introduced by Mandelstam.[9] The first step is to introduce a basis of the multistring state space

$$\Psi^{\dagger}_{P_1^+}\left[\mathbf{x}_1(\sigma)\right]\Psi^{\dagger}_{P_2^+}\left[\mathbf{x}_2(\sigma)\right]\cdots\Psi^{\dagger}_{P_N^+}\left[\mathbf{x}_N(\sigma)\right]|0\rangle \tag{3.8}$$

so that the identity operator has the resolution

$$\begin{aligned} I = \sum_N \frac{1}{N!}\sum_{\{P_i^+\}}\int\prod_k \mathcal{D}\mathbf{x}_k \\ \Psi^{\dagger}_{P_1^+}\left[\mathbf{x}_1(\sigma)\right]\cdots\Psi^{\dagger}_{P_N^+}\left[\mathbf{x}_N(\sigma)\right]|0\rangle\langle 0|\,\Psi_{P_N^+}\left[\mathbf{x}_N(\sigma)\right]\cdots\Psi_{P_1^+}\left[\mathbf{x}_1(\sigma)\right]. \end{aligned} \tag{3.9}$$

The next step is to consider the matrix element of (3.7) between basis states (3.8) and to insert the identity operator (3.9) between every pair of operators on the RHS. The matrix element of each free evolution operator is simply

$$\begin{aligned} \langle 0|\,\Psi_{R_N^+}\left[\mathbf{y}_N(\sigma)\right]\cdots\Psi_{R_1^+}\left[\mathbf{y}_1(\sigma)\right]e^{-iTP_0^-}\Psi^{\dagger}_{Q_1^+}\left[\mathbf{x}_1(\sigma)\right]\cdots\Psi^{\dagger}_{Q_M^+}\left[\mathbf{x}_M(\sigma)\right]|0\rangle = \\ \delta_{M,N}\sum_P\prod_i\left\langle \mathbf{y}_{P_i}(\sigma),Q_i^+\right|e^{-iTh}\left|\mathbf{x}_i(\sigma),Q_i^+\right\rangle\delta_{R^+_{P_i},Q_i^+} \end{aligned} \tag{3.10}$$

where the sum on the RHS is over all permutations of the N labels describing the chosen basis states. Each factor on the RHS of (3.10) is a matrix element $\langle\mathbf{y},P^+|\,e^{-ihT}\,|\mathbf{x},P^+\rangle$ in the single string state space of the first quantized evolution operator. It can be represented as a Feynman path integral over all trajectories $\mathbf{x}(\sigma,\tau)$ defined on a rectangle of length T and width πP^+ with initial and final conditions $\mathbf{x}(\sigma,0)=\mathbf{x}(\sigma)$ and $\mathbf{x}(\sigma,T)=\mathbf{y}(\sigma)$ as illustrated in Figure 1.

The operator V must be chosen so that its matrix elements between basis states converts (3.7) into the sum of interacting string diagrams postulated by Mandelstam. It should be understood that this is just a problem of transcription. For example, it is immediately obvious that open string diagrams will be generated by the cubic terms

$$
\begin{aligned}
V_3^{0pen} = g \sum_{P_1^+, P_2^+} \int \mathcal{D}\mathbf{x}_1 \mathcal{D}\mathbf{x}_2 \Big(& \Psi^\dagger_{P_1^+ + P_2^+}[\mathbf{x}_1 \circ \mathbf{x}_2] \Psi_{P_1^+}[\mathbf{x}_1] \Psi_{P_2^+}[\mathbf{x}_2] \\
& + \Psi^\dagger_{P_1^+}[\mathbf{x}_1] \Psi^\dagger_{P_2^+}[\mathbf{x}_2] \Psi_{P_1^+ + P_2^+}[\mathbf{x}_1 \circ \mathbf{x}_2] \Big).
\end{aligned}
\tag{3.11}
$$

Here we have introduced the symbol $\mathbf{x}_1 \circ \mathbf{x}_2$ to denote the curve composed from the two curves $\mathbf{x}_1$ and $\mathbf{x}_2$ through the rule

$$
\mathbf{x}_1 \circ \mathbf{x}_2(\sigma) = \begin{cases} \mathbf{x}_1(\sigma) & \text{for } 0 < \sigma < \pi P_1^+ \\ \mathbf{x}_2(\sigma - \pi P_1^+) & \text{for } \pi P_1^+ < \sigma < \pi (P_1^+ + P_2^+). \end{cases}
\tag{3.12}
$$

We are also presuming orientable strings for simplicity. For the nonorientable case, one either requires the symmetry

$$
\Psi_{P^+}\big[\mathbf{x}(\pi P^+ - \sigma)\big] = \Psi_{P^+}\big[\mathbf{x}(\sigma)\big]
\tag{3.13}
$$

in which case the commutation relations (3.2) acquire an extra term involving $\delta\big[\mathbf{x}(\pi P^+ - \sigma) - \mathbf{y}(\sigma)\big]$, or one retains (3.2) and adds extra terms to V so that Ψ_{P^+} enters only in the combination $\Psi_{P^+}\big(\mathbf{x}(\pi P^+ - \sigma)\big) + \Psi_{P^+}\big(\mathbf{x}(\sigma)\big)$.

A point to notice in (3.11) is that there are no powers of P^+ in the definition of the vertex. This is in contrast to the light cone vertex for three point particles which would be

$$
V_3^{points} = \sum_{P_1^+, P_2^+} \int d\mathbf{x} \frac{1}{\sqrt{P_1^+ P_2^+ (P_1^+ + P_2^+)}} \big(\Psi^\dagger(\mathbf{x}) \Psi(\mathbf{x}) \Psi(\mathbf{x}) + h.c.\big).
\tag{3.14}
$$

The square roots of P^+ are well known consequences of Lorentz invariance. They arise in (3.11) not because they are put in by hand, but because the matrix element of V_3^{open} between string states of definite P^- scale by a power of P^+ determined by the functional integral over $\mathbf{x}(\sigma)$. This power has the correct value for Lorentz invariance only in the critical dimension[39].

To see that V_3^{open} reproduces Mandelstam's interacting string diagrams, one simply examines its matrix elements between states of the basis (3.8). This matrix element only involves three strings at a time, two from the ket (or bra) and one from the bra (or ket). Each of these strings has been associated with the end of one of the rectangular strips associated with the matrix elements of the free evolution operators, and the completeness sums simply glue the ends of two rectangles of widths πP_1^+ and πP_2^+ to the end of a larger rectangle of width $\pi P_1^+ + \pi P_2^+$ (see Figure 2). This is just the three open string vertex of the Mandelstam diagrams. The gluing prescription is nothing other than the fundamental closure property of Feynman path integrals.

When the terms (3.11) of V are inserted into the perturbation expansion (3.7) one immediately reproduces the sum of all of the interacting open string diagrams containing only three string vertices. This includes, of course, the prescription of integrating over the locations and heights of all points where strings separate or join. These latter integrals are just the integrals over the moduli of the Riemann surface associated with each diagram. Mandelstam showed that, provided the space-time dimension was critical, for generic values of these joining points the integrand is just that of the Koba-Nielsen representation of the corresponding dual resonance amplitude. However, the full range of integration is not covered by these diagrams with cubic vertices only.

To understand what is missing, Mandelstam studied the Lorentz transformation properties of the three string vertex[71] within a given diagram. The crucial generator is, of course, just M^{i-} where i denotes a transverse component. In the critical dimension he found that the change of V_3 was just a derivative with respect to the joining location. To show the Lorentz invariance of the S-matrix, one must show that these derivatives don't contribute. Individual time ordered diagrams cover only a part of the integration region, and so these derivative terms will not cancel for each diagram. In many cases there are groups of diagrams which together cover the whole integration range of time from $-\infty$ to $+\infty$, and the integrand is obviously continuous at the interface of the integration regions for separate diagrams. An example is illustrated in Figure 3. But it is also clear that there are diagrams with cubic vertices only where the integration region terminates in the middle of moduli space with no diagrams to complete the integration region. This would cause a nonvanishing surface term when integrating total derivatives. The simplest example at the one loop level is the correction to the open string two point function shown in Figure 4a. This can be remedied by adding the diagram of Figure 4b, which clearly involves an intermediate closed string which couples to an open string through the same breaking joining mechanism by which two open strings couple to a single open string. This shows that a theory of open strings must of necessity include closed strings.

There are even open string tree diagrams with missing regions of the Koba-Nielsen integrals. The four string function includes the diagrams illustrated in Figure 5a. But the zero time limits of the two diagrams are not equal to each other, which would lead to a violation of Lorentz invariance, because the integral of total derivatives would get hung up on the discontinuity. An interpolating diagram, illustrated in Figure 5b, must be included. This diagram would arise from a new term added to V, quartic in the open string fields:

$$V_4^{open} = \frac{g^2}{2} \int \mathcal{D}\mathbf{x}_L \mathcal{D}\mathbf{x}_R \mathcal{D}\mathbf{y}_L \mathcal{D}\mathbf{y}_R dP_L^+ dP_R^+ dQ_L^+ dQ_R^+ \\ \Psi^\dagger_{P_L^+ + P_R^+}[\mathbf{x}_L \circ \mathbf{x}_R] \Psi^\dagger_{Q_L^+ + Q_R^+}[\mathbf{y}_L \circ \mathbf{y}_R] \Psi_{Q_L^+ + P_R^+}[\mathbf{y}_L \circ \mathbf{x}_R] \Psi_{P_L^+ + Q_R^+}[\mathbf{x}_L \circ \mathbf{y}_R]. \tag{3.15}$$

Here we have adopted a hopefully self-explanatory notation where each of the string field arguments is divided into two sets associated with the left ($0 \leq \sigma < \pi P_L^+$) and right ($\pi P_L^+ \leq \sigma < \pi P_L^+ + \pi P_R^+$) portions of string. The mechanism by which two open strings are changed by (3.15) is locally (on the world sheet) identical to that by which two closed strings join to form a single closed string, and indeed one can use this to see that the cubic closed string interaction is necessary for Lorentz invariance. It can be shown[12,72] that the light cone string field theory including both open and closed string fields and with a V which includes five terms

$$V = V_3^{open} + V_4^{open} + V_3^{closed} + V_{1,1}^{open-closed} + V_{2,1}^{open-closed} \tag{3.16}$$

generates a set of light cone interacting string diagrams with no missing integration region *of finite measure*. Moreover one can consistently delete the open string field from the theory thereby obtaining a purely closed string field theory where only the third term is retained in (3.16).

The specter of *zero measure* missing regions (i.e. contact interactions) is not ruled out by the above analysis. The bosonic string field theory apparently is free of such problems. But the light cone field theory of spinning strings requires the overlap vertices to have local prefactors. For example, (3.11) in the NSR formulation looks like

$$\begin{aligned} V_3^{open-spin} = g \sum_{P_1^+, P_2^+} \int & \mathcal{D}\mathbf{x}_1 \mathcal{D}\mathbf{x}_2 \; \mathbf{x}_1'(\pi P_1^+) \cdot \psi(\pi P_1^+) \\ & \Big(\Psi^\dagger_{P_1^+ + P_2^+}(\mathbf{x}_1 \circ \mathbf{x}_2) \Psi_{P_1^+}(\mathbf{x}_1) \Psi_{P_2^+}(\mathbf{x}_2) \\ & + \Psi^\dagger_{P_1^+}(\mathbf{x}_1) \Psi^\dagger_{P_2^+}(\mathbf{x}_2) \Psi_{P_1^+ + P_2^+}(\mathbf{x}_1 \circ \mathbf{x}_2) \Big) \end{aligned} \tag{3.17}$$

with similar modifications to the other vertices. Greensite and Klinkhamer[73] (see also [74]) have shown that in diagrams such as those in Figure 5a where joining points are forced to collide, these extra factors can cause unwanted zero measure contributions which can only be removed by adding new counter terms to V. For the case of Figure 5, this can be regarded as a refinement of the definition of Figure 5b. However similar effects in the closed string vertices for the spinning string can only be cancelled by entirely new terms in V.[73] We shall find that the gauge invariant formulation of spinning string field theory due to Witten is also plagued by the need for contact counter terms. Actually the occurrence of extra contact terms for the spinning string is a blessing in disguise. The absence of them for the bosonic string would seal its doom: the closed string hamiltonian would then be purely cubic and therefore absolutely unstable. Contact terms are at least quartic in the fields, so their presence offers hope that the superstring theory is not total nonsense.

In contemplating these contact interactions, one should bear in mind that ordinary gauge field theories must also be patched up by apparently gauge invariance violating counter terms when one uses a physical cutoff. These are to compensate violations of gauge invariance induced by ultraviolet divergences. This may be seen by studying, for example, a point splitting regulation of the product two gauge variant operators with a Wilson phase inserted to insure gauge invariance.[75] Alternatively, a less physical regularization scheme such as dimensional regularization, may be used to ensure gauge invariant results of perturbative calculations. But this is presumably not a satisfactory way to define the gauge theory nonperturbatively. What is disturbing for superstring field theory, is that these types of modification are apparently necessary at tree (classical) level.

3.2. Toward a Gauge Invariant Interacting String Field Theory

In the previous section we have sketched the light cone gauge formulation of string field theory and have seen that, up to open questions about contact interactions, the program succeeds. It should be stressed that the program is little more than a literal translation of the first quantized interacting string diagrams into the language of second quantization. One accordingly directly constructs the operator formulation of the quantum string field theory without going through the step of developing an action principle. One can always construct an action principle from an operator quantum theory by following Dirac's prescription of constructing the path history formulation from the operator hamiltonian formulation. But this is not really necessary since one can obtain the full fledged operator string field equations from Heisenberg's equations of motion.

The action formulation of quantum theory is almost necessary for a Lorentz covariant and gauge invariant treatment. Indeed, in the gauge invariant approach to string field theory, it is not yet clear how to set up an operator formalism. In this section we approach the problem of extending the gauge invariant linear string field equation discussed in Chapter 2 to a nonlinear field equation. We recall that for any free string theory, the field equation

$$QA = 0$$

is invariant under a gauge transformation

$$\Delta A = Q\Lambda.$$

The importance of gauge invariance resides in the irrelevance of deformations of field configurations which are pure gauge. Such deformations are not at all constrained by gauge invariant equations of motion and therefore have no dynamical significance. These irrelevant field deformations include ones such as the time components of gauge fields that can't be consistently interpreted as true physical degrees of freedom. If it were not for gauge invariance, such degrees of freedom would contribute negatively to the energy density rendering the theory unstable.

In the quantum theory one has a choice of using a state space of indefinite metric and a positive energy operator or a positive definite state space with energy operator unbounded below. In the first case, which we adopt, the unphysical degrees of freedom correspond to ghosts with imaginary couplings which would violate causality. If they did not decouple from all physical amplitudes the theory would be inconsistent. Thus, when we generalize the field equation, we must continue to have the same number of irrelevant degrees of freedom as in the linear theory. However, it is not necessary that the irrelevant degrees of freedom correspond to the same field deformations. For example, only abelian gauge fields have the same pure gauge deformations in the linear and nonlinear versions of the theory. Thus we can expect the pure gauge deformations in the nonlinear version of string field theory to have contributions which depend on the background value of the field.

Our experience with light cone string theory gives us insight into the manner in which nonlinearities enter string field theory. Consider, for example, the term that V_3^{open} would contribute to the string field equation for Ψ

$$\begin{aligned}\left[\Psi_{P^+}\big[\mathbf{x}(\sigma)\big], V_3^{open}\right] = g \sum_{0<P_1^+<P^+} \Psi_{P_1^+}[\mathbf{x}_1]\Psi_{P^+-P_1^+}[\mathbf{x}_2] + \\ g\sum_{Q^+>0}\int \mathcal{D}\mathbf{y}\big(\Psi^\dagger_{Q^+}[\mathbf{y}]\Psi_{P^++Q^+}[\mathbf{x}\circ\mathbf{y}] + \\ \Psi^\dagger_{Q^+}(\mathbf{y})\Psi_{P^++Q^+}[\mathbf{y}\circ\mathbf{x}]\big)\end{aligned} \tag{3.18}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two parts of the curve $\mathbf{x}$ in the sense that $\mathbf{x} = \mathbf{x}_1 \circ \mathbf{x}_2$. If we interpret the expression on the RHS as a rule for multiplying two string fields, we see that multiplication of string fields involves a much more complicated operation than ordinary multiplication. The second and third terms even involve a functional integral over $\mathbf{y}(\sigma)$.

In spite of the complexity of the product law implied by (3.18), it possesses one very fundamental, very stringy feature. To describe this feature let us first extend the definition to a product of two different string fields Ψ and Ψ':

$$\begin{aligned}(\Psi * \Psi')\big[\mathbf{x}(\sigma)\big] = \sum_{0<P_1^+<P^+} \Psi_{P_1^+}[\mathbf{x}_1]\Psi'_{P^+-P_1^+}[\mathbf{x}_2] + \\ g\sum_{Q^+>0}\int \mathcal{D}\mathbf{y}\big(\Psi'^\dagger_{Q^+}[\mathbf{y}]\Psi_{P^++Q^+}[\mathbf{x}\circ\mathbf{y}] + \\ \Psi^\dagger_{Q^+}[\mathbf{y}]\Psi'_{P^++Q^+}[\mathbf{y}\circ\mathbf{x}]\big).\end{aligned} \tag{3.19}$$

Now suppose we consider acting on a string field with a hermitian linear operation Ω which is an integral with respect to σ of a local polynomial $\omega(\sigma)$ of $\mathbf{x}(\sigma)$ and

$\delta/\delta\mathbf{x}(\sigma)$,

$$\Omega = \int_0^{\pi P^+} d\sigma\omega(\sigma). \tag{3.20}$$

Then (3.19) *formally* satisfies

$$\Omega\big(\Psi * \Psi'\big) = \big(\Omega\Psi\big) * \Psi' + \Psi * \big(\Omega\Psi'\big). \tag{3.21}$$

This is the property underlying Mandelstam's analysis of the behavior of the light cone three string vertex under Lorentz transformations.[71] In that analysis the role of Ω is played by the Lorentz generator. We stress that the fundamental property (3.21) may not hold exactly because of the singular nature of the joining point. Indeed, if it held exactly for all Ω we could use it to derive the absurd result that the vertex only connects states of equal P^-, since P^- can be expressed as the integral of a local density. The singular nature of the joining point allows violations of (3.21) which must be *localized at the joining point*.[71] Thus in the case of P^-, the violation is just a derivative with respect to the location of the joining point. Similarly for the case of Lorentz transformations, a literal application of (3.21) would lead one to conclude that the vertex is Lorentz *invariant*, when in reality it is *covariant*, i.e. it changes by a derivative with respect to the location of the joining point.

While the process of gauge fixing typically wrecks the gauge invariant structure of the linear term of the field equation, we can expect that less damage is done to the nonlinear terms. Thus we expect the nonlinearities in a gauge invariant field equation to bear some resemblance to the RHS of (3.18). Indeed one group of workers[21,76,77] take nonlinear terms to be exactly a covariantized version of this. Without deciding in advance which version of nonlinearity to pick, we shall nonetheless follow Witten and interpret the nonlinearities in terms of a new kind of string field product which associates with any pair of string fields A and B a third string field $A * B$. It is clear from the light cone example that this product need not be commutative, or even associative. Then we would write a nonlinear string field equation in the form

$$QA = g_1 A * A + g_2 A * A * A + \cdots \tag{3.22}$$

It is certainly clear that such an equation can have only a nonlinear gauge invariance, if any at all.

3.3. Witten's Open String Field Theory

In analogy with Yang Mills gauge invariance, Witten sought a string field equation with an invariance under the infinitesimal transformation

$$\Delta A = Q\Lambda + A * \Lambda - \Lambda * A. \tag{3.23}$$

He realized from the analogy that a quadratic field equation would be covariant under (3.23) if the product operation $*$ could be chosen to be associative and chosen so that Q behaves as a derivation with respect to it:

$$Q(A * B) = (QA) * B + (-)^A A * (QB). \tag{3.24}$$

Here the sign $(-)^A$ allows for the possibility that a string field be Grassmann odd; it appears here because Q is an odd operator. Gauge covariance is shown by a one line calculation

$$\begin{aligned}\Delta(QA + A * A) &= Q(A * \Lambda - \Lambda * A) \\ &\quad + Q\Lambda * A + A * Q\Lambda \\ &\quad + A * (A * \Lambda - \Lambda * A) + (A * \Lambda - \Lambda * A) * A \\ &= (QA + A * A) * \Lambda - \Lambda * (QA + A * A)\end{aligned} \tag{3.25}$$

during which we also learn that A must have an odd grading and Λ an even grading:

$$(-)^A = -1 \qquad\qquad (-)^\Lambda = +1. \tag{3.26}$$

Equation (3.25) shows that the field equation

$$QA + A * A = 0 \tag{3.27}$$

respects the gauge symmetry (3.23). From the light cone example and the fact that Q can be expressed as an integral of a local density, the derivation property is fairly easy to arrange with perhaps some delicacy at joining points, provided the product is some kind of overlap. Associativity is trickier and has so far been satisfactorily arranged only for open string fields. It is widely believed that the underlying gauge invariance for closed string fields is quite different from the open string one, possibly based on a nonassociative string field product.[78,70,79] For the most part the remainder of this chapter will be devoted to realizations of gauge invariant open string field equations with some discussion of why the same ideas run into difficulty for closed strings.

The most straightforward way to construct product laws obeying the derivation property is to exploit the fact that Q generates world-sheet BRST transformations. To this end we associate a world sheet with the product law by inserting an evolution operator $\exp\{-T\hat{L}_0\}$ on each string field and considering

$$e^{-T_3\hat{L}_0}\big((e^{-T_1\hat{L}_0}A) * (e^{-T_2\hat{L}_0}B)\big). \tag{3.28}$$

Then each evolution operator can be represented as a world-sheet path integral over a rectangular strip and the product operation joins the three strips together into a single world sheet. The Polyakov path integral on this world sheet possesses BRST invariance, which means that the sum of the three path integrals with Q applied in turn to each of the external strips vanishes. But this is precisely what is needed for the derivation property.

To construct a suitable product, one simply defines the path integral on the world sheet of (3.28) to be BRST invariant and then takes $T_i \to 0$. Proper inclusion of the world sheet ghosts is, of course, essential. It is simplest to employ the reparametrization ghosts in their bosonized form.[80] The world-sheet action for bosonized ghosts includes a term coupling the world-sheet ghost field $g(\sigma, \tau)$ to the world-sheet curvature.

$$S_{ghost} = \frac{1}{2\pi} \int_0^\pi dA G^{ab} \partial_a g \partial_b g + \frac{3i}{2\pi} \left[\frac{1}{2} \int dA R g + \int_\partial d\ell k g \right] \tag{3.29}$$

In this formula, G_{ab} is the metric tensor on the curved world sheet, dA is the element of area, $d\ell$ is the element of length on the boundary, R is the scalar curvature normalized to be 2 for the unit sphere, and k is the extrinsic geodesic curvature of the boundary. Note that for constant $g(\sigma)$ the coefficient of the linear term in g is just proportional to the Euler character

$$\chi = \frac{1}{2\pi} \left[\frac{1}{2} \int dA R g + \int_\partial d\ell k g \right]$$

of the world sheet. As discussed in Appendix A, this feature ensures that the energy momentum tensor is defined at the boundary. The couplings of g to curvature are crucial for world-sheet BRST invariance. Because the world sheet of (3.28) has a Euclidean metric everywhere except at points where the three rectangles come together, the curvature R will be proportional to a delta function, i.e. these points will have conical singularities. The coefficient of the delta function is determined by the strength of the conical singularity, which in turn is determined by the angular deficit or excess one acquires in circumnavigating the singularity.

Let us first consider a product of open string fields patterned after the light cone example. In this case the width of one of the three rectangles is the sum of the widths of the other two. In the context of covariant string field theory the values of the widths are spurious parameters which should disappear from any physical quantity. The world sheet used in (3.28) is just that of Figure 6. The only nonzero curvature of this diagram is the sharp bend in the boundary at the interaction point. The bend forms an angle of 2π radians, an excess of π over the unbent boundary. A generalization of this type of product ties three rectangles of arbitrary width together as in Figure 7. In this case there are three spurious length parameters. Now the only nonzero curvature is a conical singularity where the three strings meet. In circumnavigating this point one subtends an angle of 3π radians, an excess of π radians over the zero curvature situation.

More generally, one can note that the Euler character, for the surface constructed by gluing three rectangles together so that there is only one boundary, is the same as a rectangle or disk, namely one. But there are $3/2$ times as many right angle corners at the outgoing ends of the three rectangles as for a single rectangle. The excess contribution to χ must be cancelled by the curvature contribution in the internal part of the diagram. If this is a single conical singularity or bent boundary, its contribution to the Euler character must be $-1/2$, i.e. an angular excess of π. As one can see from (3.29) the net effect for all styles of vertices is an insertion in the path integral of $e^{3ig(P)/2}$ at the point where the three rectangles come together. This is for the ordinary reparametrization ghosts. For the spinning string one also has superghosts, described in "bosonized" language by two world sheet scalar fields ϕ and χ. They also have couplings to the world sheet curvature

$$
\begin{aligned}
S_{superghosts} = \frac{1}{2\pi} \int dA G^{ab} (\partial_a \chi \partial_b \chi - \partial_a \phi \partial_b \phi) \\
+ \frac{1}{2\pi} \left[\frac{1}{2} \int dA \ R(\chi + 2\phi) + \int d\ell \ k(\chi + 2\phi) \right]
\end{aligned}
\tag{3.30}
$$

which lead to the additional insertion $e^{i\phi(P) + i\chi(P)/2}$.

We finally specialize to Witten's proposal for the product of open string fields. As already mentioned, he was motivated to choose a product which was associative. The product represented by Figure 7 can only be associative if either all widths are integrated over, or all widths are equal. It is more efficient to refrain from introducing a whole lot of spurious parameters in the theory, which argues for the second alternative, and this was Witten's proposal (see Figure 8). The more general product law of Figure 7 might be viable in a theory with an enlarged gauge invariance to handle the extra spurious parameters. Some progress along these lines has been made[81,77,82,83].

We have seen that a product law for which Q is a derivation inevitably requires an insertion of the exponential of bosonized ghost fields. In the case of the open bosonic string this insertion adds 3/2 units of ghost number. Thus the ghost number of $A * B$ is the sum of those of A and B *plus* 3/2. As we have already noted in Section 2, this is a precisely what is needed to make the ghost numbers of the linear and nonlinear terms in the field equation match if the ghost number of A is $-1/2$. This is desirable since we have shown that the cohomology of Q lies entirely in this ghost number sector. For the case of the open spinning string the insertion carries total ghost number $+1/2$. This would seem to dictate that the ghost number of the boson component of the string field be $+1/2$. This sector contains the cohomology of Q in the 0 picture, which would therefore seem to be preferred. However, one can, as Witten did, work in the simpler -1 picture if an additional insertion of ghost number $+1$ is included in the product law. This insertion must be BRST invariant, and as Witten pointed out, the picture changing operator X (defined in Appendix A) fits the bill.

The spinning string field is described by a pair (A, Ψ) with A the bosonic and Ψ the fermionic component. Let us first consider A in the 0 picture so A has ghost number $+1/2$. We shall also take Ψ in the $-1/2$ picture so it has ghost number 0. To generalize the linear equation for A to one at most quadratic in the string fields, we must add to QA quadratic terms of ghost number 3/2. As we have mentioned, the simple Witten product involves a ghost number $+1/2$ insertion so $A * A$ has the right ghost number. On the other hand, $\Psi * \Psi$ has ghost number $+1/2$, and its contribution must include a ghost number $+1$ insertion.

The natural choice is $X(\frac{\pi}{2})$ at the vertex midpoint. Thus the nonlinear field equation becomes

$$QA + A * A + X(\Psi * \Psi) = 0.$$

For the fermionic component, we can add $A * \Psi$ or $\Psi * A$ to $Q\Psi$:

$$Q\Psi + A * \Psi + \Psi * A = 0.$$

The gauge transformations for these field equations are summarized by

$$\begin{aligned}\Delta A &= Q\Lambda + A * \Lambda - \Lambda * A + X(\Psi * \chi - \chi * \Psi) \\ \Delta\Psi &= Q\chi + A * \chi - \chi * A + \Psi * \Lambda - \Lambda * \Psi\end{aligned} \tag{3.31}$$

under which

$$\begin{aligned}&\Delta\big(QA + A * A + X(\Psi * \Psi)\big) \\ &\ = \big[QA + A * A + X(\Psi * \Psi), \Lambda\big] + X\big([Q\Psi + A * \Psi + \Psi * A, \chi]\big) \\ &\Delta(Q\Psi + A * \Psi + \Psi * A) \\ &\ = [Q\Psi + A * \Psi + \Psi * A, \Lambda] + [QA + A * A + X(\Psi * \Psi), \chi]\end{aligned} \tag{3.32}$$

so the field equations are gauge covariant. To obtain this it is essential that X is inserted at the midpoint because otherwise it could not be moved around in the cubic terms. Even so, the manipulations are somewhat formal because X collides with the ghost insertions included in the definition of the $*$ product. In addition, associativity of the $*$ product must be assumed to obtain gauge covariance.

In his original paper, Witten chose to put A in the -1 picture, for which it carries ghost number $-1/2$. Then $A*A$ carries ghost number $-1/2$ which doesn't agree with the ghost number of QA. On the other hand, $\Psi * \Psi$ does have ghost number +1/2 so in this picture the field equation for A reads

$$QA + X(A * A) + \Psi * \Psi = 0. \tag{3.33}$$

Further $A * \Psi$ and $\Psi * A$ have the ghost number of Ψ so an insertion of X is again required in the field equation for Ψ:

$$Q\Psi + X(A * \Psi + \Psi * A) = 0. \tag{3.34}$$

The gauge transformation law similarly has X inserted in different terms:

$$\begin{aligned} \Delta A &= Q\Lambda + X(A * \Lambda - \Lambda * A) + \Psi * \chi - \chi * \Psi \\ \Delta\Psi &= Q\chi + X(A * \chi - \chi * A) + X(\Psi * \Lambda - \Lambda * \Psi). \end{aligned} \tag{3.35}$$

In this picture, the manipulations required to demonstrate gauge covariance of the string field equations are even more problematic, because, as Wendt has pointed out,[84] the product of two X's is singular and the gauge transform of the field equations puts two X's on top of each other.

3.4. Action Principles

So far we have only considered the nonlinear string field equations. To quantize them we must have an action principle from which they can be derived. We wrote down action principles for the linear field equations in Chapter 2. Here we consider how to construct action principles in the nonlinear case.

We begin by defining integration for string fields. That is, we want a linear rule for turning a string field into a complex number:

$$A \to \int A.$$

To motivate the choice, let us note that for the free bosonic string field we found the action $\langle \mathcal{V}_2 \mid A\rangle \, Q \, |A\rangle$ which we would like to interpret as

$$\int A * QA = \langle \mathcal{V}_2 \mid A\rangle \, Q \, |A\rangle$$

Since $\langle \mathcal{V}_2 \mid A\rangle \, |B\rangle$ is defined to be essentially

$$\langle B \mid A\rangle = \int dx(\sigma) B^*(x(\sigma)) A(x(\sigma))$$

and $A * B$ identifies and integrates only half of the $x(\sigma)$'s $(0 \leq \sigma \leq \frac{\pi}{2})$ we see that $\int$ should identify and integrate the other half of the $x(\sigma)'s$. Thus, for a string field C that is not a product, $\int$ should identify $x(\pi - \sigma)$ with $x(\sigma)$ for $0 < \sigma < \frac{\pi}{2}$. It should also be BRST invariant in the sense that

$$\int QC = 0 \qquad \text{for all } C. \tag{3.36}$$

To assure this latter property we again use the BRST invariant world sheet path integration to define $\int e^{-T\hat{L}_0} C$ in terms of a path integral over the rectangle shown in Figure 9. After identification of A and A' the surface has a conical singularity at V with deficit angle π. This corresponds to a curvature delta function of opposite strength to that associated with the $*$ product. Thus the bosonized ghost lagrangian dictates a ghost number -3/2 insertion $e^{-3ig(V)/2}$ at V. We then define $\int C$ as the $T \to 0$ limit of this path integral. For $\int A * B$, we see that the ghost insertion associated with $\int$ exactly cancels that associated with $*$, so we have the desired identity $\int A * B = \langle \mathcal{V}_2 \mid A\rangle \, |B\rangle$ Further, the insertion $e^{-3ig(V)/2}$ implies that $\int C = 0$ unless C has ghost number $+3/2$.

We have found, in discussing gauge invariance of the bosonic string field, that the classical string field A must have odd grading in (3.23). Since A is a linear combination of single string states with ghost number -1/2 with c-number coefficients, we must assign odd grading to all kets with ghost number -1/2. Then since in the bosonic string only b's and c's carry ghost number, all kets with ghost number $2n-1/2$ can be obtained from ghost number -1/2 states by applying an even number of b's and/or c's and so will have odd grading. Similarly those with ghost number in $2n+1/2$ have even grading. In summary for the bosonic string, the grading of each ket is given by $(-)^{G-1/2}$. For the spinning string the superghosts β and γ carry ghost number and are bosons, so there is no longer a rigid linkage between total ghost number and grading. In addition, the spin variables $\psi^\mu(\sigma)$ are Grassmann odd, so the relative grading of kets is determined by the sign of

$$(-)^{g_0+1/2+N_\psi} = (-)^{G+1/2-\phi_0-N_\psi}.$$

In the NS sector

$$(-)^{N_\psi} = (-)^{\sum_{r>0}\psi_{-r}\cdot\psi_r},$$

but in the R sector it is

$$(-)^{\sum_{n>0}\psi_{-n}\cdot\psi_n}\Gamma_{11},$$

where $\Gamma_{11}=\gamma^0\gamma^1\cdots\gamma^9$ is the ten dimensional version of γ_5. The coefficients in the expansion of string fields in kets are generally allowed to be either c-numbers or anticommuting Grassmann numbers (a-numbers). So again the total grading of a given string field is not limited by the ghost number constraint.

A fundamental property of the integration operation, needed to prove gauge invariance of action principles, is

$$\int A*B = (-)^{AB}\int B*A \tag{3.37}$$

For the bosonic string field this follows directly from the symmetry $\langle\mathcal{V}_2 \mid s_1\rangle\,|s_2\rangle = \langle\mathcal{V}_2 \mid s_2\rangle\,|s_1\rangle$ for basic kets $\mid s_1\rangle$ and $\mid s_2\rangle$. The sign is right because $\int B*A\neq 0$ only if the ghost number of $\mid s_1\rangle$ is minus that of $\mid s_2\rangle$ which means that $\mid s_1\rangle$ and $\mid s_2\rangle$ have opposite gradings. For the classical spinning string field both A and Ψ must have odd grading, but A has c-number coefficients and Ψ has a-number coefficients (since it is a space-time fermion) so the kets in A and Ψ must have opposite grading.

Since the kets contributing to a given string field have relative grading

$$(-)^{G-1/2+\phi_0+N_\psi},$$

the condition that A and Ψ have odd grading requires this sign to be constant for all contributing kets. Since the classical string fields are restricted to have fixed total ghost number it follows that contributing kets to the classical component must have constant $(-)^{\phi_0+N_\psi}$. In the NS sector this translates to kets with *either* integral total mode number or half odd integral total mode number, but not both. In the R sector it translates to a restriction on the chirality of the ground state spin 1/2 state such at $(-)^{\phi_0+\sum_{n=1}^{\infty}\psi_{-n}\cdot\psi_n}\Gamma_{11}$ has a fixed sign. These restrictions precisely correspond to the GSO projection[85] necessary for a supersymmetric spectrum. The overall grading of the kets contributing to A and Ψ is fixed by requiring that the ket corresponding to the gauge boson in A has odd grading and the ket corresponding to the chosen spinor ground state chirality in Ψ has even grading.

The integration operation for the spinning string is associated with a path integral over the same region (see Figure 9) as in the bosonic case. However, now the ghost insertion factor is $e^{-3ig/2-i\chi/2-i\phi}$ which carries total ghost number -1/2. Thus this definition would only give a nonvanishing result when the integrand has ghost number + 1/2. In the 0 picture, we would like a nonzero result for $\int$(insertion)$A * QA$, but $A * QA$ has ghost number + 5/2. Thus we require a ghost number -2 insertion Y_{-2} to get a nonzero action:

$$\int : \qquad C \to \int Y_{-2}(\frac{\pi}{2})C$$

Of course this vanishes unless C has ghost number +5/2.

In the simpler -1 picture, $X(A*QA)$ and $\Psi * Q\Psi$ have ghost number $+3/2$, so to get a nonzero free action one has to supply a ghost number -1 insertion Y_{-1}. As defined in Appendix A, $Y_{-1}X \to 1$ so the free action for spinning string fields in the -1 picture is just

$$\frac{1}{2}\int A * QA + \frac{1}{2}\int Y_{-1}\Psi * Q\Psi.$$

We are now in a position to write down the string field actions for non-linear open string field theory. For the open bosonic string we have

$$S = \frac{1}{2}\int A * QA + \frac{1}{3}\int A * A * A. \tag{3.38}$$

To check the equations of motion we substitute $A \to A + \delta A$ and require that S be stationary

$$\begin{aligned} O = \delta S &= \frac{1}{2}\int \delta A * QA + \frac{1}{2}\int A * Q\delta A \\ &\quad + \frac{1}{3}\int (\delta A * A * A + A * \delta A * A + A * A * \delta A) \\ &= \int \delta A * (QA + A * A), \end{aligned}$$

where use has been made of associativity, the symmetry (3.37), and (3.36). Using these same properties one can also easily show that S is invariant under nonlinear gauge transformations.

Similarly, putting aside problems with the X insertions, the spinning string field actions should be

$$S_0 = \frac{1}{2}\int Y_{-2}A*QA + \frac{1}{3}\int Y_{-2}A*A*A + \frac{1}{2}\int Y_{-1}\Psi*Q\Psi + \int Y_{-1}A*\Psi*\Psi \tag{3.39}$$

and

$$S_{-1} = \frac{1}{2}\int A * QA + \frac{1}{3}\int XA * A * A + \frac{1}{2}\int Y_{-1}\Psi * Q\Psi + \int A * \Psi * \Psi \tag{3.40}$$

for the A field in 0 and -1 pictures respectively. These actions are formally gauge invariant if one assumes that the insertions can be moved around freely and if one ignores the fact that $X \cdot X$ has a singular short distance expansion. In the 0 picture these manipulations require $Y_{-2}X = Y_{-1}$ to get formal gauge invariance, whereas in -1 picture the required identity is $Y_{-1}X = 1$.

In considering the requirement of odd grading for the spinning string fields A and Ψ we were naturally led to the GSO restriction on the kets contributing to these string fields. Thus the string field theory described by S_0 or S_{-1} should correspond to the superstring, i.e. S_0 and S_{-1} should process supersymmetry. In his original paper, Witten identified, at least formally, this symmetry. In the -1 picture in which he worked, his supersymmetry transformation is

$$\begin{aligned} \delta A &= \epsilon^\alpha Q_\alpha \Psi \\ \delta\Psi &= X(\frac{\pi}{2})\epsilon^\alpha Q_\alpha A \end{aligned} \tag{3.41}$$

where Q_α is the integral of the zero momentum fermion emission vertex operator

$$\begin{aligned} Q_\alpha &= \int_0^\pi d\sigma V_\alpha(\sigma; k=0) \\ &= \int_0^\pi d\sigma e^{i\phi/2} S_\alpha \end{aligned}$$

where S_α, the "matter" part of the vertex operator, is just the spin variable part of the fermion vertex operator constructed long ago,[27,86] and has the intertwining property of changing the boundary condition obeyed by ψ^μ from Neveu-Schwarz to Ramond and *vice versa*. S_α is a Lorentz spinor with conformal weight + 5/8. The ghost factor $e^{i\phi/2}$ carries ghost number -1/2 and conformal weight 3/8, and is the necessary factor to make the emission vertex BRST invariant.[68,87] The emission vertex V_α is Grassmann odd as can be inferred from the operator product expansion $V_\alpha(z)V_\beta(0)$.[68] Thus ϵ^α must be odd as is appropriate for a supersymmetry transformation. The picture changing operator in the second of (3.41) is to balance ghost numbers on both sides: To use X for this purpose, it is essential that it commute with Q_α which holds becausee Q_α is BRST invariant and does not contain any η operators.

The proof that (3.41) is indeed a symmetry of S_{-1} utilizes the fact that the linear operator $\epsilon^\alpha Q_\alpha$, being the integral of a conserved local BRST invariant current operator on the world sheet, enjoys the properties of a derivation for the $*$ algebra

$$\int \epsilon^\alpha Q_\alpha B = 0$$

and

$$\epsilon^\alpha Q_\alpha (B * C) = (\epsilon^\alpha Q_\alpha B) * C + B * \epsilon^\alpha Q_\alpha C.$$

The rigor of this proof is clouded by the problems mentioned earlier with the operator insertions at the midpoint. Although these seem formally harmless since X commutes with Q_α, singular operator products could spoil the argument.

In the 0 picture the supersymmetry transformation is similar to (3.41) but to balance ghost numbers the picture changing insertion appears in a different place:

$$\begin{aligned}\delta A &= X(\frac{\pi}{2})\epsilon^\alpha Q_\alpha \Psi \\ \delta\Psi &= \epsilon^\alpha Q_\alpha A.\end{aligned}$$

In all of our discussion of open superstring field theory we have ignored subtleties associated with the presence of operator insertions at the midpoint. Recent work[73,84] indicates that these insertions cause problems which suggest the need for the introduction of singular counter terms even at the tree (classical) level. These problems have not yet been resolved definitively, and there is clearly need for more study of them. We shall return to these issues in the concluding chapter where we outline the open problems for string field theory.

4. Quantization of Interacting Open String Field Theory

In the previous chapter we discussed the problem of formulating a gauge invariant action principle for non-linear string field equations. In this chapter we turn to the problem of quantizing such theories. We shall restrict attention to Witten's open string field theory, since it is the simplest and most definitive classical string field theory proposed so far. Most of the detailed discussion will be further restricted to the bosonic open string, with an abbreviated discussion of the open spinning string at the end. Our approach to quantization will be via path integrals. Then the central problem is defining a gauge-fixing procedure which allows path integrals to be well-defined and doesn't alter the physical content of the gauge invariant theory. Once a suitable gauge fixed action has been derived, standard manipulations yield the Feynman rules for the loop expansion of the theory. This perturbation theory will be extensively discussed in the following chapter.

From the work of chapter 3 we have seen that Witten's action for the bosonic open string field can be written

$$S = \frac{1}{2}\int A * QA + \frac{1}{3}\int A * A * A \tag{4.1}$$

where A is a string field carrying ghost number -1/2, and S is invariant under the gauge transformation

$$\Delta A = Q\Lambda + A * \Lambda - \Lambda * A. \tag{4.2}$$

At first sight, this gauge invariance resembles Yang-Mills gauge invariance so closely that one might expect the standard Fadeev-Popov gauge fixing procedure to be applicable. However, even the linearized gauge transformation suffers from the problem of redundancy. Because $Q^2 = 0$, the gauge function $\Lambda = QK$ does not change A in the linearized approximation. This means that the gauge invariance is somewhat smaller than would appear from (4.2), and this makes the problem of gauge fixing subtle. Normally, a gauge fixing condition on A,

$$\phi(A) = 0$$

would be imposed by inserting

$$1 = \Delta_{FP}(A) \int \mathcal{D}\Lambda \delta(\phi(A_\Lambda))$$

into the path integral, changing variables $A \to A_{\Lambda^{-1}}$, and factoring out the volume of the gauge group $\int \mathcal{D}\Lambda$. This procedure fails because Λ gives a redundant parametrization of the gauge group. Moreover, it is not simple to characterize this redundancy. Even at the linear level the redundant gauge transformation $\Lambda = QK$, is redundantly parameterized. For $K = QK'$, $QK = 0$, so these K's do not give a redundancy. Clearly this redundancy of description never ends because at each stage the redundancy is parameterized by a gauge transformation of the same form as the previous stage.

4.1. Quantization of Free String Fields

Before discussing the nonlinear case it is instructive to first resolve the difficulty at the linearized level. For this purpose, we specialize to the simple gauge fixing condition advocated by Siegel:

$$b_0 A = 0. \tag{4.3}$$

We can isolate the redundancy problem by examining the first stage Fadeev-Popov determinant

$$\det(b_0 Q) = \int dB' dC \exp \int B' * b_0 QC \tag{4.4}$$

where we have introduced a ghost field C corresponding to Λ and an auxiliary "anti-ghost" field B' to represent the determinant by a path integral. The path integral may be regarded as an integral over the coefficients in the state representations of the string fields C and B':

$$\begin{aligned} C &= \sum_{G_s=-3/2} |s\rangle\, c_s \\ B' &= \sum_{G_s=+1/2} |s\rangle\, b_s. \end{aligned} \tag{4.5}$$

Of course c_s and b_s should be Grassmann anti-commuting numbers so that the path integral does indeed reproduce the determinant of b_0Q. According to the grading assignments discussed in the previous chapter, $|s\rangle$ has grading $(-)^{G_s-1/2}$ so the kets appearing in C have even and those in B' have odd grading. But since the c_s, b_s are odd, C has odd grading and B' has even grading.

The problem of redundancy is apparent in (4.4). The functional integral over B' and C is divergent because the exponent is invariant under $\Delta C = QK$ and $\Delta B' = b_0 L$. The problem with the B' integral is harmless: the integrals over the b_s for $b_0\,|s\rangle = 0$ simply give a divergent overall constant. Thus, one need only restrict the states appearing in B' to be those annihilated by c_0, or equivalently, write $B' = c_0 B$ with $b_0 B = 0$, so that the ghost Lagrangian is simply

$$\int c_0 B * b_0 QC = \int B * QC$$

where we have used the identity $\int c_0 A * B = -(-)^A \int A * c_0 B$ and the odd grading of B.

The invariance under $\Delta C = QK$ is a gauge invariance, since Q is a dynamical operator. This invariance is directly connected with the redundancy of the original gauge invariance. But the interpretation of the redundancy as a gauge invariance provides a resolution to the problem. One fixes this gauge invariance in the standard way by introducing further ghost for ghost fields. But the ghost for ghost Lagrangian then has a gauge invariance and so on. One ends up by introducing an infinite tower of ghost fields. To describe them, we need a new notation. The input string field A will be denoted $\Phi_{-1/2}$, the subscript indicating the ghost number. Then B and C will be named $\Phi_{+1/2}$ and $\Phi_{-3/2}$ respectively. Then, one clearly needs to introduce a string field Φ_n for each ghost number $-\infty < n < +\infty$. Each Φ_n can be given an expansion

$$\Phi_n = \sum_{G_s=n} |s\rangle\, \phi_s^n.$$

We have noted that $\phi_s^{-1/2}$ is an ordinary c number, whereas $\phi_s^{+1/2}, \phi_s^{-3/2}$, are anti-commuting c numbers (a numbers). The ghost fields for the gauge fixing of $\Phi_{-3/2}$ are $\Phi_{-5/2}$ and $\Phi_{+3/2}$, so, $\phi_s^{-5/2}$ and $\phi_s^{+3/2}$ are c numbers. The grading of ϕ_s^n alternates in step with the grading of the kets $|s\rangle$, so each Φ_n has an overall odd grading. Each Φ_n is subject to the condition

$$b_0 \Phi_n = 0. \tag{4.6}$$

For negative n these are just gauge conditions. For positive n they simply reflect the presence of b_0 in the F-P ghost Lagrangian.

We can summarize this heuristic discussion by unifying all the Φ_n in a single field

$$\Phi = \sum_{n=-\infty}^{\infty} \Phi_n$$

containing all ghost numbers, with odd grading. Then all of the FFP Lagrangians are included in the compact expression

$$\frac{1}{2}\int \Phi * Q\Phi.$$

The gauge condition (4.6) can be implemented by Lagrange multipliers

$$S_{G.F.} = \frac{1}{2}\int \Phi * Q\Phi + \int b_0 \beta * \Phi. \tag{4.7}$$

Integrating $e^{-S_{G.F.}}$ over all Φ and β then prescribes the free quantum string field theory.

4.2. The Batalin-Vilkovisky Formalism

To understand why the above procedure is correct, it is helpful to recall the general problem of defining functional integrals in the presence of gauge invariance. The following discussion reviews the work of Ref.[45]. One is typically given an input classical action S_0 depending on a number of field variables ϕ_a (with a labelling space-time cordinates as well as any internal quantum numbers),[88] and the problem is to give meaning to an integral

$$\int d\phi_a \exp\{-\frac{1}{\hbar} S_0(\phi_a)\}, \tag{4.8}$$

even though S_0 is independent of an infinite number of the ϕ_a. The goal is to replace (4.8) with an alternative one with the same physical content (i.e. giving the same correlators of gauge invariant dynamical variables) but which is convergent.

We are accustomed to introducing auxiliary variables in gauge theories, so augment the set of ϕ_a to include a host of ghost variables not present in the input classical system. For the free string field theory discussed earlier the ϕ_a would include all of the Φ_n for $n < 0$, whereas the input classical theory only had $\Phi_{-1/2}$. For each ϕ_a one introduces an "antifield" ϕ_a^* with opposite statistics to ϕ_a. Then we replace S_0 with a function $\Gamma(\phi, \phi^*)$, with the crucial property

$$\Gamma(\phi, 0) = S_0(\phi) \tag{4.9}$$

where the auxiliary ϕ_a are not present in S_0.

The idea is to arrange the dependence of $\Gamma(\phi, \phi^*)$ on ϕ^* so that when ϕ^* is fixed by a gauge condition of the form

$$\phi_a^* = \frac{\partial \Psi}{\partial \phi_a}$$

with Ψ the so called gauge fermion, then

$$Z_\Psi = \int d\phi_a \exp\{-\frac{1}{\hbar}\Gamma(\phi, \frac{\partial\Psi}{\partial\phi})\} \tag{4.10}$$

is independent of Ψ. For $\Psi = 0$, this expression is, of course, just an (infinite) constant times the ill defined (4.8). This, together with the independence of Z_Ψ on Ψ, justifies the claim that Z_Ψ has the same physical content as (4.8). One hopes that for a wide class of Ψ the resulting path integral is well-defined.

We now turn to the condition on Γ that ensures that (4.10) is independent of Ψ. We calculate

$$\begin{aligned}\delta Z_\Psi &= \int d\phi_a \sum_b \frac{\partial\delta\Psi}{\partial\phi_b}\left(\frac{\partial}{\partial\phi_b^*}e^{-\Gamma(\phi,\phi^*)/\hbar}\right)_{\phi^*=\frac{\partial\Psi}{\partial\phi}} \\ &= -\int d\phi_a \delta\Psi \sum_b (-)^{\phi_b}\left(\frac{\partial}{\partial\phi_b}\frac{\partial}{\partial\phi_b^*}e^{-\Gamma/\hbar} + \sum_c (\frac{\partial}{\partial\phi_b}\frac{\partial\Psi}{\partial\phi_c})\frac{\partial}{\partial\phi_c^*}\frac{\partial}{\partial\phi_b^*}e^{-\Gamma/\hbar}\right)\end{aligned} \tag{4.11}$$

Now examine

$$\sum_{b,c}(-)^{\phi_b}(\frac{\partial}{\partial\phi_b}\frac{\partial\Psi}{\partial\phi_c})\frac{\partial}{\partial\phi_c^*}\frac{\partial}{\partial\phi_b^*}e^{-\Gamma/\hbar}.$$

If ϕ_b and ϕ_c have the same statistics, ϕ_c^* and ϕ_b^* have the opposite statistics, so $\frac{\partial^2}{\partial\phi_b\partial\phi_c}$ and $\frac{\partial^2}{\partial\phi_c^*\partial\phi_b^*}$ have opposite symmetry under $b \leftrightarrow c$ so all these terms drop out. If ϕ_b and ϕ_c have opposite statistics, $\frac{\partial^2}{\partial\phi_b\partial\phi_c}$ and $\frac{\partial^2}{\partial\phi_c^*\partial\phi_b^*}$ are both symmetric under $b \leftrightarrow c$ but $(-)^{\phi_b} = -(-)^{\phi_c}$ so these terms all cancel in pairs. Thus

$$\delta Z_\Psi = -\int d\phi_a \delta\Psi \sum_b (-)^{\phi_b}\left(\frac{\partial}{\partial\phi_b}\frac{\partial}{\partial\phi_b^*}e^{-\Gamma/\hbar}\right)_{\phi^*=\frac{\partial\Psi}{\partial\phi}}$$

vanishes if Γ satisfies the equation

$$\sum_b (-)^{\phi_b}\frac{\partial}{\partial\phi_b}\frac{\partial}{\partial\phi_b^*}e^{-\Gamma/\hbar} = 0$$

or

$$\sum_b (-)^{\phi_b} \frac{\partial \Gamma}{\partial \phi_b} \frac{\partial \Gamma}{\partial \phi_b^*} = \hbar \sum_b (-)^{\phi_b} \frac{\partial}{\partial \phi_b} \frac{\partial}{\partial \phi_b^*} \Gamma \tag{4.12}$$

This equation is frequently too difficult to solve in closed form. In that case one can try expanding Γ in a power series in $\hbar$:

$$\Gamma = \sum_{n=0}^{\infty} \hbar^n \Gamma_n. \tag{4.13}$$

The zeroth order approximation Γ_0 then must satisfy the master equation

$$\sum_b (-)^{\phi_b} \frac{\partial \Gamma_0}{\partial \phi_b} \frac{\partial \Gamma_0}{\partial \phi_b^*} = 0. \tag{4.14}$$

Then one can try to determine the Γ_n iteratively. For example Γ_1 satisfies

$$\sum_b (-)^{\phi_b} \left(\frac{\partial \Gamma_1}{\partial \phi_b} \frac{\partial \Gamma_0}{\partial \phi_b^*} + \frac{\partial \Gamma_0}{\partial \phi_b} \frac{\partial \Gamma_1}{\partial \phi_b^*} \right) = \hbar \sum_b (-)^{\phi_b} \frac{\partial}{\partial \phi_b} \frac{\partial}{\partial \phi_b^*} \Gamma_0.$$

If one is lucky the RHS of this equation is zero and $\Gamma_n = 0$ for $n > 0$ gives a solution. Then the quantization procedure outlined above would at least be gauge invariant, although there is no guarantee that the Γ_n *must* be zero.

We have described a very general formulation of quantum gauge theories. The gauge fixing procedure is guaranteed to be gauge independent for any Γ solving (4.12). We have not given a method for finding Γ. Nor have we given a general prescription for what auxiliary ghost fields must be adjoined to the input classical variables to render the path integral convergent. Batalin and Vilkovisky have given a thorough analysis of these issues for most situations that arise in quantum field theory. Their methods apply to situations with open gauge algebras as well as those with redundant parametrizations of the gauge symmetry, such as we have encountered in string field theory. These methods are so general that they do not take advantage of the simplifying features of any specific dynamical system. For the case of string field theory, it is not hard to guess that the ϕ_a should include the coefficients ϕ_s^n in the string fields Φ_n for $n < 0$. Then one can make an ansatz for Γ_0, the $\hbar \to 0$ limit of Γ, which can be easily shown to solve the master equation.

Before turning to string field theory, we pause to make contact with the BRST method of quantization. We have already reviewed in section 2.1 the basic ingredients of the BRST method within the operator formulation of quantum mechanics. The central feature is the existence of a nilpotent fermionic Q which commutes with all physical observables, and it therefore generates a global fermionic symmetry of the theory. In the Lagrangian path integral formulation of quantum mechanics there should be a corresponding global fermionic symmetry respected by the path integral. That is, a transformation $\phi_a \to s\phi_a$ which leaves the path integral unchanged. This symmetry is given by

$$s\phi_a = (-)^{\phi_a} \frac{\partial \Gamma}{\partial \phi_a^*}\bigg|_{\phi^* = \partial\Psi/\partial\phi.} \tag{4.15}$$

To see this, simply compute

$$\begin{aligned} se^{-\Gamma/\hbar} &= -\frac{1}{\hbar} e^{-\Gamma/\hbar} \sum_a s\phi_a \left(\frac{\partial \Gamma}{\partial \phi_a} + \sum_b \frac{\partial^2 \Psi}{\partial \phi_a \partial \phi_b} \frac{\partial \Gamma}{\partial \phi_b^*} \right) \\ &= -\frac{1}{\hbar} e^{-\Gamma/\hbar} \sum_a (-)^{\phi_a} \frac{\partial \Gamma}{\partial \phi_a^*} \frac{\partial \Gamma}{\partial \phi_a} - \frac{1}{\hbar} e^{-\Gamma/\hbar} \sum_{a,b} (-)^{\phi_a} \frac{\partial \Gamma}{\partial \phi_a^*} \frac{\partial^2 \Psi}{\partial \phi_a \partial \phi_b} \frac{\partial \Gamma}{\partial \phi_b^*}. \end{aligned}$$

The second term vanishes because the summand is odd under $a \leftrightarrow b$, and the first term can be rewritten using (4.12):

$$se^{-\Gamma/\hbar} = -e^{-\Gamma/\hbar} \sum_a (-)^{\phi_a} \frac{\partial}{\partial \phi_a} \frac{\partial}{\partial \phi_a^*} \Gamma.$$

Thus the integrand of the functional integral transforms under

$$\phi_a \to \phi_a + \theta s \phi_a, \tag{4.16}$$

where θ is a constant Grassmann number, as

$$e^{-\Gamma/\hbar} + \theta s e^{-\Gamma/\hbar} = \Big(1 - \theta \sum_a (-)^{\phi_a} \frac{\partial}{\partial\phi_a}\frac{\partial}{\partial\phi_a^*}\Gamma\Big) e^{-\Gamma/\hbar}.$$

The prefactor is just the inverse of the Jacobian for the change of variables (4.16):

$$\begin{aligned}
\mathrm{s}\det\big(\frac{\partial(\phi_a + \theta s\phi_a)}{\partial\phi_b}\big) &= 1 + \sum_a (-)^{\phi_a}\frac{\partial}{\partial\phi_a}(\theta s\phi_a) \\
&= 1 + \theta \sum_a (-)^{\phi_a}\frac{\partial^2\Gamma}{\partial\phi_a\partial\phi_a^*}
\end{aligned}$$

where $\mathrm{s}\det$ stands for the superdeterminant. Thus, writing $\bar\phi_a = \phi_a + \theta s\phi_a$, the path integral satisfies the invariance

$$\int d\bar\phi_a e^{-\Gamma(\bar\phi_a)/\hbar} = \int d\phi_a e^{-\Gamma(\phi_a)/\hbar}. \tag{4.17}$$

Contrary to common belief, s need not be nilpotent off shell. Indeed, calculating s^2 on ϕ_a we find

$$s^2\phi_a = s(s\phi_a) = (-)^{\phi_a}\sum_b (-)^{\phi_b}\frac{\partial\Gamma}{\partial\phi_b^*}\Big(\frac{\partial^2\Gamma}{\partial\phi_b\partial\phi_a^*} + \sum_c \frac{\partial^2\Psi}{\partial\phi_b\partial\phi_c}\frac{\partial^2\Gamma}{\partial\phi_c^*\partial\phi_a^*}\Big).$$

Differentiating (4.12) with respect to ϕ^* yields the identity

$$\sum_b (-)^{\phi_b}\frac{\partial\Gamma}{\partial\phi_b^*}(-)^{\phi_a^*\phi_b^*}\frac{\partial^2\Gamma}{\partial\phi_a^*\partial\phi_b} + \sum_b (-)^{\phi_b}\frac{\partial^2\Gamma}{\partial\phi_a^*\partial\phi_b^*}\frac{\partial\Gamma}{\partial\phi_b} = \hbar\frac{\partial}{\partial\phi_a^*}\sum_b (-)^{\phi_b}\frac{\partial^2\Gamma}{\partial\phi_b\partial\phi_b^*}.$$

Using $(-)^{\phi_a^*\phi_b^*}\frac{\partial^2\Gamma}{\partial\phi_a^*\partial\phi_b} = (-)^{\phi_a^*}\frac{\partial^2\Gamma}{\partial\phi_b\partial\phi_a^*}$ we can use this identity to obtain

$$s^2\phi_a = \sum_b (-)^{\phi_b}\frac{\partial^2\Gamma}{\partial\phi_a^*\partial\phi_b^*}\Big(\frac{\partial\Gamma}{\partial\phi_b} + \sum_c \frac{\partial^2\Psi}{\partial\phi_b\partial\phi_c}\frac{\partial\Gamma}{\partial\phi_c^*}\Big) - \hbar\sum_b (-)^{\phi_b}\frac{\partial}{\partial\phi_a^*}\frac{\partial^2\Gamma}{\partial\phi_b\partial\phi_b^*}.$$

Notice that the terms enclosed in parentheses in the first sum vanish if the equations of motion derived from Γ are satisfied. Of course inside the path integral, the equations of motion are not satisfied. Instead they can be replaced by a derivative of $e^{-\Gamma/\hbar}$. The term proportional to $\hbar$ (which is present because of the RHS of (4.12)) is then just what is needed so that

$$\begin{aligned} s^2\phi_a e^{-\Gamma/\hbar} &= -\hbar\sum_b(-)^{\phi_b}\frac{\partial^2\Gamma}{\partial\phi_a^*\partial\phi_b^*}\frac{\partial}{\partial\phi_b}e^{-\Gamma/\hbar} - \hbar e^{-\Gamma/\hbar}\sum_b(-)^{\phi_b}\frac{\partial}{\partial\phi_a^*}\frac{\partial^2\Gamma}{\partial\phi_b\partial\phi_b^*} \\ &= -\hbar\sum_b(-)^{\phi_b(1+\phi_a^*)}\frac{\partial}{\partial\phi_b}\left(\frac{\partial^2\Gamma}{\partial\phi_a^*\partial\phi_b^*}e^{-\Gamma/\hbar}\right) \end{aligned}$$

is a total derivative. Integrating over all ϕ_a then confirms nilpotency of s on shell:

$$\left\langle s^2\phi_a\right\rangle = 0.$$

If $s^2\phi_a$ is inserted in a multi-field correlation function, the integration by parts produces a sum of correlators with a smaller number of fields. This is not inconsistent with the nilpotency of the second quantized BRST operator $\mathbf{Q}$. This is because the functional average of higher time derivatives of a field times a string of other fields is, in the operator formalism, what is sometimes called the vacuum expectation of the T^* product, which is that of the time ordered product with all time derivatives *outside* the time ordering symbol. When these time derivatives are brought inside the time ordering symbol they produce a sum of equal time commutators. Thus, although $s^2\phi$ corresponds to the operator $[\mathbf{Q}^2, \phi]$, the functional average of $s^2\phi$ with other fields, involves the T^* product of that operator. If $\mathbf{Q}^2 = 0$, this T^* product is a sum of equal time commutators, which are not necessarily zero.

4.3. Quantization of Witten's String Field Theory

Now we are ready to apply this formalism to Witten's open string field theory. We shall find a solution of (4.14) with a system of ghost fields which together with the input classical field $\Phi_{-1/2}$ comprise a string field Φ_- containing all negative ghost numbers:

$$\Phi_- = \sum_{G_s<0} |s\rangle \, \phi_s. \tag{4.18}$$

The anti-fields are then taken to comprise a string field Φ_+ containing all positive ghost numbers

$$\Phi_+ \equiv \sum_{G_s<0} |\bar{s}\rangle \, \phi_s^* (-)^{\phi_s^*} \tag{4.19}$$

where $|\bar{s}\rangle$ is dual to $|s\rangle$ in the sense that

$$\int |\bar{s}\rangle * |s'\rangle = \delta_{ss'}.$$

Referring to the definition of $\langle \mathcal{V}_2|$, (2.52), we can deduce that

$$|\bar{s}\rangle = (-)^{R+\frac{1}{2}(G^2-\frac{1}{4})} \, |s(c \leftrightarrow b, \downarrow\leftrightarrow\uparrow, \alpha_\mu \leftrightarrow -\alpha^\mu)\rangle \,, \tag{4.20}$$

where $R = \sum_{n=1}^{\infty}\big(\alpha_{-n}\cdot\alpha_n + n(c_{-n}b_n + b_{-n}c_n)\big)$. The condition that Φ_+ and Φ_- are overall odd enforces the opposite statistics of ϕ_s and ϕ_s^*. Call $\Phi = \Phi_+ + \Phi_-$ and consider the ansatz

$$\Gamma_0(\phi, \phi^*) = \frac{1}{2}\int \Phi * Q\Phi + \frac{1}{3}\int \Phi * \Phi * \Phi. \tag{4.21}$$

If we set all the anti-fields to zero, then because of ghost number conservation, only the contribution of the component $\Phi_{-1/2}$ survives

$$\Gamma_0(\phi, 0) = \frac{1}{2}\int \Phi_{-1/2} * Q\Phi_{-1/2} + \frac{1}{3}\int \Phi_{-1/2} * \Phi_{-1/2} * \Phi_{-1/2}$$

which is just Witten's classical string field action so Γ_0 satisfies the boundary condition (4.9).

To check the master equation we must compute $\frac{\partial \Gamma_0}{\partial \phi_s}$. It is helpful to display the dependence on ϕ_s and $\phi_s^* \equiv (-)^{\phi_s^*} \phi_{\bar{s}}$

$$\Gamma_0 = \frac{1}{2} \int |s\rangle * Q\,|s'\rangle\, \phi_{s'} \phi_s + \frac{1}{3} \int |s_1\rangle * |s_2\rangle * |s_2\rangle\, \phi_{s_3} \phi_{s_2} \phi_{s_3} (-)^{\phi_{s_2}}.$$

Then

$$\begin{aligned}
\frac{\partial \Gamma_0}{\partial \phi_t} &= (-)^{\phi_t} \int |t\rangle * (Q\Phi + \Phi * \Phi) \\
\frac{\partial \Gamma_0}{\partial \phi_t^*} &= \frac{\partial \Gamma_0}{\partial \phi_{\bar{t}}} (-)^{\phi_t^*} = \int |\bar{t}\rangle * (Q\Phi + \Phi * \Phi),
\end{aligned}$$

which we can write alternatively as

$$Q\Phi + \Phi * \Phi = \sum_{G_t < 0} \left(|t\rangle \frac{\partial \Gamma_0}{\partial \phi_t^*} + |\bar{t}\rangle \frac{\partial \Gamma_0}{\partial \phi_t} (-)^{\phi_t} \right).$$

Then

$$2 \sum_t (-)^{\phi_t} \frac{\partial \Gamma_0}{\partial \phi_t^*} \frac{\partial \Gamma_0}{\partial \phi_t} = \int (Q\Phi + \Phi * \Phi) * (Q\Phi + \Phi * \Phi) = 0.$$

The last equality follows by inspection of the individual terms. $\int Q\Phi * Q\Phi = 0$ after integrating by parts and using the nilpotency of Q.

$$\int (\Phi * \Phi) * (\Phi * \Phi) = \int \Phi * (\Phi * \Phi * \Phi) = -\int (\Phi * \Phi * \Phi) * \Phi = 0 \qquad (4.22)$$

by associativity of $*$ and the odd grading of Φ. And, finally,

$$\int Q\Phi * \Phi * \Phi = \frac{1}{3} \int Q(\Phi * \Phi * \Phi) = 0 \qquad (4.23)$$

by the derivation property of Q with respect to $*$, the symmetry property $\int C * D = (-)^{CD} \int D * C$, and associativity.

To determine Γ_1 we must compute

$$\sum_t (-)^{\phi_t} \frac{\partial^2}{\partial\phi_t^* \partial\phi_t}\Gamma_0 = \sum_t (-)^{\phi_{\bar t}} \frac{\partial}{\partial\phi_{\bar t}} \int |t\rangle * (Q\Phi + \Phi * \Phi)$$
$$= \sum_{G_t<0} (-)^{\phi_{\bar t}} \big[\int |t\rangle * Q\,|\bar t\rangle + \int (|t\rangle * |\bar t\rangle - |\bar t\rangle * |t\rangle) * \Phi\big].$$

The first term vanishes because $|t\rangle * Q\,|\bar t\rangle$ has ghost number $+5/2$. The second term can give a nonvanishing contribution because of the component $\Phi_{-3/2}$ in Φ. Thus the linearized theory is gauge independent with $\Gamma_n = 0$ for $n > 0$. For the nonlinear theory the equation for Γ_1 reads

$$\begin{aligned} &\sum_{G_t<0} \left[\frac{\partial\Gamma_1}{\partial\phi_t^*} \int |t\rangle * (Q\Phi + \Phi * \Phi) + (-)^{\phi_t} \frac{\partial\Gamma_1}{\partial\phi_t} \int |\bar t\rangle * (Q\Phi + \Phi * \Phi)\right] \\ &\quad = \hbar \sum_{G_t<0} (-)^{\phi_{\bar t}} \int \big(|t\rangle * |\bar t\rangle - |\bar t\rangle * |t\rangle\big) * \Phi \\ &\quad = \hbar \sum_t (-)^{\phi_{\bar t}} \int (|t\rangle * |\bar t\rangle) * \Phi, \end{aligned} \tag{4.24}$$

where, in the final form, the sum over t includes all ghost numbers.

To put (4.24) in a neater form, define the Grassmann even string field

$$d\Gamma_1 = \sum_{G_t<0} \left(|t\rangle \frac{\partial\Gamma_1}{\partial\phi_t^*} + (-)^{\phi_t}\,|\bar t\rangle \frac{\partial\Gamma_1}{\partial\phi_t}\right) \tag{4.25}$$

Then (4.24) reads

$$\int d\Gamma_1 * (Q\Phi + \Phi * \Phi) = \hbar \sum_t \int (|t\rangle * (-)^{g_0 + \frac{1}{2}}\,|\bar t\rangle) * \Phi \tag{4.26}$$

which should hold for arbitrary string fields Φ. It will be useful to associate with $d\Gamma_1$ the bra defined by

$$\langle d\Gamma_1 | s\rangle = \int d\Gamma_1 * |s\rangle\,.$$

Although (4.26) is too difficult to solve in closed form, an expansion of $d\Gamma_1$ in powers of the string field should be tractable. The term with $n-1$ powers of Φ

would give n powers of Φ in Γ_1 and so would contribute to the n string function. The equation for the $\Phi \to 0$ limit of $d\Gamma_1$ is simply

$$\left\langle d\Gamma_1^0 \right| Q = \hbar \left\langle \mathcal{V}_3 \right| \sum_t \left| t \right\rangle (-)^{g_0+\frac{1}{2}} \left| \bar{t} \right\rangle . \tag{4.27}$$

It is important for consistency that the RHS of this equation is annihilated by Q, and this follows from identities discussed in the next chapter. There are many solutions of (4.27), but a particular one is

$$\left\langle d\Gamma_1^0 \right| = \hbar \left\langle \mathcal{V}_3 \right| \sum_t \left| t \right\rangle (-)^{g_0+\frac{1}{2}} \left| \bar{t} \right\rangle \frac{b_0}{\hat{L}_0},$$

which has the virtue that it doesn't contribute in Siegel gauge. The corresponding contribution to Γ_1 is simply

$$\Gamma_1^0 = \hbar \sum_t \left\langle \mathcal{V}_3 | t \right\rangle (-)^{g_0+\frac{1}{2}} \left| \bar{t} \right\rangle \frac{b_0}{\hat{L}_0} \sum_s \left| s \right\rangle \phi_s. \tag{4.28}$$

One can add to $\langle d\Gamma_1^0|$ any term of the form $\langle X|\, Q$ and still get a solution of (4.27). This would alter the off-shell but not the on-shell dependence of the string field diagrams.

The source term in (4.27)

$$\sum_t \left| t \right\rangle * (-)^{g_0+\frac{1}{2}} \left| \bar{t} \right\rangle$$

is clearly highly divergent. Diagrammatically it resembles a tadpole without an internal propagator. The sum over t includes an integral over all space time momenta so it involves expressions like

$$\int dp\ f(p) \tag{4.29}$$

with $f(p)$ an entire function of p. In a point field theory $f(p)$ would be a polynomial in p so (4.29) would be zero in dimensional regularization. Thus in point field theory all the higher Γ_n can be set to zero provided one uses dimensional regularization. With a physical cutoff, their role is to cancel very severe ultraviolet divergences which arise in the Feynman rules derived from Γ_0 alone and which violate the Ward identities derived from gauge invariance. In string field theory dimensional regularization is certainly problematic. When one calculates with the Feynman rules with Γ_0 alone, it is natural to use the Schwinger representation

$$\frac{1}{\hat{L}_0} = \int_0^\infty dT_i e^{-T_i \hat{L}_0}$$

for each propagator and calculate the integrand of the T_i integrals. For all T_i finite, one is calculating finite quantites, so the vertices derived from the higher Γ_n only furnish terms which look like finite end point contributions with $T_i \to 0$. For example, they presumably cancel boundary contributions which arise from contributions to the integrand which are total derivatures in the various T_i, and which must not contribute to physical amplitudes. A complete study of these issues has not yet been carried out, but it is clearly important to do so. In chapter 5 we shall derive the Feynman rules from Γ_0 alone.

Let us now return to the problem of fixing a specific gauge in string field theory. This is done by specifying a gauge fermion Ψ. The Siegel gauge (4.3) is probably the most convenient for perturbation theory. We would like to impose the Siegel gauge condition on all of the string fields, $b_0\Phi_- = 0$. The anti-fields Φ_+ will be specified in terms of the gauge fermion. To get a tractable gauge fixed Lagrangian it is better to extend the field content of the theory[89,90] by introducing a Grassmann even string field

$$\bar{\Phi} = \sum_{G_s \geq 3/2} |s\rangle\, \bar{\phi}_s$$

together with anti-fields

$$\bar{\Phi}^* = \sum_{G_s \geq 3/2} |\bar{s}\rangle\, \bar{\phi}^*_s.$$

If we add a term to Γ of the form

$$-\int \beta * \bar{\Phi}^*$$

where

$$\beta = \sum_{G_s \leq -3/2} |s\rangle \, \beta_s$$

is a Grassmann odd Lagrange multiplier string field, we continue to have a solution of (4.9) and (4.12). Integrating over β will impose the Siegel condition on Φ_- if we choose the gauge fixing condition

$$\bar{\Phi}^* = b_0 \Phi_-$$

This corresponds to the gauge fermion

$$\Psi = \int b_0 \Phi_- * \bar{\Phi}$$

which in turn implies that Φ_+ is gauge fixed to

$$\Phi_+ = \sum_{G_s < 0} |\bar{s}\rangle \, (-)^{\phi_s^*} \frac{\partial \Psi}{\partial \phi_s} \propto b_0 \bar{\Phi}.$$

Since $\bar{\Phi}$ is unconstrained, this condition is completely equivalent to the side condition

$$b_0 \Phi_+ = 0. \tag{4.30}$$

Thus, writing the gauge fixed Γ in terms of Φ_+ instead of $\bar{\Phi}$, and introducing further Lagrange multipliers to impose (4.30), the gauge fixed Γ is simply

$$\Gamma^{G.F.} = \Gamma - \int \beta * b_0 \Phi$$

where now the Lagrange multiplier field β contains all ghost numbers. In the limit $\hbar \to 0$ we get the explicit expression

$$\Gamma_0^{G.F.} = \frac{1}{2}\int \Phi * Q\Phi + \frac{1}{3}\int \Phi * \Phi * \Phi - \int \beta * b_0\Phi. \tag{4.31}$$

The gauge condition $b_0\Phi = 0$ means that the kinetic energy term can be rewritten in the nondegenerate form

$$\frac{1}{2}\int \Phi * Q\Phi = \frac{1}{2}\int \Phi * c_0\hat{L}_0\Phi. \tag{4.32}$$

The gauge condition is enforced by simply restricting the states entering Φ

$$\Phi = \sum_{b_0|s\rangle=0} |s\rangle\, \phi_s$$

so after integrating over β we can replace

$$\Gamma_0^{G.F.} \to \frac{1}{2}\int \Phi * c_0\hat{L}_0\Phi + \frac{1}{3}\int \Phi * \Phi * \Phi \tag{4.33}$$

which is a very convenient form for deriving the Feynman rules.

The BRST invariance of (4.31) is according to (4.15) just given by

$$\begin{aligned} s\Phi_- &= (Q\Phi + \Phi * \Phi)_- \\ s\Phi_+ &= (b_0\beta)_+ \\ s\beta &= 0 \end{aligned} \tag{4.34}$$

where the subscripts signify that the expression is only the part with that sign of ghost number. Historically, the form of the gauge fixed action (4.33) was guessed[91−93] by interpreting the Polyakov path integrals as gauged fixed string field theory Feynman rules. Then it was noted[94] that the equivalent expression (4.31) had the global BRST symmetry (4.34). Independently, Bocchicchio[95] arrived at the same gauge fixed action, although in a less transparent form by applying the machinery developed by Batalin and Vilkovisky. He later showed the equivalence of his result to the compact form (4.31).

The Batalin-Vilkovisky machinery was not used in Ref.[94]. Instead, (4.31) was obtained by imposing BRST invariance stage by stage as in our preliminary discussion of the linearized theory. It is instructive to show the BRST invariance of $\Gamma_0^{G.F.}$ directly as in Ref.[94]. One simply calculates

$$\begin{aligned}
s\Gamma_0 &= \int s\Phi * (Q\Phi + \Phi * \Phi - b_0\beta) \\
&= \int (Q\Phi + \Phi * \Phi)_- * (Q\Phi + \Phi * \Phi - b_0\beta)_+ \\
&\quad + \int (b_0\beta)_+ * (Q\Phi + \Phi * \Phi - b_0\beta)_- \\
&= \int (Q\Phi + \Phi * \Phi)_- * (Q\Phi + \Phi * \Phi)_+ - \int (b_0\beta)_+ * (b_0\beta)_- .
\end{aligned}$$

Now since $Q\Phi + \Phi * \Phi$ and $b_0\beta$ are even, and ghost number conservation forbids terms like $\int ()_\pm * ()_\pm$ involving like sign ghost number, the above equation is just

$$s\Gamma_0 = \frac{1}{2}\int (Q\Phi + \Phi * \Phi) * (Q\Phi + \Phi * \Phi) + \frac{1}{2}\int (b_0\beta) * (b_0\beta)$$

The last term vanishes since $b_0^2 = 0$ and the first term vanishes by the identities (4.22). It is also easy to check nilpotency directly:

$$\begin{aligned}
&s^2\Phi_+ = s^2\beta = 0 \\
&s^2\Phi_- = (-Qs\Phi + s\Phi * \Phi - \Phi * s\Phi)_-
\end{aligned} .$$

Now write

$$s\Phi = (Q\Phi + \Phi * \Phi) - (Q\Phi + \Phi * \Phi)_+ + (b_0\beta)_+ .$$

Then because

$$Q(Q\Phi + \Phi * \Phi) - (Q\Phi + \Phi * \Phi) * \Phi + \Phi * (Q\Phi + \Phi * \Phi) = 0,$$

the first term in $s\Phi$ doesn't contribute to $s^2\Phi_-$:

$$\begin{aligned} s^2\Phi_- &= \big(Q[(Q\Phi + \Phi * \Phi - b_0\beta)_+] \\ &\quad + \Phi * (Q\Phi + \Phi * \Phi - b_0\beta)_+ \\ &\qquad - (Q\Phi + \Phi * \Phi - b_0\beta)_+ * \Phi\big)_- \\ &= \big([\Phi, (Q\Phi + \Phi * \Phi - b_0\beta)_+]\big)_- \end{aligned} \tag{4.35}$$

where the first term vanished because of the $-$ projection and the commutator is with respect to the $*$ product. We see that, although not identically zero, $s^2\Phi_-$ does vanish for Φ satisfying the field equations derived from Γ_0.

We close this chapter with a brief discussion of gauge fixing in the open superstring field theory. Since the algebraic structure of the bosonic string is largely preserved, the general machinery can be applied in an analogous fashion. In the -1 picture the boson component of the superstring field enters in almost identical fashion. The presence of a picture changing operator at the vertex, can wreck the fundamental axioms satisfied by $*$ and $\int$ even at the classical level.[84,73] Since the axioms are used to solve the master equation, [see (4.22) and (4.23)], one cannot expect to have a decent gauge fixing prescription until these issues are cleared up.

Another difference with the bosonic case is the presence of Y_{-1} in the kinetic term for the fermion string field. In addition, as discussed in Ref.[78], it is desirable to impose (two) gauge conditions on the fermion string field

$$b_0\Phi = \beta_0\Phi = 0,$$

as only then can one get a recognizable form for the fermion propagator. It has been observed[96,97] that the condition $\beta_0\Phi = 0$ can only be imposed if there are extra gauge symmetries beyond those associated with Q. The presence of Y_{-1} accounts for these extra symmetries. This is because Y_{-1} can be represented as a delta function of ghost variables

$$Y_{-1} = c\delta'(\gamma)$$

and so is annihilated by c and γ^2. It is argued in these references that these extra gauge symmetries are sufficient to impose $\beta_0\Phi = 0$. However, for a proper treatment one must apply the methods described in the present chapter to these new symmetries, and this has not been done.

Finally, if one wants to define superstring field theory using the 0 picture for the boson component, the kinetic energy of the boson component carries a ghost number -2 insertion Y_{-2}. In light of the foregoing remarks on the fermion kinetic term, one should anticipate a similar complication of the gauge fixing procedure for the boson kinetic term. There may be extra gauge symmetries to be fixed. If so, it is obviously very important to identify them and to explore their implications.

5. Perturbation Theory

We devote this chapter to a derivation of the Feynman rules for open string field theory. We shall also explain in detail how the covariant prescription for integrating over world sheets due to Polyakov is predicted by perturbation theory. As we have shown in chapter 4, we can define quantum string field theory by Lagrangian path integrals, as in (4.10), where the fields ϕ_s are simply the coefficients of an expansion of the string field Φ in a basis of the single string state space and where Γ satisfies (4.9) and (4.12). Γ has the expansion in powers of $\hbar$ (4.13) and we have verified the explicit solution (4.21) for Γ_0. We shall derive the Feynman rules implied by Γ_0 and we shall show that they reproduce the Koba-Nielsen[98] integrands for the Polyakov path integrals for generic values of the Koba-Nielsen variables (moduli). We shall then illustrate the role played by the Γ_n for $n > 0$ in assuring BRST invariance by studying the simplest diagram where Γ_1 provides an essential contribution, the one loop tadpole. When one examines the BRST invariance of the diagrams generated by Γ_0, he encounters non vanishing "cancelled propagator" terms which can alternatively be viewed as boundary contributions from total derivatives in the moduli. For the one loop tadpole this contribution will be seen to be cancelled by the lowest order contribution from Γ_1.

5.1. The Feynman Rules

We shall derive the Feynman rules for the N string correlation functions

$$\langle \phi_{s_1}\phi_{s_2}\cdots\phi_{s_N}\rangle = \frac{\displaystyle\int \prod d\phi_s \phi_{s_1}\phi_{s_2}\cdots\phi_{s_N} e^{-\Gamma_0^{G.F.}/\hbar}}{\displaystyle\int \prod d\phi_s e^{-\Gamma_0^{G.F.}/\hbar}} \tag{5.1}$$

with $\Gamma_0^{G.F.}$ given by (4.33) with the fields scaled so that the bilinear term is of order 1:

$$\frac{1}{\hbar}\Gamma_0^{G.F.} = \frac{1}{2}\int \Phi * c_0\hat{L}_0\Phi + \frac{g}{3}\int \Phi * \Phi * \Phi \tag{5.2}$$

and we have named the effective coupling $\sqrt{\hbar} = g$. Here we have integrated out the Lagrange multiplier field β so that Φ is an expansion in single string states annihilated by b_0

$$\Phi = \sum_{b_0|s\rangle=0} |s\rangle\, \phi_s. \tag{5.3}$$

We remind the reader that ϕ_s are c-numbers for $G_s = -\frac{1}{2} + 2n$ and anti-commuting Grassmann numbers for $G_s = +\frac{1}{2} + 2n$.

The Feynman rules are obtained in standard fashion by first calculating the generating function for multi string correlation functions in the linearized theory

$$Z^0(J) = \frac{\displaystyle\int d\phi \exp\Big\{-\frac{1}{2}\int \Phi * c_0\hat{L}_0\Phi + \int J * \Phi\Big\}}{\displaystyle\int d\phi \exp\Big\{-\frac{1}{2}\int \Phi * c_0\hat{L}_0\Phi\Big\}} \tag{5.4}$$

where we have introduced a Grassmann even source string field

$$J = \sum_{c_0|\bar{s}\rangle=0} |\bar{s}\rangle\, J_s$$

so that $\int J * \Phi = \sum_s J_s\phi_s$. $Z^0(J)$ is simply evaluated by completing the square:

$$Z^0(J) = \exp\{+\frac{1}{2}\int J * \Phi_J\}$$

with

$$c_0\hat{L}_0\Phi_J = J$$

which has the solution (since $b_0\Phi_J = 0$)

$$\Phi_J = \frac{b_0}{\hat{L}_0}J$$

so we have

$$Z^0(J) = \exp\{\frac{1}{2}\int J * \frac{b_0}{\hat{L}_0} J\}$$

from which we read off the free propagator

$$\begin{aligned} D_{s_2 s_1} &= \langle \phi_{s_1} \phi_{s_2} \rangle_0 = \frac{\partial}{\partial J_{s_1}} \frac{\partial}{\partial J_{s_2}} Z^0(J) \Big|_{J=0} \\ &= \int |\bar{s}_2\rangle * \frac{b_0}{\hat{L}_0} |\bar{s}_1\rangle \\ &= \langle \mathcal{V}_2 | \bar{s}_2 \rangle \frac{b_0}{\hat{L}_0} |\bar{s}_1\rangle \\ &= (-)^{\phi_{s_2}} D_{s_1 s_2}. \end{aligned} \tag{5.5}$$

Note that the explicit presence of the insertion b_0 projects out the states s violating the gauge condition $b_0 |s\rangle = 0$ or equivalently $c_0 |\bar{s}\rangle = 0$. This is very fortunate for the following discussion because it means that all state sums can be taken unrestricted by the gauge condition.

Given the explicit expression for Z^0, the Wick expansion for the free correlator

$$\langle \phi_{s_1} \cdots \phi_{s_n} \rangle_0 = \sum_{pairings} \pm \prod_{pairs} \langle \phi_{s_i} \phi_{s_j} \rangle_0$$

follows, where the sign in front of each term is fixed in the usual way by the number of interchanges of Grassmann odd ϕ's needed to bring each contracted pair together. The correlators for the nonlinear theory can now be expanded in powers of g by expanding

$$\exp\{-\frac{g}{3}\int \Phi * \Phi * \Phi\} = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \Big(\frac{1}{3}\int \Phi * \Phi * \Phi\Big)^n$$

and writing

$$\begin{aligned} \int \Phi * \Phi * \Phi &= \sum_{s_1 s_2 s_3} \langle \mathcal{V}_3 | s_1 \rangle |s_2\rangle |s_3\rangle (-)^{\phi_{s_2}} \phi_{s_3} \phi_{s_2} \phi_{s_1} \\ &= \sum_{s_1 s_2 s_3} V_{s_1 s_2 s_3} (-)^{\phi_{s_2}} \phi_{s_3} \phi_{s_2} \phi_{s_1}. \end{aligned} \tag{5.6}$$

Clearly one has all the Feynman diagrams of a cubic field theory where the three string vertex is obtained by computing

$$
\begin{aligned}
-\frac{g}{3}\left\langle \phi_{t_1}\phi_{t_2}\phi_{t_3}\sum_{s_1 s_2 s_3} V_{s_1 s_2 s_3}(-)^{\phi_{s_2}}\phi_{s_3}\phi_{s_2}\phi_{s_1}\right\rangle = \\
-g\sum_{s_1 s_2 s_3}(-)^{\phi_{s_2}}\big(V_{s_1 s_2 s_3}+(-)^{\phi_{s_1}-\phi_{s_2}+\phi_{s_2}\phi_{s_1}}V_{s_2 s_1 s_3}\big)D_{s_3 t_3}D_{s_2 t_2}D_{s_1 t_1}.
\end{aligned}
\tag{5.7}
$$

There is an illuminating way to rewrite the sign multiplying the second term of (5.7).

$$
(-)^{\phi_{s_1}-\phi_{s_2}+\phi_{s_2}\phi_{s_1}} = (-)^{(\phi_{s_1}^2+\phi_{s_2}^2+\phi_{s_3}^2)/2} = \begin{cases} + & \text{if all } \phi\text{'s are even} \\ - & \text{if a pair of } \phi\text{'s is odd,} \end{cases} \tag{5.8}
$$

where we used $(-)^{\phi} = (-)^{\phi^2}$ and $(-)^{\phi_{s_3}} = (-)^{\phi_{s_1}+\phi_{s_2}}$. But recalling the correlation between the grading of ϕ_s and the ghost number of s, the same sign can be coded

$$
(-)^{[G_{s_1}^2+G_{s_2}^2+G_{s_3}^2-3/4]/2} \tag{5.9}
$$

if we bear in mind that $G_{s_1}+G_{s_2}+G_{s_3} = -3/2$. The fundamental definition of $V_{s_1 s_2 s_3}$ makes it obvious that it is cyclically symmetric in the three indices. However the interchange of a pair of indices is accomplished by reversing the orientation of the three strings coming into the vertex. Since the vertex is defined in bosonized ghost language this change of orientation corresponds to applying $(-)^{\sum_{n=1}^{\infty}(g_{-n}g_n+\alpha_{-n}\cdot\alpha_n)}$ to each state $|s_i\rangle$. Then the additional sign (5.9) combines with this so that the total vertex is

$$
-g(-)^{\phi_{s_2}}V_{s_1 s_2 s_3}\big(1+(-)^{R_{s_1}+R_{s_2}+R_{s_3}}\big) \tag{5.10}
$$

where R_s is the eigenvalue on the state $|s\rangle$ of the mode number operator

$$
\begin{aligned}
R &= \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty} g_{-n}g_n + \frac{g_0^2-1/4}{2} \\
&= \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty} n(c_{-n}b_n + b_{-n}c_n).
\end{aligned}
\tag{5.11}
$$

Thus we see that $(-)^R$ reverses the orientation of the fermionized ghost variables as well as of the coordinate variables. The signs arising from the Grassman odd character of some of the ϕ_s are precisely what is needed so that "twisting" a vertex corresponds to reversing the orientation of the c's and b's! This is crucial for the equivalence of the Feynman diagrams to the world sheet approach.

In addition to the relative sign in (5.10) associated with the interchange of two legs in the vertex, there is an overall sign multiplying each graph determined by keeping track of interchanges of odd operators. If all the external lines of the diagram are even, then the odd lines of a given diagram from internal disjoint closed loops. This is because the vertex conserves grading. If we express the vertex for the emission of an even particle from an odd line as a matrix

$$(V_{oeo})_{ss'} = -g\phi^e_{s''}V_{ss''s'}(1+(-)^{R_s+R_{s'}+R_{s''}})$$

and similarly for the propagator, a closed odd line will contribute the factor

$$-Tr\{V_{oeo}D_{oo}V_{oco}D_{oo}.......V_{oeo}D_{oo}\}.$$

For example, this extra minus sign for odd lines is responsible for the extra $(-)^{g_0+1/2}$ in the ghost factor of the partition function. (Recall that the string states with $g_0 = \frac{1}{2}+2n$ are associated with odd fields.)

$$Tr(-)^{g_0+1/2}b_0c_0q^{2\hat{L}_0} = \prod_{n=1}^{\infty}(1-q^{2n})^2\prod_{n=1}^{\infty}(1-q^{2n})^{-D}$$

which otherwise would involve factors $(1+q^{2n})^2$ in place of $(1-q^{2n})^2$. Thus this extra minus sign is necessary for the ghosts to change the factors of $(1-q^{2n})^{-D}$ from the coordinate degrees of freedom to $(1-q^{2n})^{-(D-2)}$, which gives the correct counting of physical states which are in 1-1 correspondence with the transverse coordinate degrees of freedom.

5.2. Diagrams as World Sheet Path Integrals

Our next task is to interpret the Feynman diagrams as world sheet path integrals. The first step is to write the Schwinger representation for the factor $\frac{1}{\hat{L}_0}$ in the propagator

$$\frac{1}{\hat{L}_0} = \int_0^\infty dT e^{-\hat{L}_0 T}.$$

so that the propagator becomes

$$D_{s_2 s_1} = \int_0^\infty dT \left\langle \mathcal{V}_2 | \bar{s}_2 \right\rangle e^{-T\hat{L}_0} b_0 \left| \bar{s}_1 \right\rangle .$$

Next, one interprets $e^{-T\hat{L}_0}$ as the (imaginary time) evolution operator for the BRST first quantized string which has the path integral representation, using bosonized ghosts,

$$\begin{aligned} &\left\langle g_f, x_f^\mu | e^{-T\hat{L}_0} b_0 | g_i, x_i^\mu \right\rangle = \\ &\int dg dx \exp\left(-\frac{1}{2\pi} \int_0^T d\tau \int_0^\pi d\sigma \left(\dot{x}^2 + (x')^2 + \dot{g}^2 + (g')^2 \right) \right) \int_0^\pi d\sigma b(\sigma, \tau_0) \end{aligned} \tag{5.12}$$

where the trajectories $g(\sigma,\tau), x^\mu(\sigma,\tau)$ are constrained to be $g_i(\sigma), x_i^\mu(\sigma)$ at $\tau = 0$ and to be $g_f(\sigma), x_f^\mu(\sigma)$ at $\tau = T$. The parameter space (σ,τ) on which the variables are defined is a rectangle of length T and width π. If T is identified with one of the moduli, then the factor $b_0 = \int_0^\pi d\sigma b(\sigma,\tau_0)$, where $b(\sigma,\tau) = b^{(+)} + b^{(-)}$ with $b^{(\pm)}$ the bosonized representation of the chiral world-sheet anti-ghost field, is just the insertion prescribed to produce the correct measure for this modulus.[99,100] Each propagator of the Feynman diagram contributes a rectangular strip to the eventual Riemann surface. These rectangular strips are glued together by the vertices: the state sums just amount to the closure property of path integrals.

When we use bosonized ghosts, the state sums do not quite impose continuity of the bosonized ghost field g. The nonzero modes of g are forced to be continuous but the zero modes are not. This is simply because the sum over ghost number produces either a factor of

$$\sum_n e^{i(n+1/2)(q-q')} = 2\pi \sum_m (-)^m \delta(q - q' + 2m\pi) \tag{5.13}$$

or a factor

$$\sum_n (-)^n e^{i(n-1/2)(q-q')} = 2\pi \sum_m e^{i\pi(m+1/2)} \delta(q - q' + (2m+1)\pi). \tag{5.14}$$

Thus, depending on the presence or absence of the grading factor $(-)^{g_0+\frac{1}{2}}$, the bosonized ghost field g is only required to be continous *modulo* $(2m+1)\pi$ or $2m\pi$. (According to (5.9) the presence or absence of the grading factor depends on the number of twisted vertices participating in the loop.) However, the translational invariance in $g \to g + const$ of the representation of each vertex and propagator allows an adjustment to make g continous as long as there are no closed loops in the diagram. So trees can be represented with g continous over the whole world sheet. For multi-loop diagrams the above formulae show that upon circumnavigating a noncontractible loop g only returns to itself modulo a suitable period:

$$\phi^f_m - \phi^i_m = \begin{cases} (2m+1)\pi & \text{for an odd number of } (-)^{g_0+1/2} \\ 2m\pi & \text{for an even number of } (-)^{g_0+1/2}. \end{cases}$$

There is also a prescribed phase $e^{\pi i(m+1/2)}$ or $(-)^m$ associated with each winding number m. The vertices are determined by Witten's choice for the $*$ product to identify the ends of the three strips as in Figure 8.

To see how this works, let us consider how the state sum glues a single rectangular strip to a vertex:

$$\sum_s V_{s_1 s_2 s} D_{st} = \sum_s \langle \mathcal{V}_3 | s_1 \rangle \, |s_2\rangle \, |s\rangle \, \langle \mathcal{V}_2 | \bar{t} \rangle \, \frac{b_0}{\hat{L}_0} \, |\bar{s}\rangle \, (-)^{\phi_t} \tag{5.15}$$

Because the sum is essentially a completeness sum, one can transfer the $b_0/\hat{L}_0$ in the D to the vertex matrix element. The relevant identities that enable this are given by

$$\begin{aligned} \sum_s |\bar{s}\rangle \, b_n \, |s\rangle &= \sum_s \bigl(b_{-n} (-)^{n+g_0-1/2} \, |\bar{s}\rangle \bigr) \, |s\rangle \\ \sum_s |\bar{s}\rangle \, c_n \, |s\rangle &= \sum_s \bigl(c_{-n} (-)^{n+g_0+1/2} \, |\bar{s}\rangle \bigr) \, |s\rangle \\ \sum_s |\bar{s}\rangle \, \alpha_n \, |s\rangle &= \sum_s \bigl(\alpha_{-n} (-)^{n+1} \, |\bar{s}\rangle \bigr) \, |s\rangle \, . \end{aligned} \tag{5.16}$$

These identities are proved by using the formula for $|\bar{s}\rangle$ (4.20):

$$|\bar{s}\rangle = (-)^{\frac{1}{2}(g_0^2 - \frac{1}{4}) + R} \, |s(c \leftrightarrow b, \downarrow \leftrightarrow \uparrow, \alpha_n \leftrightarrow -\alpha_n)\rangle$$

to show

$$\begin{aligned} \overline{c_{-n} \, |s\rangle} &= b_{-n} (-)^{n+g_0-1/2} \, |\bar{s}\rangle \\ \overline{b_{-n} \, |s\rangle} &= c_{-n} (-)^{n+g_0+1/2} \, |\bar{s}\rangle \\ \overline{\alpha_{-n} \, |s\rangle} &= \alpha_{-n} (-)^{n+1} \, |\bar{s}\rangle \end{aligned} \tag{5.17}$$

and then changing summation variables. For example,

$$\sum_s |\bar{s}\rangle \, b_n \, |s\rangle = \sum_t \overline{c_{-n} \, |t\rangle} b_n c_{-n} \, |t\rangle = \sum_t \bigl(b_{-n} (-)^{n+g_0-1/2} \, |\bar{t}\rangle \bigr) \, |t\rangle \, .$$

Here we used the fact that $b_n \, |s\rangle \neq 0$ only if $|s\rangle$ contains c_{-n} in which case $|\bar{s}\rangle$ contains b_{-n} so $b_{-n} \, |\bar{s}\rangle = 0$. This establishes the first of (5.16); the others are proved similarly.

Now we can use the first of (5.16) for $n = 0$ to rewrite (5.15) as

$$\begin{aligned}
\sum_s V_{s_1 s_2 s} D_{st} &= \sum_s \langle \mathcal{V}_3 | s_1 \rangle \, | s_2 \rangle \, \frac{b_0}{\hat{L}_0} \, | s \rangle \, \langle \mathcal{V}_2 | \bar{t} \rangle \, (-)^{g_0 + 1/2} \, | \bar{s} \rangle \, (-)^{\phi_t} \\
&= \sum_s \langle \mathcal{V}_3 | s_1 \rangle \, | s_2 \rangle \, \frac{b_0}{\hat{L}_0} (-)^{g_0 - 1/2} \, | s \rangle \, \delta_{\bar{t} s} (-)^{\phi_t} \\
&= \sum_s \langle \mathcal{V}_3 | s_1 \rangle \, | s_2 \rangle \, \frac{b_0}{\hat{L}_0} \, | \bar{t} \rangle
\end{aligned} \tag{5.18}$$

where we used the equality $(-)^{\phi_t} = (-)^{G_{\bar{t}} - 1/2}$. Using the Schwinger representation, the RHS of (5.18) becomes

$$\int_0^\infty dT \, \langle \mathcal{V}_3 | s_1 \rangle \, | s_2 \rangle \, e^{-T \hat{L}_0} b_0 \, | \bar{t} \rangle$$

which shows that the attached propagator inserts the factor $e^{-T\hat{L}_0}$ which reintroduces the regulator used to define the $*$ product in terms of the world sheet path integral (see Figure 8). By construction, the vertex has a ghost insertion ($e^{3ig/2}$ for the bosonic string) at the midpoint, which gives precisely the curvature dependence of the world sheet action in the bosonized ghost formulation of the world sheet path integral.

In summary, every Feynman diagram can be interpreted as a world sheet path integral defined on a Riemann surface constructed by gluing together the rectangular strips of width π and variable length T_i associated with each propagator by Witten-style vertices. The b_0 which comes along with each propagator is precisely the insertion required by the Polyakov formalism so that dT_i gives the correct measure for integrating over the moduli T_i. As an illustration we draw the one loop tadpole diagram in Figure 10.

The interpretation of the Feynman diagrams of string field theory as world sheet path integrals facilitates the proof that they reproduce the dual resonance amplitudes. The world sheet path integrals are obtained in a particular conformal frame which is awkward for explicit computations. But the slice independence of the Polyakov path integrals allows one to transform to a more convenient conformal frame. This of course induces a complicated slice dependence on the external states which only disappears for physical on shell amplitudes. For such amplitudes the external line strips are of effectively infinite length, so their ends can be mapped to points in the new frame. Thus for S matrix elements one can show[91] that the integrand over the T_i can be changed to one over standard Koba-Nielsen variables. However the region of integration must still be shown to be the correct one. Giddings, Martinec, and Witten[92] were able to show (in essence) that summing over all diagrams and integrating each T_i from 0 to ∞ gives a complete single cover of moduli space. This does not rule out the presence of missed contributions at points in moduli space, i.e. delta function contributions. The Γ_n for $n > 0$ we encountered in our gauge fixing procedure are just such contributions. We shall study the role of Γ_1 in the one loop tadpole diagram.

Let us calculate the tadpole diagram from scratch. Thus we want to evaluate

$$\begin{aligned}\langle\phi_t\rangle_0 &= -\frac{g}{3}\left\langle\phi_t V_{s_1s_2s_3}\phi_{s_3}\phi_{s_2}\phi_{s_1}(-)^{\phi_{s_2}}\right\rangle_0 \\ &= -\frac{g}{3}D_{st}D_{s_1s_2}[V_{s_1s_2s}(-)^{\phi_{s_2}} + V_{s_1ss_2}(-)^{\phi_t\phi_{s_2}+\phi_s} + \\ &\qquad V_{ss_1s_2}(-)^{\phi_t\phi_{s_2}+\phi_t\phi_{s_1}+\phi_{s_1}}] \\ &= -\frac{g}{3}D_{st}D_{s_1s_2}V_{s_1s_2s}[(-)^{\phi_{s_2}} + (-)^{\phi_{s_2}+\phi_t\phi_{s_2}+\phi_s} + (-)^{\phi_t\phi_{s_1}+\phi_t\phi_{s_2}+\phi_{s_1}}]\end{aligned} \tag{5.19}$$

where in the last line we used the cyclic symmetry of V and the symmetry property $D_{s_1s_2} = (-)^{\phi_{s_2}}D_{s_2s_1}$. Clearly $(-)^{\phi_t} = (-)^{\phi_s} = 1$ and $(-)^{\phi_{s1}} = (-)^{\phi_{s2}}$, so all three terms are equal and we have

$$\langle\phi_t\rangle_0 = -gV_{s_1s_2s}D_{st}D_{s_2s_1} = -g\sum_s\langle\mathcal{V}_3|s_1\rangle\,\frac{b_0}{\hat{L}_0}\,|\bar{s}_1\rangle\,\frac{b_0}{\hat{L}_0}\,|\bar{t}\rangle\,. \tag{5.20}$$

Amputating the external propagator, we see that the tadpole diagram can be represented by the single string bra

$$\langle T| = -g \sum_s \langle \mathcal{V}_3|s\rangle \frac{b_0}{\hat{L}_0} |\bar{s}\rangle \tag{5.21}$$

where we must bear in mind that this formula is, strictly speaking, ambiguous up to a term of the form $\langle\Lambda| b_0$ because our gauge choice has restricted the fields we can work with to those annihilated by b_0. We resolve this ambiguity by appealing to the fact that the vertex is fundamentally a string field "overlap" which has a geometrical meaning for all string fields whether or not they satisfy the gauge condition. The result (5.21) presumes this definition of the vertex for fields violating the gauge condition.

We can now investigate the gauge invariance of this result. This amounts⋆ to testing whether $\langle T|$ is annihilated by Q. So we calculate

$$\begin{aligned}
\langle T| Q &= -g \sum_s \langle \mathcal{V}_3|Q_3|s\rangle \frac{b_0}{\hat{L}_0} |\bar{s}\rangle \\
&= g \sum_s \langle \mathcal{V}_3|(Q_1 + Q_2)|s\rangle \frac{b_0}{\hat{L}_0} |\bar{s}\rangle \\
&= g \sum_s \langle \mathcal{V}_3|(Q_1|s\rangle \frac{b_0}{\hat{L}_0} |\bar{s}\rangle + (-)^{g_0-1/2} |s\rangle\, Q_2 \frac{b_0}{\hat{L}_0} |\bar{s}\rangle) \\
&= g \sum_s \langle \mathcal{V}_3|(|s\rangle \frac{b_0}{\hat{L}_0} Q_2 (-)^{g_0+1/2} |\bar{s}\rangle + (-)^{g_0-1/2} |s\rangle\, Q_2 \frac{b_0}{\hat{L}_0} |\bar{s}\rangle) \\
&= g \sum_s \langle \mathcal{V}_3|s\rangle\, (-)^{g_0+1/2} |\bar{s}\rangle
\end{aligned} \tag{5.22}$$

which is apparently nonvanishing. But looking back to the equation for Γ_1^0, (4.27)

⋆ The second quantized BRST invariance discussed in Chapter 4 implies

$$\langle Q\Phi + \Phi * \Phi + d\Gamma_1\rangle = 0.$$

But the second and third terms are cancelled by the anticommutator of Q with the propagator on the external line so the condition simplifies to $(\langle T| + \langle T_{\Gamma_1}|)Q = 0$.

and (4.28), we see that Γ_1 would contribute

$$\langle T_{\Gamma_1}|\, Q = -g \sum_s \langle 0\mathcal{V}_3|s\rangle\, (-)^{g_0+1/2}\, |\bar{s}\rangle$$

to the one string function times Q so that $\langle T| + \langle T_{\Gamma_1}|$ is gauge invariant. So we confirm in this simplest case the role of the Γ_n in arranging the gauge invariance of the perturbation expansion. Nonetheless, if we choose the solution (4.28) the presence of the factor of b_0 means that the tadpole contribution to Γ_1 will not contribute in Siegel gauge, so it is still possible that the Γ_n can be arranged not to contribute to the Feynman amplitudes in this gauge.

5.3. Calculational Methods

We next turn to a more detailed discussion of the calculational methods provided by the relationship of diagrams to world sheet path integrals. As we have described, each Feynman diagram is associated with a Riemann surface constructed by gluing together a number of rectangular strips of width π and variable length T_i with Witten vertices. The path integral defined on each rectangular strip reproduces $b_0/\hat{L}_0$ if we take the metric on the world sheet to be the standard Euclidean one $G_{ij} = \delta_{ij}$ for the natural orthonormal coordinates $\rho = \tau + i\sigma$ with $\sigma\epsilon[0,\pi]$ and $\tau\epsilon[0,T_i]$ for the i^{th} strip. Then, using bosonized ghosts, the action for the world sheet path integral for the i^{th} propagator is just

$$S_i = \frac{1}{2\pi}\int_0^{T_i} d\tau \int_0^{\pi} d\sigma \big(\dot{x}^2 + x'^2 + \dot{g}^2 + g'^2\big).$$

Since we are using bosonized ghosts, the factor b_0 is represented by a line integral crossing the strip at constant τ:

$$b_0 = \frac{1}{\pi}\int_0^{\pi} d\sigma [b^{(+)}(\rho) + b^{(-)}(\rho)]$$

where

$$b^{(\pm)}(\rho) =: \exp\Big\{ -i\big[g(\rho) \pm i \int\limits_0^{\sigma} d\sigma' \dot{g}(\tau, \sigma')\big]\Big\} :$$

The normal ordering prescription is implemented in path integral language by deleting the singular factor

$$\lim_{\rho' \to \rho} e^{\frac{1}{2}\ln(\rho - \rho')}$$

which arises from each self contraction in the Wick expansion for the world sheet correlators.

At a vertex three strips are glued together. Away from the common midpoint, the coordinates of any of the strips can be extended across the join, with the constant metric $G^{ij} = \delta_{ij}$ obviously continuous at the join. As we have discussed in Chapter 3, the midpoint is a conical singularity, so the curvature dependence of the bosonized ghost action supplies the insertion

$$e^{3ig(\pi/2)/2}$$

which has been built into the definition of the Witten vertex.

In the formulation of world-sheet path integrals using bosonized ghosts, one functionally integrates over coordinates $x^\mu(\sigma, \tau)$ and a world sheet scalar bosonized ghost field $g(\sigma, \tau)$. For a world sheet metric $G_{ab}(\sigma, \tau)$ the world sheet action is given by [see Appendix A].

$$S = \frac{1}{2\pi}\int dA\big[G^{ab}\partial_a x_\mu \partial_b x^\mu + G^{ab}\partial_a g \partial_b g\big] + \frac{3i}{2\pi}\big(\frac{1}{2}\int dARg + \int d\ell k g\big).$$

The path integral has additional dependence on g through the anti-ghost insertions (to get the correct measure for moduli), and additional dependence on both g and x^μ from the specification of the states of external strings entering the diagram. All of this extra depedence can be worked out from the generating function

$$Z(J_g, J_x) = \int \mathcal{D}g\mathcal{D}x \ \exp\{-S + i\int dA(J_g g + J_x \cdot x)\}. \tag{5.23}$$

As we have discussed, for multi-loop diagrams g is only periodic up to a winding piece. The dependence of the path integral on winding can be extracted by writing

$$g = g_w + g_P$$

where g_w satisfies Laplace's equation $\partial_a \sqrt{G} G^{ab} \partial_b g_w = 0$ and boundary condition $n \cdot \partial g_w = 0$ on open string boundaries, and g_P is single valued. Then

$$Z(J_g, J_e) = \sum_w e^{i\delta_w} e^{-S(g_w)+i\int J_g g_w} Z_P(J_g, J_x).$$

Here the phase $e^{i\delta_w}$ associated with each winding sector is prescribed by the Feynman rules via the equations (5.13) and (5.14). One such sum appears for each noncontractible loop that can be drawn on the world sheet representing the diagram.

For the calculation of string field theory diagrams, one takes $\int dAJ_x \cdot x = \sum_e \int_0^\pi d\sigma p_e(\sigma) \cdot x(\sigma, T_e)$ where $p_e(\sigma)$ is just the eigenvalue of momentum density specified at the end of each external strip. The ghost source is more complicated:

$$\begin{aligned} \int dAJ_g \cdot g = \sum_e \left[\int_0^\pi d\sigma \Pi_e(\sigma) g(\sigma, T_e) + \frac{3}{4}\big(g(0, T_e) + g(\pi, T_e)\big) \right] \\ - \sum_i \left[g(\rho_i) \pm i \int_0^{\sigma_i} d\sigma' \dot{g}(\sigma', \tau_i) \right] \end{aligned} \tag{5.24}$$

where the last term produces the $b^+(\rho)$ or $b^-(\bar{\rho})$ insertions and $\Pi_e(\sigma)$ is the eigenvalue of ghost number density specified at the end of each external strip. Notice that Π_e is the *total* incoming ghost number density, which explains the

$\frac{3}{4}(g(0)+g(\pi))$ term which cancels off the $\int d\ell k$ contribution due to the corners at the end of each strip. Thus the total linear term in g in the exponent of the path integrand is

$$\sum_c i\frac{3}{2}g(\rho_c) - i\sum_i \Big[g(\rho_i) \pm i\int_0^{\sigma_i} d\sigma' \dot{g}(\sigma',\tau_i)\Big] + i\sum_e \int_0^{\pi} \Pi_e(\sigma)g(\sigma,T_e).$$

where ρ_c are the locations of the vertex midpoints on the world sheet.

Since the exponent in (5.23) is at most quadratic in the fields, the functional integral can be evaluated using the Neumann function for the Laplacian on the world sheet $\mathcal{S}$ to complete the square

$$\begin{aligned} \frac{1}{\sqrt{G}}\partial_a G^{ab}\sqrt{G}\partial_b N(\rho,\rho') &= 2\pi\delta(\rho,\rho') - 2\pi\mu(\rho) \\ n^a\partial_a N(\rho,\rho') &= 0 \qquad \text{for} \quad \rho\epsilon\partial\mathcal{S}. \end{aligned} \tag{5.25}$$

Here we have denoted (τ,σ) by the two component ρ^a, and μ is a background charge density, with $\int dA\mu = 1$, necessary for the consistency of the Neumann boundary conditions. We also impose the symmetry of N under interchange of its arguments, which is always possible. Then

$$\begin{aligned} Z_G(J_g,J_x) =& (\det\Delta_G)^{-(D+1)/2}\exp\Big\{\frac{9}{16\pi^2}\Big[\frac{1}{4}\int dAdA'R(\rho)R(\rho')N(\rho,\rho') \\ &+ \int dAd\ell' R(\rho)k(\rho')N(\rho,\rho') + \int d\ell d\ell' k(\rho)k(\rho')N(\rho,\rho')\Big] \\ &- \frac{3}{4\pi}\int dAJ_g(\rho)\Big(\frac{1}{2}\int dA'R(\rho')N(\rho,\rho') + \int d\ell' k(\rho')N(\rho,\rho')\Big) \\ &+ \frac{1}{4}\int dAdA'\big(J_g(\rho)J_g(\rho') + J_x(\rho)\cdot J_x(\rho')\big)N(\rho,\rho')\Big\}. \end{aligned} \tag{5.26}$$

Here $dA \equiv d^2\rho\sqrt{G}$ is the element of area and $d\ell$ is the proper element of length on the boundary. There is of course an implicit charge conserving delta function in the world sheet path integral which imposes

$$\int dAJ_g = \frac{3}{2\pi}\left(\frac{1}{2}\int dAR + \int d\ell k\right)$$
$$\int dAJ_x = 0.$$

For string field theory diagrams we can use (5.26) to extract the dependence of Z_P on $p_e(\sigma)$ and $\Pi_e(\sigma)$:

$$\begin{aligned} Z_P(p_e,\Pi_e) = Z_P(0,0)\exp\Bigg\{\frac{1}{4}\sum_{e,e'}\int_0^\pi d\sigma\int_0^\pi d\sigma'\big(\Pi_e(\sigma)\Pi_e(\sigma')+ \\ p_e(\sigma)\cdot p_e(\sigma')\big)N(T_e+i\sigma,T_{e'}+i\sigma') \\ +\frac{1}{2}\sum_e\int_0^\pi d\sigma\Pi_e(\sigma)\Big[\sum_c\frac{3}{2}N(T_e+i\sigma,\rho_c)-\sum_i\big(N(T_e+i\sigma,\rho_i) \\ \pm i\int_0^{\sigma_i} d\sigma'\frac{\partial}{\partial\tau_i}N(T_e+i\sigma,\tau_i+i\sigma')\big)\Big]\Bigg\}. \end{aligned}$$

A convenient way to present the dependence on external state data is through the oscillator basis. That is one interprets $Z_P(p_e,\Pi_e)$ as a bracket $\langle\Psi_P|p_e(\sigma),\Pi_e(\sigma)\rangle$ and shows that

$$\begin{aligned} \langle\Psi_P| = \mathcal{M}\,\langle\mathbf{0}|\exp\Bigg\{\frac{1}{2}\sum_{rsmn}N^{rs}_{mn}(\alpha^r_m\cdot\alpha^s_n+g^r_mg^s_n)+\frac{3}{2}\sum_{crm}g^r_mN^r_m(\rho_c)- \\ \sum_{irm}g^r_m\big(N^r_m(\rho_i)\pm i\int_0^\sigma d\sigma'\frac{\partial}{\partial\tau_i}N^r_m(\tau_i+i\sigma')\big)\Bigg\} \end{aligned} \tag{5.27}$$

where N^{rs}_{mn}, $N^r_m(\rho)$ are expansion coefficients as τ, $\tau'\to\infty$ of the Neumann function for the world sheet with all $T_e \to \infty$:

$$
\begin{aligned}
N^{\infty}(\rho,\rho') = &-2\delta_{rs}\mathrm{min}(\tau,\tau') - 2\delta_{rs}\sum_{m=1}^{\infty}\frac{1}{m}e^{-m|\tau-\tau'|}\cos m\sigma\cos m\sigma' \\
&+ 2\sum_{m,n=0}^{\infty} N_{mn}^{rs}e^{-m\tau-n\tau'}\cos m\sigma\cos n\sigma' \qquad (5.28) \\
N^{\infty}(\rho,\rho_0) = &\, 2\sum_{m=0}^{\infty} N_m^r(\rho_0)e^{-m\tau}\cos m\sigma.
\end{aligned}
$$

There is also dependence on the external state data J_g due to the winding part of g which for the case $T_e \to \infty$ has the large time expansions

$$
g_w(\sigma,\tau) = \sum_{m=0}^{\infty} w_m^r e^{-m\tau}\cos m\sigma
$$

where $\tau \to \infty$ on the r^{th} string. Then the complete state dependence of the generating function can be summarized by

$$
\begin{aligned}
\langle Z| = &\mathcal{M}\sum_w e^{i\delta_w}e^{-S(g_w)}\langle\mathbf{0}|\exp\Bigg\{\frac{1}{2}\sum_{rsmn} N_{mn}^{rs}(\alpha_m^r\cdot\alpha_n^s + g_m^r g_n^s) \\
&+ \frac{3}{2}\sum_{crm} g_m^r N_m^r(\rho_c) - \sum_{irm} g_m^r(N_m^r(\rho_i) \pm i\int_0^{\sigma} d\sigma'\frac{\partial}{\partial\tau_i}N_m^r(\tau_i + i\sigma')) \qquad (5.29) \\
&+ i\sum_{rm} w_m^r g_m^r\Bigg\}e^{-\sum_r T_r\hat{L}_0^r}
\end{aligned}
$$

and we remind the reader that the coefficients w_m^r depend on the winding sector w. The measure factor $\mathcal{M}$ is proportional to $(\det\Delta_G)^{-(D+1)/2}_{,}$ a state independent factor depending on the detailed geometry of the diagram. Because the state dependence is known, this factor can be inferred by evaluating $\langle Z|$ on a single convenient state, say external strings in the unexcited state $\left|0,0,-\frac{1}{2}\right\rangle$ with $p =$

$g_0 + \frac{1}{2} = 0$:

$$\begin{aligned}\langle Z| \prod_r |0,0,-1/2\rangle_r =& \mathcal{M} \sum_w e^{i\delta_w} e^{-S(g_w) - i\sum_r w_0^r/2} \\ & \exp\Big\{ \frac{1}{8} \sum_{rs} N_{00}^{rs} - \frac{3}{4} \sum_{rc} N_0^r(\rho_c) + \\ & \frac{1}{2} \sum_{ri} \big(N_0^r(\rho_i) \pm i \int_0^\sigma d\sigma' \frac{\partial}{\partial \tau_i} N_0^r(\tau_i + i\sigma') \big) \Big\} e^{\sum_e T_e}.\end{aligned}$$

A good way to find the Neumann function on a complicated surface is to conformally map that surface to one for which the Neumann function is known. For example, all tree (zero loop) diagrams with infinitely long external strips can be mapped to the upper half plane with the ends of the external strips mapped to semicircles of infinitesimal radii centered on points on the real axis. And one loop diagrams with infinite length external strips can be mapped to right circular cylinders or equivalently to a rectangle in the complex plane with, say, opposite vertical boundaries identified and the ends of the external strips mapped to infinitesimal semicircles centered on points on the horizontal boundaries.

The complicated geometry of the Witten diagram still enters $\det \Delta_G$ and so it is necessary to know how this factor transforms. The metric for the world sheet associated with a string field theory diagram is δ_{ij} away from the vertex midpoints in the original coordinates ρ. In a new set of coordinates z obtained by a conformal map $z(\rho)$ from the original coordinates the same metric is

$$\bar{G}_{ab} = \left| \frac{d\rho}{dz} \right|^2 \delta_{ab} \equiv e^{2\Sigma} \delta_{ab}. \tag{5.30}$$

So we would like to know how $\det \Delta_G$ transforms under the substitution $G_{ab} \to \bar{G}_{ab} = e^{2\Sigma} G_{ab}$. For smooth world sheets with smooth boundaries the answer is given by[101–103]

$$(\det \Delta_{\bar{G}})^{-1/2} = e^{-S_{conf}} (\det \Delta_G)^{-1/2} \tag{5.31}$$

with

$$S_{conf} = -\frac{1}{12\pi}\left[\frac{1}{2}\int dA G^{ab}\partial_a\Sigma\partial_b\Sigma + \int\limits_{\partial} d\ell k\Sigma + \frac{1}{2}\int dAR\Sigma\right].$$

However, the world sheets corresponding to string field theory diagrams are not smooth everywhere. There are sharp corners at the end of each external strip, and there is a sharp conical singularity at each vertex midpoint. In order to apply (5.31) to these world sheets, it is convenient to first slightly round each corner and conical vertex with, say, a radius of curvature of order r. In the limit $r \to 0$, $\det \Delta^r$ is then a power of r times the object we want to calculate, namely $\det \Delta$. Specifically, this limit produces a factor[101,102]

$$r^{(\alpha/\pi+\pi/\alpha-2)/12} \to r^{1/24} \qquad \text{for } \alpha = \frac{\pi}{2} \tag{5.32}$$

for each 90° corner and a factor

$$r^{(\alpha/2\pi+2\pi/\alpha-2)/6} \tag{5.33}$$

for each conical singularity subtending the angle α (*i.e.* the deficit angle is $2\pi-\alpha$). These singular factors are multiplied (in the limit $r \to 0$) by an r independent factor proportional to the determinant $\det \Delta$ that provides the measure of the world sheet path integral.

Under a conformal transformation, $\det \Delta^r$ transforms according to (5.31), but because r is scaled under this transform, we infer that $\det \Delta$ has an anomalous transformation law given by multiplying the RHS of (5.31) by the factor[104,105]

$$\exp\{\sum_{corners}\Sigma(z_i)/48 + \sum_{cones}\Sigma(z_c)(\alpha/2\pi + 2\pi/\alpha - 2)/12\}. \tag{5.34}$$

But we can avoid these awkward factors by calculating $\det \Delta^r$ instead. Then, taking $r \to 0$, the singular factors (5.32) and (5.33) could be removed or, alternatively, simply absorbed in the wave function normalization constant and coupling constant respectively. The normalization constant is determined by demanding the closure property of the path integral for strips, and the coupling constant is, in any case, a free parameter of the theory.

For clarity we stress that in the limit $r \to 0$, k and R entering S_{conf} contain effective delta function contributions for each corner and conical singularity, so that the Euler character of the world sheet is given by

$$\chi = \frac{1}{2\pi}\Big[\int_{\partial} d\ell k + \frac{1}{2}\int dAR\Big].$$

χ is, of course, a topological invariant independent of r.

Denoting quantities in the initial conformal frame with an over bar, so $\bar{k}, \bar{R}$ are the curvatures in the original frame and k, R those in the new conformal frame, one can show[91,104] that for D=26

$$\begin{aligned}
Z_{\bar{G}}^{\bar{r}}(J_g, J_x) =& (\det \Delta_G^r)^{-27/2} \\
& \exp\Big\{ \frac{9}{16\pi^2}\Big[\frac{1}{4}\int dAdA'R(z)R(z')N(z,z') \\
& + \int dAd\ell' R(z)k(z')N(z,z') + \int d\ell d\ell' k(z)k(z')N(z,z')\Big] \\
& - \frac{3}{4\pi}\int d\bar{A}\bar{J}_g(z)\Big(\frac{1}{2}\int dA'R(z')N(z,z') + \int d\ell' k(z')N(z,z')\Big) \\
& + \frac{1}{4}\int d\bar{A}d\bar{A}'\big(\bar{J}_g(z)\bar{J}_g(z') + \bar{J}_x(z)\cdot\bar{J}_x(z')\big)N(z,z') \\
& + \frac{3}{2}\int d\bar{A}\Sigma\bar{J}_g(z)\Big\}.
\end{aligned} \tag{5.35}$$

We have denoted the rounding radius as $\bar{r}$ (r) for the initial (final) conformal frames. Of course, $\bar{r}$ and r are related by $\bar{r}_i = e^{\Sigma(z_i)} r_i$. Since the rounding is specified on the initial world sheet, it is $\bar{r}$ that is held fixed during the conformal transformation. It is convenient to leave the description of the external source dependence untransformed: this is indicated by the bars that remain on the J's and corresponding area elements.

We now consider how (5.35) depends on the external state data. The infinitesimal semicircles which are the images of the propagator ends make right angles where they attach to the smooth boundary in the new frame. Because these corners have been rounded, there is, for fixed (small) $\bar{r}$, a smooth limit in which these radii go to zero (corresponding to the T_e tending to infinity)

$$\det \Delta_G^r \underset{\epsilon_i \to 0}{\widetilde{\quad}} \det \Delta_G^0 \prod_{corners} \bar{r}^{1/24}$$

where Δ_G^0 is the Laplacian on the target figure stripped of the semicircles. This is explained in Ref.[104]. We want to use (5.35) to infer the measure factor $\mathcal{M}$. For this purpose we specialize to unexcited states with ghost number $-1/2$ and also take all $T_e \to \infty$. (No information is lost in this limit since the T_e dependence can be restored.) Then the contribution to k from the extrinsic curvature of the semi-circles cancels that from the corners where they attach to the boundary. Then the RHS can be identified as the calculation of a correlation function in the new conformal frame of an insertion of $b(z_i)$ $\big(b(\bar{z}_i)\big)$ for each propagator and the ghost number $+1$ insertion : $e^{+ig(z_e)}$: for each external line. Each external line inserts effective ghost number $+1$ because the extrinsic curvature of the corners, which has been cancelled, contributed $-3/2$ to the total ghost number $-1/2$ of the incoming string. In addition the $3/2 \int \Sigma \bar{J}$ term combines with the self contractions from the $\int \bar{J} N \bar{J}$ term to produce the additional factors

$$\left(\frac{dz}{d\rho_i}\right)^2 \quad \text{or} \quad \left(\frac{d\bar{z}}{d\bar{\rho}_i}\right)^2$$

for each $b(z_i)$ or $\bar{b}(\bar{z}_i)$ insertion and

$$\frac{e^{T_e}}{|\xi_e|}$$

for each external line insertion. Here ξ_e is defined by analyzing the $\tau \to \infty$ limit of the mapping function:

$$z(\tau + i\sigma) \underset{\tau \widetilde{\rightarrow \infty}_e}{} z_e + \xi_e e^{-\tau - i\sigma} + \cdots.$$

The factor of e^{T_e} just supplies the -1 contribution to $\hat{L}_0^e$ in the factor

$$e^{-\sum_e T_e \hat{L}_0^{(e)}},$$

which restores the T_e dependence of the diagram. Thus we can summarize our result for $\mathcal{M}$ by writing

$$\begin{aligned}
\langle Z| = &\left\langle \prod_e \frac{1}{|\xi_e|} : e^{ig(z_e)} : \prod_i \left[\left(\frac{dz}{d\rho_i} \right)^2 b(z_i) + \left(\frac{d\bar{z}}{d\bar{\rho}_i} \right)^2 b(\bar{z}_i) \right] \right\rangle_{g,P} \\
&\sum_w e^{i\delta_w} e^{-S(g_w)} \langle \mathbf{0}| \exp\Bigg\{ \frac{1}{2} \sum_{rsmn} N_{mn}^{rs} (\alpha_m^r \cdot \alpha_n^s + g_m^r g_n^s - \frac{\delta_{m0}\delta_{n0}}{4}) \\
&\qquad + \frac{3}{2} \sum_{crm} (g_m^r + \frac{\delta_{m0}}{2}) N_m^r(\rho_c) \\
&\qquad - \sum_{irm} (g_m^r + \frac{\delta_{m0}}{2}) (N_m^r(\rho_i) \pm i \int_0^{\sigma_i} d\sigma' \frac{\partial}{\partial \tau_i} N_m^r(\tau_i + i\sigma')) \\
&\qquad + i \sum_{rm} w_m^r g_m^r \Bigg\} e^{-\sum_r T_r \hat{L}_0^r}
\end{aligned} \tag{5.36}$$

where the expectation $< \ >_{g,P}$ is a world sheet path integral with bosonized ghosts in the zero winding sector in the new conformal frame. This expression must be integrated over the moduli T_i and the anti-ghost insertion locations σ_i:

$$\langle \Gamma| = \int_0^\infty \prod_i dT_i \int_0^\pi \prod_i d\sigma_i \, \langle Z|$$

The state dependence of $\langle Z|$ includes coefficients which depend on the z_i and on the winding sector w. This complicates the sum over w and the insertion line integrals. However for states with ghost number -1/2 and no ghost excitations, some simplifications are possible. First of all, it has been shown up to one loop[104] that the winding number sum and phases $e^{i\delta_w}$ are just right to combine with the expectation value $\langle \quad \rangle_{g,P}$ to produce the fermion ghost expectation

$$\left\langle \prod_e \frac{1}{|\xi_e|} c(z_e) \prod_i \left[\left(\frac{dz}{d\rho_i} \right)^2 b(z_i) + \left(\frac{d\bar{z}}{d\bar{\rho}_i} \right)^2 b(\bar{z}_i) \right] \right\rangle_{fermion\ ghosts} .$$

It had been conjectured[91] that this replacement happens to all orders, but the proof at higher genera is not complete. It is important to appreciate that the detailed sign prescription deduced from (5.13) and (5.14) exactly accounts for the replacement of the nonchiral insertion $: e^{ig(z_e)} :$ by the chiral insertion $c(z_e)$[104].

Further, the integrals

$$\int d\sigma_i \left[\left(\frac{dz}{d\rho_i} \right)^2 b(z_i) + \left(\frac{d\bar{z}}{d\bar{\rho}_i} \right)^2 b(\bar{z}_i) \right] = \frac{1}{i} \left[\int_C dz_i \frac{dz}{d\rho_i} b(z_i) - \int_{\bar{C}} d\bar{z}_i \frac{d\bar{z}}{d\bar{\rho}_i} \bar{b}(\bar{z}_i) \right]$$

are just the coordinate transformations of the insertions appropriate to the choice of the T_i as moduli. If we change the integration variables to moduli m_i more natural to the new conformal frame, for which the correct measure for moduli is

$$\frac{1}{i} \left[\int_{C'} dz_i b(z_i) - \int_{\bar{C}'} d\bar{z}_i \bar{b}(\bar{z}_i) \right],$$

with the contours C', $\bar{C}'$ determined by the quasiconformal deformation produced by a change in the new modulus m_i,[100] then the slice independence of the world sheet path integral dictates

$$\frac{\partial(T_i)}{\partial(m_i)} \left[\int_C dz_i \frac{dz}{d\rho_i} b(z_i) - \int_{\bar{C}} d\bar{z}_i \frac{d\bar{z}}{d\bar{\rho}_i} \bar{b}(\bar{z}_i) \right] = \left[\int_{C'} dz_i b(z_i) - \int_{\bar{C}'} d\bar{z}_i \bar{b}(\bar{z}_i) \right]. \quad (5.37)$$

These two facts show that the integrands of the integrals defining on shell scattering amplitudes of unexcited strings are precisely the standard Koba-Nielsen integrands.[91] The integration region over moduli has also been essentially shown[92] to be completely covered exactly once — up to possible δ functions in the moduli.

At tree level, the gauge fixing procedure indicates no extra contributions are needed to maintain gauge invariance. In this case the amplitudes calculated from $\langle Z|$ should be the whole story, and indeed, given the validity of the substitution (5.37) the standard dual amplitudes are obtained. The remarkable cancellation required to confirm (5.37) has only been checked in detail for the four string function.[106] For multi-loop diagrams the gauge fixing procedure indicates that delta function contributions from the Γ_n may be needed for gauge independence, and we have seen how Γ_1 assures BRST invariance of the one loop tadpole.

Equation (5.36) summarizes the dependence of the path integral, defined on a Riemann surface, on boundary data specified on the ends of the strips. For us it simply summarizes in a neat form the value of a string field theory Feynman diagram. But the idea of coding information about world sheet path integrals by states in the first quantized string Fock space has been exploited in a wider context. Thus the authors of Ref.[107] have developed an operator formalism for describing Riemann surfaces in which such states play a central role. They also are central to the approach to string field theory advocated by Leclair, Peskin, and Preitschopf[108] in which the various terms in the string field action are represented by correlation functions of operators defined on the Riemann sphere or disk. In this approach, one can contemplate nonlinearities in the string field theory which do not necessarily have a direct "overlap" interpretation.

5.4. Some Sample Calculations

The simplest application of the conformal mapping technique is to obtain the oscillator representation for integration and for the $*$ product. For the integration we want to evaluate the path integral on the region of Figure 9. The state at T must have ghost number $+3/2$ to give a nonzero amplitude. The map

$$z(\rho) = \sinh \rho$$

maps Figure 9 to the upper half plane Figure 11. By composing the map with the Neumann function for the upper half plane we obtain the $N^\infty(\rho,\rho')$ for Figure 9 for $T\to\infty$, we find

$$
\begin{aligned}
N^\infty(\rho,\rho') &= \ln|\sinh\rho-\sinh\rho'||\sinh\rho-\sinh\rho'^*| + \text{sink terms} \\
&= 2\max\{\tau,\tau'\} - 2\ln 2 - 2\sum_{n=1}^{\infty}\frac{1}{n}e^{-n|\tau-\tau'|}\cos n\sigma\cos n\sigma' \\
&\quad + 2\sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n}e^{-n(\tau+\tau')}\cos n\sigma\cos n\sigma' + \text{sink terms} \qquad (5.38)\\
N^\infty(\rho,\frac{i\pi}{2}) &= \ln|\sinh\rho - i||\sinh\rho+i| + \text{sink terms} \\
&= 2\tau - 2\ln 2 + 2\sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n}e^{-n\tau}\cos n\sigma + \text{sink terms.}
\end{aligned}
$$

Thus the state dependence of $\int$ is given by (note that the sink terms do not contribute because of momentum and ghost number conservation):

$$
\left\langle \int \right| = \left\langle 0,0,-\frac{3}{2}\right| \exp\left\{-\frac{1}{2}\sum_{m=1}^{\infty}\frac{(-)^m}{m}(\alpha_m\cdot\alpha_m + g_m g_m) + \frac{3}{2}\sum_{m=1}^{\infty}\frac{(-)^m}{m}g_{2m}\right\} \qquad (5.39)
$$

where an overall multiplicative constant has been fixed by convention.

To calculate the $*$ product, we write

$$
A * B = \sum_s |\bar{s}\rangle \int |s\rangle * A * B,
$$

so it is enough to evaluate the vertex function

$$
\int e^{-T_1\hat{L}_0^1}\,|s_1\rangle\, e^{-T_2\hat{L}_0^2}\,|s_2\rangle\, e^{-T_3\hat{L}_0^3}\,|s_3\rangle\,,
$$

which means we have to do a path integral on the domain Figure 12. We shall obtain the vertex function in terms of bosonized ghosts.[109,110] Both $\int$ and $*$ have been cast in an oscillator representation using fermionized ghosts b, c in Ref.[110,111].

The problem of constructing a useful conformal map from any Witten diagram to a simpler figure can be approached systematically.[106,91] It is usually desirable that the target figure not have the curvature singularities at the images of the vertex midpoints. Calling z_c one such point, this singularity will not be present in the new figure if

$$\rho(z) \underset{z \to z_c}{\sim} (z - z_c)^{3/2}$$

or equivalently

$$\frac{d\rho}{dz} \underset{z \to z_c}{\sim} (z - z_c)^{1/2}.$$

Thus $(d\rho/dz)^2$ should have a simple zero at each vertex midpoint. We always think of the external strips of the diagram of interest extended to infinity. In the new frame it is desirable to have at least all but one of these strip ends at infinity mapped to finite values of z. In order that these points z_e lie on smooth boundaries of the new figure, we require

$$\rho(z) \underset{z \to z_e}{\sim} \pm \ln(z - z_e) + \text{finite}$$

or

$$\frac{d\rho}{dz} \underset{z \to z_e}{\sim} \frac{\pm 1}{z - z_e}$$

so $(d\rho/dz)^2$ has a double pole of unit strength at z_e. If one of the strip ends maps to ∞, $d\rho/dz$ should go like $1/z$, otherwise like $1/z^2$, at ∞. Since the open string boundaries are at constant imaginary part in $\rho, d\rho/dz$ should be real on the images of the open string boundaries. This can be arranged by doubling the diagram about the open string boundaries and imposing symmetry requirements. Thus $\varphi(z) \equiv (d\rho/dz)^2$ is an analytic function with specified simple zeros and double poles defined on the double of the new figure.

As a simple example of these principles, we map the vertex diagram (with the external strips extended to $T_e \to \infty$) to the upper half plane, with the strip ends mapped to 0,1, and ∞ so that the open string boundaries map to the real axis. Call the image of the vertex midpoint V. Then $(d\rho/dz)^2$ must be real analytic in the whole plane. The above requirements fix

$$\varphi(z) = \frac{(z-V)(z-V^*)}{z^2(z-1)^2}.$$

Unit strength of the double pole at $z=0$ and 1 requires

$$|V|^2 = |1-V|^2 = 1$$

or

$$V = e^{\pi i/3}$$

so $d\rho/dz$ is fixed up to a sign to be

$$\frac{d\rho}{dz} = -\frac{\sqrt{z^2-z+1}}{z(z-1)}.$$

This differential equation can be explicitly solved[109] to give

$$z = \frac{1}{2} + \frac{i\sqrt{3}}{4}(\zeta + 1/\zeta) + \frac{i\sqrt{3}}{4}(\zeta - 1/\zeta)[Y_+(\zeta)e^{-i\pi/3} + Y_-(\zeta)e^{i\pi/3}] \tag{5.40}$$

where we have defined $\zeta \equiv -ie^{\rho}$ and the $Y_\pm$ are defined by

$$Y_\pm \equiv \left(\frac{1\mp\zeta}{1\pm\zeta}\right)^{1/3}. \tag{5.41}$$

The branch of the cube root in (5.41) is selected to be real and positive for ζ real and of magnitude less than unity. Referring to Figure 12, we note that (5.40) maps string 1 to $z = 1$, string 2 to $z = \infty$ and string 3 to 0. By studying the large ρ expansion of the Neumann function

$$N(\rho, \rho') = \ln |z(\rho) - z(\rho')||z(\rho) - z(\rho')^*| + \text{sink terms}, \tag{5.42}$$

the authors of Ref.[109,110] were able to determine the N^{rs}_{mn} needed to construct the vertex. Again momentum and ghost number conservation ensure that the contribution of the sink terms will cancel out, so we only need the expansion coefficients of the explicitly displayed terms in (5.42):

$$\begin{aligned}
N^{rr}_{mn} &= (-)^{(m+n)/2}\left\{\frac{(-)^m + (-)^n}{3(m+n)}(2W_{mn} - y_m y_n)\frac{\delta_{mn}}{3m}\right\} \\
N^{r,r+1}_{mn} &= (-)^{m+n} N^{r+1,r}_{mn} \\
&= -\left\{\left[(-)^{(m+n)/2}\frac{(-)^m + (-)^n}{6(m+n)}\right.\right. \\
&\quad \left.\left. + (-)^{(m+n+1)/2}\sqrt{3}\frac{(-)^m - (-)^n}{6(m+n)}\right][2W_{mn} - y_m y_n] + (-)^n \frac{\delta_{mn}}{3m}\right\}
\end{aligned} \tag{5.43}$$

for $m \neq 0$ and $n \neq 0$, and one can express all of the coefficients involving one zero and one nonzero mode in terms of

$$N^{11}_{n0} = \begin{cases} \frac{y_n}{n}(-)^{n/2} & n \text{ even} \\ \frac{1}{\sqrt{3}}\frac{y_n}{n}(-)^{(n+1)/2} & n \text{ odd.} \end{cases}$$

They are given by

$$N^{11}_{0n} = \begin{cases} N^{33}_{n0} = N^{33}_{0n} = N^{11}_{n0} & n \text{ even} \\ -N^{33}_{n0} = -N^{33}_{0n} = N^{11}_{n0} & n \text{ odd} \end{cases}$$

$$N^{22}_{n0} = N^{22}_{0n} = N^{32}_{n0} = N^{23}_{0n} = N^{12}_{n0} = N^{21}_{0n} = 0$$

$$N^{23}_{n0} = N^{32}_{0n} = -N^{11}_{n0}$$

$$N^{21}_{n0} = N^{12}_{0n} = (-)^{n+1} N^{11}_{n0}$$

$$N^{13}_{n0} = N^{31}_{0n} = -N^{13}_{0n} = -N^{31}_{n0} = \begin{cases} 0 & n \text{ even} \\ 2N^{11}_{n0} & n \text{ odd.} \end{cases}$$

The asymmetrical values for these coefficients are due to the asymmetrical way the flux exits the diagram when the sink terms are dropped. In any case they would depend intimately on the choice of background charge. However momentum and ghost number conservation ensure that these asymmetries disappear in the final answer. Of course for the ghost part of the vertex this only happens if the proper ghost insertion is included. The effect of the insertion can be evaluated by the large time expansion of $\big(N(\rho,0) - \text{sink terms}\big)$. This gives the coefficients (for $n \neq 0$)

$$
\begin{aligned}
N_n^1(0) &= \delta_{n,even} \frac{(-)^{n/2}}{3} \frac{y_n - 2}{2} + \delta_{n,odd} N_{n0}^{11} \\
N_n^2(0) &= \delta_{n,even} \left[\frac{(-)^{n/2}}{3} \frac{y_n - 2}{2} + N_{n0}^{11} \right] \\
N_n^3(0) &= \delta_{n,even} \frac{(-)^{n/2}}{3} \frac{y_n - 2}{2} - \delta_{n,odd} N_{n0}^{11}.
\end{aligned}
$$

In these formulas y_n and W_{mn} are defined by

$$
\left(\frac{1-x}{1+x} \right)^{1/3} = \sum_{n=0}^{\infty} y_n x^n
$$

$$
W_{mn} = \begin{cases} -\dfrac{3}{2} \dfrac{m(n+1) y_m y_{n+1} - n(m+1) y_{m+1} y_n}{m-n} & m \neq n, \\ \displaystyle\sum_{k=0}^{n} (-)^{k+n} y_k^2 & m = n. \end{cases}
$$

When all the ingredients are put together, all the asymmetries cancel out giving the explicit formula for the three string vertex function

$$
\begin{aligned}
\langle \mathcal{V}_3 | &= \delta(p_1 + p_2 + p_3) \delta_{G_1+G_2+G_3,-3/2} \\
&\quad \langle 0 | \exp \Bigg\{ \frac{1}{2} \sum_{rs} \sum_{m,n=1}^{\infty} (\alpha_m^r \cdot \alpha_n^s + g_m^r g_n^s) N_{mn}^{rs} \\
&\qquad + \sum_r \sum_{odd\ n>0} \big(\alpha_n^r \cdot (p_{r+1} - p_{r+2}) + g_n^r (G_{r+1} - G_{r+2}) \big) N_{n0}^{11} \\
&\qquad + \sum_r \sum_{even\ n>0} \big([\alpha_n^r \cdot p_r + g_n^r (G_r + \frac{1}{2})] N_{n0}^{11} - (-)^{n/2} \frac{g_n^r}{2} \big) \Bigg\} \\
&\quad \times \left\{ \frac{4}{3\sqrt{3}} \right\}^{\sum_r [p_r \cdot p_r + (G_r^2 - 1/4)]/2}.
\end{aligned}
\tag{5.44}
$$

In Ref.[109] this formula was obtained by the method outlined above except that the measure, being a purely numerical constant, was fixed by convention (i.e. it was absorbed in the coupling constant). The authors of Ref.[110] employ the "overlap" method which avoids, at least initially, definition of the world sheet path integral. In this method, one seeks to determine vertices by imposing the continuity conditions implied by them, in this case, for the coordinates,

$$\begin{aligned} \langle\mathcal{V}_3|\left(x^a_\mu(\pi-\sigma)-x^{a+1}_\mu(\sigma)\right)&=0\\ \langle\mathcal{V}_3|\left(\mathcal{P}^a_\mu(\pi-\sigma)+\mathcal{P}^{a+1}_\mu(\sigma)\right)&=0 \qquad \text{for } 0<\sigma<\frac{\pi}{2}. \end{aligned} \tag{5.45}$$

The dependence on bosonized ghosts is obtained by applying the ghost insertion factor $:e^{3ig(\pi/2)/2}:$ to the solution of (5.45) with $g_{nr}+\delta_{n0}/2$ substituted for α_{nr}. Although this method appears very different from that using world sheet path integrals, it eventually uses the same mathematical machinery of Neumann functions, as explained, for example, in Ref.[112].

The overlap method is the most efficient for deriving the Ward-Takahashi identities, satisfied by vertices, which ensure gauge invariance. This is because (5.45) entails the identities

$$\langle\mathcal{V}_3|\left[:\left(x^{a\prime}_\mu(\pi-\sigma)\pm\pi\mathcal{P}^a_\mu(\pi-\sigma)\right)^2:-:\left(x^{a+1\prime}_\mu(\sigma)\pm\pi\mathcal{P}^{a+1}_\mu(\sigma)\right)^2:+\ \ const\ \ \right]=0$$

which imply, with a careful choice of moment functions, identities of the form

$$\langle\mathcal{V}_3|\,(L^r_{-m}+\sum_{n\geq0}M^{rs}_{mn}L^s_n+\ \ const\ \ )=0. \tag{5.46}$$

As advocated in Ref.[113], it is easiest to derive these identities for the matter sector, and then one can *infer* the dependence on ghosts b,c by requiring BRST invariance

$$\langle\mathcal{V}_3|\,(Q^1+Q^2+Q^3)=0.$$

This gives an alternative derivation[112] of the fermion ghost form of the Witten vertex found in Ref.[110,111].

This strategy generalizes easily to the case of the spinning string. Then one supplements (5.45) with conditions on the spin degrees of freedom

$$\begin{aligned} \langle \mathcal{V}_3| \left(\psi^a_\pm(\pi-\sigma) \pm i\psi^{a+1}_\pm(\sigma)\right) &= 0 \qquad \text{NS Sector} \\ \langle \mathcal{V}_3| \left(\psi^a_\pm(\pi-\sigma) + i\psi^{a+1}_\pm(\sigma)\right) &= 0 \qquad \text{R Sector} \end{aligned} \tag{5.47}$$

where the subscripts $(\pm)$ denote right and left moving modes respectively. Explicit oscillator formulas for the vertex and integration operations for Witten's superstring field theory have been worked out using these methods in Ref.[114, 111]. These works confirm that the insertions of picture changing opeators discussed in Chapter 3 are essential for BRST invariance, a necessary prerequisite for order g gauge invariance[115] of the superstring field theory. The issue of order g^2 and higher gauge invariance was not addressed until very recently with the calculation of four string amplitudes,[84] and it appears that extra contact interactions must be introduced in the gauge fixed action, at least when the boson component of the string field is put in the -1 picture.

As a final application of perturbation theory we discuss the role of closed strings in open string field theory. It has long been known that nonplanar one loop amplitudes[116,117] violate perturbative unitarity of the open string scattering amplitudes unless extra states are introduced. For subcritical dimensions, the one loop amplitudes have unphysical cuts[117] not interpretable in terms of particle states. The first suggestion that the critical dimension is necessary for consistency was made by Lovelace[118] who noted that, for D=26 in the bosonic model, the unphysical cuts might change to poles which could be given a particle interpretation. By including these extra particle states in the theory, one loop unitarity could conceivably be restored. Subsequently it was shown, by factorizing the nonplanar one loop amplitude on these poles,[119,120] that these extra states are just the excitations of closed strings. In particular, the graviton is included among them, so, remarkably, a theory constructed without gravity turned out to imply it. We conclude from this discussion that if open string field theory is to be consistent, it must predict at one loop level extra "composite" closed string states.

We have argued that the Feynman diagrams of Witten's open string field theory can be interpreted as the Polyakov world sheet formulas for the multi loop dual amplitudes, which would imply that this does happen. This issue has been investigated by Shapiro and Thorn[121] and Freedman, Giddings, Shapiro, and Thorn.[104] Consider the nonplanar two string diagram shown in Figure 13. This diagram is an integral over the moduli T_1 and T_2. The corner of moduli space responsible for closed string poles is, in these coordinates,[91] $T_1, T_2 \to 0$. In this limit the integrand of Figure 13 becomes Figure 13b which evidently factorizes into two pieces. Each piece has a simple world sheet interpretation: a rectangular strip with the two halves of one end identified in precisely the manner of the Witten integration operation. Of course, the dependence on external states must be funneled through the midpoint V, so in fact the integrand is really a sum of factored terms, each factor having various local operator insertions at V. In Ref.[121] it was shown that these insertions could be associated with states in the closed string Fock space, and a BRST invariant representation for each coupling was presented:

$$\begin{aligned}\left\langle\hat{\Upsilon}\right| = 2^{p^2-5}\langle 0|\exp\Bigg[-\sum_{m=1}^{\infty}\Bigg((-)^m\frac{1}{m}(\frac{1}{2}\alpha_m\cdot\alpha_m - p\cdot\alpha_{2m}) + (-)^m c_m b_m \\ + \frac{1}{m}A_m\cdot\bar{A}_m + C_m\bar{B}_m + \bar{C}_m B_m + (-)^m(\bar{C}_m - C_m)(\bar{B}_0 - B_0)\Bigg) \\ - \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\Bigg(\frac{(-)^m}{m}\eta_n^{2m}\alpha_n\cdot(A_m i^n + \bar{A}_m i^{-n}) \\ + 2(-)^n(\eta_{2n+1}^m - \eta_{2n-1}^m)c_m[i^m\bar{B}_n + i^{-m}B_n] \\ + \frac{1}{4}(-)^m(\eta_{n+1}^{2m} - \eta_{n-1}^{2m})[i^n\bar{C}_m + i^{-n}C_m]b_n\Bigg)\Bigg],\end{aligned} \tag{5.48}$$

where η_n^k is defined by

$$\left(\frac{1+x}{1-x}\right)^k = \sum_{n=0}^{\infty} x^n \eta_n^k$$

and where upper case letters denote oscillators for the closed string. This picture of the open string — closed string coupling had emerged earlier in studies of the string field equations[122,79].

The detailed nature of the factorization of the Witten nonplanar amplitudes on closed string poles was further clarified by explicit calculation.[104] Without going into detail one can understand the results of that work by considering Figure 14, which shows how one can extract a closed string propagation kernel from the nonplanar diagram. Closed string data can be specified on the circular boundaries of the "hourglass". Unlike a cylinder, this kernel is not diagonal in the data of the two external strings: there is an interruption at roughly half of the propagation "time" $-\ln q$, so the kernel has the form

$$\left\langle s'|q^{L_0+\bar{L}_0}Vq^{L_0+\bar{L}_0}|s\right\rangle = q^{L_s+L_{s'}}\left\langle s'|V|s\right\rangle$$

compared with

$$\left\langle s'|q^{2(L_0+\bar{L}_0)}|s\right\rangle = \delta_{ss'}q^{2L_s}$$

for a cylinder. The closed string poles arise from integrating q near zero, so since $\langle s'|V|s\rangle \neq 0$ for $L_s \neq L_{s'}$, the poles are separated by half the basic spacing of closed string (mass)^2 levels.

In addition, the process of amputating the external open string propagators leaves behind external state dependent factors $e^{-t(q)\hat{L}_0^e}$ with $t(q) = 0(q^{1/2})$ as $q \to 0$. The free closed string levels (before imposing $L_0 = \bar{L}_0$) are spaced by even integers. Explicit calculation[104] reveals a spectrum

$$\tfrac{1}{2}M^2 = \frac{1}{2}n - 4 \qquad\qquad n = 0, 1, 2, ...$$

when the external open strings are off shell and unphysical. Here the half integer spacing is due to the q dependence in the factors $\exp\{-t(q)\hat{L}_0^e\}$ If the external open strings are unphysical but on shell ($\hat{L}_0^e = \hat{L}_0^{e'} = 0$), the poles at half integer values go away and the spectrum is

$$\tfrac{1}{2}M^2 = n - 4$$

in accord with the above qualitative argument. The spurious closed string poles complicate the problem of extracting closed string information from open string diagrams, but have not (at least so far) been shown to lead to consistency problems for open string field theory. The argument is that the closed string propagation kernel should be diagonal in the subspace of conformally invariant states since the hourglass shape is then equivalent to a cylinder. The detailed analysis of Ref.[104] covered only the poles at $\frac{1}{2}M^2 = -4, -7/2$, and -3, so it is important to extend the detailed state analysis at least up to the graviton level.

6. Open Questions and Conclusions

We have concentrated in this article on aspects of string field theory which can be regarded as firmly established. In this concluding chapter we shall try to outline the major open questions for string field theory. These questions center on two serious shortcomings of the existing formalism. We do not have a satisfactory field theory for superstrings — even for open superstrings. And we are even further away from an adequate theory of closed strings. The difficulty with open superstrings appears to be technical in nature: how does one properly deal with the phenomenon of picture changing in the NSR formulation or how can one even use the Green-Schwarz formulation? The difficulty with closed strings is much more fundamental. Some promising ideas have been put forward for nonheterotic closed strings,[123,78,70,79] but they have not yet been realized in a sufficiently concrete form to, for example, obtain Feynman rules for the loop expansion. Since these proposals are tied in detail to open string theories, they are not much help with heterotic closed strings.

6.1. Problems for Open Superstring Field Theory

As we have developed the formulation of bosonic open string field theory, we have discussed in parallel how the various concepts extend to the superstring. At the linearized level we found that one could succeed in developing the field equations as well as an action principle. Provided that the string field for the bosonic component A is restricted to -1 picture and ghost number $-1/2$, the bosonic part of the free action is exactly parallel to that of the bosonic string field, and picture changing is not required. But the fermionic part of the free action is another story. As we have explained, fixing a picture for the fermionic component Ψ of the string field, say $-1/2$ picture for definiteness, one must insert an "inverse picture changing operator" to get a nonvanishing free action

$$S_\Psi = \int Y_{-1} \Psi * Q\Psi. \tag{6.1}$$

Another way of putting this is to say that the natural geometrical definition of $*$

and $\int$ lead to nonvanishing bilinear forms which couple together fields belonging to different pictures:

$$\int \Psi_{-P-2} * Q\Psi_P \neq 0.$$

Only for the case $P = -1$ can one stay within a fixed picture without further insertions. This fact of life means that it is, at the very least, awkward to set up superstring field theory with fields in a definite picture. At the nonlinear level, even the bosonic component is not immune from picture changing since

$$\int A_{P_1} * A_{P_2} * A_{P_3} \neq 0 \qquad\qquad \text{only for } P_1 + P_2 + P_3 = -2.$$

The formalism seems to be telling us not to insist on restricting the picture at the outset. So far, there has not been much effort in developing a multi-picture string field theory, but perhaps that prejudice should be changed. One reason behind the prejudice was that it appeared that, by making use of picture changing operators, one could develop an action principle with the string fields in a fixed picture. This scheme, which we have sketched in Chapter 3, seems to hold together at the naive formal level. In fact, at the level of defining integration and three string vertices which are BRST invariant, all appears to be well[114,111].

But as soon as one considers four string correlation functions, one is confronted with the problem noted by Wendt[84] and presaged by Greensite and Klinkhamer.[73] The picture changing operator $X(z) = \{Q, \xi(z)\}$ has a singular operator product with ξ

$$X(z_1)\xi(z_2) \underset{z_1 \to z_2}{\sim} -\frac{1}{2}\frac{1}{(z_1 - z_2)^2} b(z_1) e^{-2i\phi(z_1)}.$$

This fact, together with the manner in which string field theory (by its very nature) chops up the Koba-Nielsen integrals for single dual amplitudes into several pieces corresponding to different diagrams spells trouble. We have drawn in Figure 15 a way of representing a four string diagram where the potential conflict between crossed channel diagrams is avoided. The two diagrams are smooth continuations of one another, and when summed together they presumably give the correct answer. But the singularity is avoided only by treating the three strings at a vertex asymmetrically, and by focusing only on a particular diagram. The asymmetric treatment of the strings is of no help in the diagrams of Figure 16. Of course, one might hope that a careful definition of the vertices might be possible for which the troublesome singularities cancel out, but Wendt's explicit calculation shows that this is difficult if not impossible to arrange. Failing that, one would be forced to add four string contact interactions designed to cancel away the offending terms.[84] At the present level of our understanding, there is no reason why the problem should be cured by a single four string contact term: one can anticipate new contact terms with any power of string fields. To underline the severity of this situation, recall that these are difficulties in *tree* amplitudes. Thus the *classical* string theory is ill defined. Even if all required contact terms could be written down, they would be divergent making the usefulness of the resulting formalism questionable.

I have suggested that there may be a glimmer of hope in single picture superstring field theory if one chooses 0 picture rather than -1 picture.[112] Then, as we discussed in Chapter 3, the bosonic component of the field equation with Ψ set to zero requires no picture changing. But, of course, the action does require insertions of -2 picture changing operators Y_{-2}. Yost and I are presently studying this possibility. Unfortunately, the candidates for Y_{-2} are very awkward and complicated. One possibility is to analyze the operator product expansion of Y_{-1} with itself and extract the term with zero conformal weight. As we have already noted, the choice of Y_{-2} is also restricted by the requirement

$$X(z_1)Y_{-2}(z_2) \underset{z_1 \to z_2}{\widetilde{\quad}} Y_{-1}(z_1).$$

The presence of Y_{-2} in the kinetic term immediately complicates the description of the free action — for example, gauge fixing probably involves similar issues to those uncovered in Ref.[96, 97] in gauge fixing the Ramond sector of Witten's superstring field theory. We have not yet seen our way through all these complications, so at the moment this way out is only a hope.

A more likely resolution will probably involve including more than one picture — perhaps an infinite number — in the fundamental formulation (see for example, Ref.[68, 124]). The big obstacle with this idea is that one will have to deal with a system of auxiliary fields (possibly infinitely) degenerate at each mass level. Until this difficulty can be dealt with at a practical level, this program will not get off the ground. Another escape route would be to abandon the NSR formalism and to seek a formulation of superstring field theory within the Green Schwarz formalism. In this connection, it is interesting that recent work on first quantizing the Green-Schwarz superstring[65–67,125,126] indicates that an infinite system of world sheet ghost fields is probably required for covariant quantization. In the language of string fields, this would entail an infinite number of fields assigned to each level, which brings us back to the technical quandary of the multi-picture NSR formalism. But perhaps, since the logic of this degeneracy is more transparent, the Green-Schwarz formalism will be more suited to deal with it.

6.2. Closed String Field Theory

Closed string field theory is at a more primitive stage than open superstring field theory. In this section we survey some of the approaches that have been tried. We first remind the reader that nothing stands in the way of a (noncovariant) light cone gauge closed string field theory. For the superstring, of course, one may have to deal with contact interactions,[73] but that is associated with the usual picture changing insertions, and is not unique to closed strings. One approach that has been tried[21,70,77,82] has been a "covariantized" light cone approach, wherein the interactions are patterned after those of the light cone (see Figure 17). The fundamental difficulty with this idea is that the lengths α of parameter space, $0 < \sigma < \alpha$, have no physical significance. Some progress has been made in interpreting this type of redundant parameter as a new gauge degree of freedom,[82,83,77] but whether this resolves all the difficulties at the multi-loop level remains to be seen.

Another approach has been to try to mimic Witten's successful treatment of the bosonic open string field theory.[69,127,128] There is a rather basic difficulty with ghost number counting that must first be dealt with. In our discussion of the BRST formalism for closed strings, we noted that there are two cohomology sectors with ghost numbers 0 and -1. Each contains all physical closed string states. Thus one can choose the classical closed string field to have either ghost number 0 or ghost number -1. Neither choice is satisfactory since only off diagonal matrix elements of Q,

$$\langle -1 \mid Q \mid 0\rangle \qquad \text{or} \qquad \langle 0 \mid Q \mid -1\rangle$$

are nonzero. This is somewhat reminiscent of the picture changing difficulty for the superstring, but there is apparently no concept of picture changing for the closed string. An *ad hoc* fix that was proposed by Lykken and Raby (and also used in the covariantized light cone approach) has been to insert a zero mode factor like $c_0 - \bar{c}_0$ in the kinetic term:

$$\int A * (c_0 - \bar{c}_0) QA \neq 0$$

for A with ghost number -1. This is accompanied with a side constraint on A

$$(b_0 - \bar{b}_0)A = 0,$$

which is only consistent with $QA = 0$ if also

$$(\hat{L}_0 - \bar{\hat{L}}_0)A = 0.$$

This amounts to a sort of partial gauge fixing from the beginning. The completely gauge fixed action in Siegel gauge would be

$$\int A * c_0\bar{c}_0(\hat{L}_0 + \hat{\bar{L}}_0)A,$$

and one can attempt to add a cubic interaction term and derive the Feynman rules. The Lykken-Raby choice for vertex is patterned after Witten's open string vertex Figure 18, and was in fact mentioned as a possibility in Witten's original paper. Unfortunately, the four string function derived from the Lykken-Raby action has a finite chunk missing from the integration over Koba-Nielsen variables. This can apparently not be fixed up by adding quartic or higher terms because the one loop tadpole (which only involves cubic couplings) is incorrect.[129,130] Finally we mention that there is an alternative proposal for a free closed string field action which avoids imposing the $L_0 = \bar{L}_0$ constaint by adding extra fields.[131] But it is not clear how to extend this to a nonlinear theory.

The above strategy is probably too closely tied to what worked for open strings. Another set of ideas has been advanced by Witten.[78] These in turn were inspired by a suggestion of Friedan[123] that, since string theory is a theory of gravity, the kinetic term should be regarded as a background dependent piece of the action which should appear only after shifting the fundamental field by a classical solution describing the background metric. In this view, the cubic term of the action is really the more fundamental. If the fundamental action is cubic,

$$S_{cubic} = \frac{1}{3}\int A * A * A, \tag{6.2}$$

the classical field equation is just

$$A * A = 0, \tag{6.3}$$

which resembles the nilpotency condition on the BRST charge. The gauge invariance of (6.2) is

$$\Delta A = A * \Lambda - \Lambda * A. \tag{6.4}$$

Witten noted that such gauge transformations include general coordinate transformations which leave the vertex midpoint $x(\pi/2)$ fixed. To see this define the string field

$$D[x] = \frac{1}{2}\int_0^{\pi/2} d\sigma\Big[\Big(\epsilon\big(x(\sigma)\big) - \epsilon\big(x(\pi/2)\big)\Big)\cdot\frac{\delta}{\delta x(\sigma)} + \frac{\delta}{\delta x(\sigma)}\cdot\Big(\epsilon\big(x(\sigma)\big) - \epsilon\big(x(\pi/2)\big)\Big)\Big]\prod_{\sigma<\pi/2}\delta[x(\sigma) - x(\pi-\sigma)]. \tag{6.5}$$

Then

$$\begin{aligned}D * A - A * D = {}& \frac{1}{2}\int_0^{\pi/2} d\sigma\Big[\Big(\epsilon\big(x(\sigma)\big) - \epsilon\big(x(\pi/2)\big)\Big)\cdot\frac{\delta}{\delta x(\sigma)} \\ &+ \frac{\delta}{\delta x(\sigma)}\cdot\Big(\epsilon\big(x(\sigma)\big) - \epsilon\big(x(\pi/2)\big)\Big)\Big]A[x] \\ &- \int\prod_\sigma dy(\sigma)A[y(\pi-\sigma)]\prod_{\sigma<\pi/2}\delta[x(\sigma) - y(\pi-\sigma)] \\ &\frac{1}{2}\int_0^{\pi/2} d\sigma\Big[\Big(\epsilon\big(y(\sigma)\big) - \epsilon\big(y(\pi/2)\big)\Big)\cdot\frac{\delta}{\delta y(\sigma)} \\ &+ \frac{\delta}{\delta y(\sigma)}\cdot\Big(\epsilon\big(y(\sigma)\big) - \epsilon\big(y(\pi/2)\big)\Big)\Big]\prod_{\sigma<\pi/2}\delta[y(\sigma) - x(\pi-\sigma)] \\ = {}& \frac{1}{2}\int_0^{\pi} d\sigma\Big[\Big(\epsilon\big(x(\sigma)\big) - \epsilon\big(x(\pi/2)\big)\Big)\cdot\frac{\delta}{\delta x(\sigma)} \\ &+ \frac{\delta}{\delta x(\sigma)}\cdot\Big(\epsilon\big(x(\sigma)\big) - \epsilon\big(x(\pi/2)\big)\Big)\Big]A[x].\end{aligned} \tag{6.6}$$

Thus a general coordinate transformation on an open string field can be written

$$\begin{aligned}\Delta A = {}& D * A - A * D \\ &+ \frac{1}{2}\Big[\epsilon\big(x(\pi/2)\Big)\cdot\int_0^{\pi} d\sigma\frac{\delta}{\delta x(\sigma)} + \int_0^{\pi} d\sigma\frac{\delta}{\delta x(\sigma)}\cdot\epsilon\big(x(\pi/2)\big)\Big]A.\end{aligned} \tag{6.7}$$

Although the second term is apparently not an open string gauge transformation, it is nonetheless a derivation of the $*$ algebra. Adjoining to (6.4) transformations that move the midpoint would give a group including all general coordinate transformations. Since general covariance is a prerequisite for gravity being dynamically induced, the fact that (6.2) "almost" possesses general covariance "explains" how gravitons could be consistently present in higher order open string processes.[120,119] These ideas were developed further by the Kyoto group[132] and Horowitz, Lykken, Rohm, and Strominger.[133] In these references a solution A_0 of (6.3) was presented such that the substitution $A = A_0 + \hat{A}$ changed (6.2) into the usual Witten action:

$$S_{cubic} = \frac{1}{3}\int A * A * A + \int A_0 * A * A$$

where A_0 is chosen so that

$$\int A_0 * A * A = \frac{1}{2}\int A * QA.$$

In Ref.[78] Witten argued that the gauge group of a theory with both closed and open strings should be the group of all derivations of the $*$ algebra since those would include all general coordinate transformations. He identified the open string field gauge group (in the context of a cubic action) as the subgroup of all inner derivations of the $*$ algebra i.e. those of the form

$$D_\Lambda(A) = A * \Lambda - \Lambda * A, \tag{6.8}$$

where it is understood that Λ is not "too singular" at the vertex midpoint. Derivations not expressible as (6.8) he termed outer derivations, including among these Q itself as well as general coordinate transformations that move the vertex midpoint.

The distinction between inner and outer derivations was clouded by the work of Ref.[133, 134] where Λ's were constructed for which the operation (6.8) produced QA in the former and pA in the latter reference. However such Λ's were not "nice": they are examples of string fields which violate the associativity axiom for the $*$ algebra.[134] Thus the distinction between the two classes of derivation is restored if such "singular inner derivations" are adjoined to the set of outer derivations.

Returning to Witten's proposal, he argues that a closed string theory without open strings should have as gauge group the quotient group of all (ghost number preserving) derivations by the (normal) subgroup of inner derivations, which we now understand as "nonsingular" inner derivations. If closed string gauge transformations are ghost number preserving derivations of the $*$ algebra, then, he suggests, the closed string field C should be a derivation (such as Q itself) which adds unit ghost number to the string field it acts on. Finally he proposed that the closed string action be of the form

$$S_{closed} = Tr\mathcal{I}C^3 \tag{6.9}$$

where the insertion $\mathcal{I}$ carries -3 units of ghost number and is such that S_{closed} is invariant under shifts of C by inner derivations. Then the equation of motion

$$C^2 = 0$$

would have as solution the BRST charge Q_B for any closed string background B, not just the flat one. Further, putting $C = Q_B + \hat{C}$, the linearized equation for the fluctuations about the background is

$$\{Q_B, \hat{C}\} = 0.$$

Thus, for example, the emission of an on-shell closed string from an open string would be given by an open string matrix element of a BRST invariant vertex operator.

If C is to be a derivation of the $*$ algebra, it must first of all be a linear transformation on open string fields. On the other hand, a closed string field should be a functional of closed curves in space-time. It is not mysterious how a functional of closed curves can be regarded as a linear transformation on open string fields. Parametrize each closed curve $x(\sigma)$ on the interval $0 \leq \sigma \leq 2\pi$ and, in the spirit of the open string $*$ product, mark two points, say $\sigma = 0$ and $\sigma = \pi$, that divide the interval into two equal segments. Then regard C as a functional of two open curves $y(\sigma)$ and $z(\sigma)$, $0 < \sigma < \pi$ with

$$x^\mu(\sigma) = \begin{cases} y^\mu(\sigma) & 0 < \sigma < \pi \\ z^\mu(\sigma - \pi) & \pi < \sigma < 2\pi \end{cases}$$

and define the linear transformation

$$C: \qquad A \to CA$$

where

$$CA[x^\mu] \equiv \int dy^\mu(\sigma) C[x^\mu(\sigma), y^\mu(\sigma)] A[y(\pi - \sigma)]. \tag{6.10}$$

The additional restriction that C be a derivation of the open string $*$ algebra limits the form of C rather sharply. Writing out the derivation condition

$$C(A * B) = C(A) * B + (-)^{AC} A * C(B) \tag{6.11}$$

leads to the restriction

$$C_D[x_L, x_R; y_L, y_R] = \delta[x_R - y_L] f[x_L, y_R] - \delta[y_R - x_L] f[y_L, x_R]. \tag{6.12}$$

Applying C_D to an open string field, we see that

$$C_D A[x_L, x_R] = \int dy_R f[x_L, y_R] A[y_R, x_R] - \int dy_L f[y_L, x_R] A[x_l, y_L]. \tag{6.13}$$

We recognize on the RHS of (6.13) an "inner" derivation of the $*$ algebra with respect to the open string field

$$f[x_L, x_R]e^{-3ig(\pi/2)/2}$$

where the ghost insertion is needed to cancel the ghost insertion in the definition of $*$. Since these considerations have glossed over the behavior at the midpoint, this "inner" derivation is not necessarily "nice", i.e. it could be an "outer" derivation. Since the closed string action is supposed to vanish for C an inner derivation, the fact that C looks formally like an inner derivation means that all the closed string dynmaics is associated with "singular" f. This presents a formidable obstacle to the development of a concrete realization of these ideas. Strominger[79] has developed them most completely. He uses a basis of string fields formed by applying various local vertex operators to the identity string field for the $*$ algebra. An unfortunate feature of his approach is that all closed string dynamics is associated with vertex operators applied to the vertex midpoint. While this formalism is adequate for obtaining correct conclusions about the closed string spectrum and on shell cubic couplings, it is awkward for developing Feynman rules for a loop expansion.

The fact that closed string dynamics formulated in terms of Witten open string fields is compressed to a point is also evident from the calculation of the one loop non-planar two string function sketched in Chapter 5. Perhaps the restriction that the closed string field be a derivation of the open string star algebra can be relaxed in theories of closed strings alone. The need to depart from open string field concepts is also underlined by the existence of heterotic closed strings with chiral asymmetry. Such closed strings can't even be coupled to open strings.

Finally we note a rather fundamental fact about the first quantized description of closed strings which has proved to be a major obstacle to even free closed string field theory. This is the role of modular invariance in the expressions for the one loop cosmological constant. Because of modular invariance, the moduli of the torus, parametized by the complex variable τ, are integrated over the fundamental region

$$F : \qquad -\frac{1}{2} \leq \mathrm{Re}\ \tau \leq \frac{1}{2}, \qquad |\tau|^2 \geq 1. \tag{6.14}$$

It is very difficult to imagine how a free closed string field theory can avoid producing an integration region over the whole strip

$$-\frac{1}{2} \leq \mathrm{Re}\ \tau \leq \frac{1}{2}, \qquad \mathrm{Im}\ \tau \geq 0.$$

This strip is tiled by an infinite number of images of (6.14) under the modular group. Woodard[135] has argued that effective integration over F can be produced if the (gauge fixed) kinetic term is nonpolynomial in the total spacetime momentum (or in $\partial/\partial x$). An explicit realization of this, based on considerations that Polchinski[136] and McClain and Roth[137] applied to finite temperature string theory, has been described in Ref.[112]. Other string field theoretic attempts to account for the integration region F have been made by Zwiebach.[138] All of these efforts to understand the cosmological constant are tied to some type of gauge fixed version of the putative theory. It remains to be seen how such prescriptions could arise from a truly fundamental formulation of closed string field theory.

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APPENDIX A

Bosonization of Ghosts and Superghosts

There is a famous formula due to Jacobi[139] which enables the possibility of representing a charged fermion field as a real bose field:

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+e^{2i\theta}q^{2n-1})(1+e^{-2i\theta}q^{2n-1}) = \sum_{m=-\infty}^{\infty} q^{m^2}e^{2im\theta}. \tag{A.1}$$

Using this formula one can rewrite the partition function for the right moving component of a half odd integer moded charged fermion field on a circle

$$\begin{aligned} Z_F^{\frac{1}{2}}(q,\theta) &\equiv Tr\{e^{2i\theta N}q^{2L_0}\} \\ &= \prod_{n=1}^{\infty}(1+e^{2i\theta}q^{2n-1})(1+e^{-2i\theta}q^{2n-1}) \\ &= \sum_{m=-\infty}^{\infty} q^{m^2}e^{2im\theta}\prod_{n=1}^{\infty}(1-q^{2n})^{-1} \end{aligned} \tag{A.2}$$

Here

$$N = \sum (b_m^\dagger b_m - d_m^\dagger d_m)$$

is the charge carried by the fermi field, and

$$L_0 = \sum_{m>0}(m-\frac{1}{2})(b_m^\dagger b_m + d_m^\dagger d_m)$$

is the Hamiltonian for the fermion system. Of course, identical formulae would apply to the left moving component with $q \to \bar{q}$ and also to a fermion field on an open interval. Including the factor $e^{2i\theta N}$ allows one to keep track of sectors of different total charge. The RHS of (A.2) can be interpreted as the partition function for an integer moded free boson field. The sum on the RHS represents the contribution of the zero mode of the boson field. For an uncompactified string coordinate this would be a continuous momentum contributing the factor

$$\int dp q^{-p^2} = \sqrt{\frac{\pi}{-\ln q}}.$$

Thus the "momentum" of the bosonized fermion system is quantized in integers, i.e. it would correspond to a compactified string coordinate. We see also from (A.2) that this integer momentum is in fact precisely the charge of the state.

For an integer moded fermion field one need only sustitute $e^{2i\theta} \rightarrow qe^{2i\theta}$ in (A.1) to obtain

$$\begin{aligned} Z_F^0(q,\theta) &= q^{1/4}(e^{i\theta}+e^{-i\theta})\prod_{n=1}^{\infty}(1+e^{2i\theta}q^{2n})(1+e^{-2i\theta}q^{2n}) \\ &= \sum_{m=-\infty}^{\infty} q^{(m+\frac{1}{2})^2}e^{2i(m+\frac{1}{2})\theta}\prod_{n=1}^{\infty}(1-q^{2n})^{-1} \end{aligned} \tag{A.3}$$

from which we see that the only difference from the previous case is that now the momentum or charge is quantized in $\frac{1}{2}$ odd integers.

The analogous formulae for a charged boson field can be easily obtained. Let's do the half integer moded charged bosons first. In this case we start with

$$Z_B^{-\frac{1}{2}}(q,\theta) = q\prod_{n=1}^{\infty}(1-q^{2n-1}e^{2i\theta})^{-1}(1-q^{2n-1}e^{-2i\theta})^{-1} \tag{A.4}$$

The desired formula is obtained by resolving (A.4) in partial fractions with respect to the variable $\cos\theta$. This leads straightforwardly to

$$\begin{aligned} Z_B^{-\frac{1}{2}}(q,\theta) &= q\prod_{m=1}^{\infty}(1-q^{2m})^{-2}\sum_{n=1}^{\infty}\frac{(-)^{n-1}q^{n(n-1)}(1-q^{2(2n-1)})}{1-2\cos 2\theta q^{2n-1}+q^{4n-2}} \\ &= q\prod_{m=1}^{\infty}(1-q^{2m})^{-2}\sum_{k}e^{2ik\theta}\sum_{n=1}^{\infty}(-)^{n-1}q^{n(n-1)+|k|(2n-1)} \end{aligned} \tag{A.5}$$

where the last equality is obtained by factorizing the denominator, expanding the inverses of the two factors in powers of q and doing a geometric sum. The absolute value sign can be removed from the k because the exponent of q can be written $(n-\frac{1}{2}+|k|)^2-k^2-\frac{1}{4}$ and in the case that k is negative we have the identity

$$\sum_{m=k+1}^{|k|}(-)^m q^{(m-\frac{1}{2})^2}=0.$$

Thus a neater way to write (A.5) is

$$Z_B^{-\frac{1}{2}}(q,\theta)=q^{\frac{3}{4}}\prod_{m=1}^{\infty}(1-q^{2m})^{-2}\sum_k e^{2ik\theta}\sum_{n=0}^{\infty}(-)^n q^{(n-k+\frac{1}{2})^2-k^2} \tag{A.6}$$

where we have shifted the n sum by one unit and have chosen to replace $|k|$ by $-k$ since this is equally valid.

For an integer moded charged boson, we define

$$Z_B^0(q,\theta)=q^{\frac{3}{4}}(e^{-i\theta}-e^{i\theta})^{-1}\prod_{n=1}^{\infty}(1-q^{2n}e^{2i\theta})^{-1}(1-q^{2n}e^{-2i\theta})^{-1}. \tag{A.7}$$

This can be obtained from (A.4) by substituting $e^{2i\theta}\to q^{-1}e^{2i\theta}$ and multiplying by an overall factor of $q^{-\frac{1}{4}}e^{i\theta}$. Making the same modification on the RHS of (A.6) then leads to

$$Z_B^0(q,\theta)=q^{\frac{3}{4}}\prod_{m=1}^{\infty}(1-q^{2m})^{-2}\sum_k e^{2i(k+\frac{1}{2})\theta}\sum_{n=0}^{\infty}(-)^n q^{(n-(k+\frac{1}{2})+1)^2-(k+\frac{1}{2})^2} \tag{A.8}$$

More generally, one can consider charged bosons in a different picture as defined by Friedan, Martinec, and Shenker(FMS).[68] A picture is defined by selecting a half integer or integer P and defining

$$\begin{aligned}Z_B^{P+\frac{1}{2}}(q,\theta)=&e^{2i(P+1)\theta}q^{-P(P+2)}\\&\prod_{n=1}^{\infty}(1-q^{2(n-P)-3}e^{2i\theta})^{-1}(1-q^{2(n+P)+1}e^{-2i\theta})^{-1}.\end{aligned} \tag{A.9}$$

Then the sustitution $e^{2i\theta}\to q^{-2P-2}e^{2i\theta}$ in (A.6) and multiplying by the prefactor in (A.9) leads to the "bosonized" form

$$
\begin{aligned}
Z_B^{P+\frac{1}{2}}(q,\theta) =& q^{\frac{3}{4}} \prod_{m=1}^{\infty}(1-q^{2m})^{-2} \\
& \sum_k e^{2i(k+P+1)\theta} \sum_{n=0}^{\infty}(-)^n q^{(n-(k+P+1)+P+\frac{3}{2})^2-(k+P+1)^2}.
\end{aligned}
\tag{A.10}
$$

The first product on the RHS of (A.10) can be identified with the nonzero mode part of the partition function for two neutral boson fields. But we see that the zero mode sectors of these fields have been projected onto a subsector of those expected for two compactified boson fields. The momentum of one of these neutral fields is identified with the charge and is quantized in integers plus P. The momentum of the other neutral boson is restricted to have a set of values correlated with the charge of the state.

For a better understanding of these zero mode restrictions consider the identity

$$
\begin{aligned}
e^{-i\phi} \prod_{n=1}^{\infty}(1+e^{2i\phi}q^{2n})(1+e^{-2i\phi}q^{2n}) = & \\
& \prod_{m=1}^{\infty}(1-q^{2m})^{-1} \sum_{m=-\infty}^{\infty} e^{2i(m+\frac{1}{2})\phi} \sum_{n=0}^{\infty}(-)^n q^{(m+n+\frac{3}{2})^2-\frac{1}{4}},
\end{aligned}
\tag{A.11}
$$

which can be easily obtained from (A.3). Then

$$
\mathcal{N}_{m+\frac{1}{2}}(q) = \prod_{m=1}^{\infty}(1-q^{2m})^{-1} \sum_{n=0}^{\infty}(-)^n q^{(m+n+\frac{3}{2})^2-\frac{1}{4}} \tag{A.12}
$$

is seen to count the number of states of an integer moded charged fermion system built by applying only nonzero mode creation operators to the charge $-\frac{1}{2}$ vacuum with total charge $m+\frac{1}{2}$. Then (A.10) can be rewritten

$$
Z_B^{P+\frac{1}{2}}(q,\theta) = q \prod_{m=1}^{\infty}(1-q^{2m})^{-1} \sum_k e^{2i(k+P+1)\theta} q^{-(k+P+1)^2} \mathcal{N}_{-k-\frac{1}{2}}. \tag{A.13}
$$

This shows that the RHS of (A.10) can also be interpreted as the partition function for a system consisting of a neutral time-like boson and a charged fermion system built on the $-\frac{1}{2}$ vacuum with the constraint that the charge of the original charged boson plus the charge of this new fermion system is $P+\frac{1}{2}$.

The physical interpretation of these "bosonization" formulae is sharply different in the fermi and bose cases. In the fermi case both forms of Z_F can be identified as partition functions for *physical* systems: the LHS of (A.2) and (A.3) with a physical charged fermion and the RHS with a physical compactified neutral boson. In the bose case, the second form (A.10) has two obvious unphysical features. The first is the $-$ in front of the $(k+P+1)^2$ in the exponent of q. It is clear that this causes the RHS of (A.10) to have arbitrarily negative powers of q, implying a spectrum of L_0 unbounded below. However, within each charge sector $k+P+1$ is fixed so L_0 is bounded below in each sector. It is therefore crucial that charge be conserved to make sense of this "bosonization". With the interpretation of the two terms in the exponent as squares of "momenta," this $-$ sign corresponds to a Minkowski signature: the boson field associated with the charge is "time-like" and the other "space-like." Thus the system described by (A.10) would be represented on a state space of indefinite metric. The second unphysical feature is the $(-)^n$ in (A.10): the sectors with n odd have *negative* degeneracy! This means that, in the interpretation of the system as compactified free boson fields, certain states are removed from the spectrum. The net number of states at a given level is never negative, of course, as is clear from the starting counting formulas. For applications to FFP ghost systems, of course, the formulae identify unphysical systems with unphysical systems, so the incongruity of the alternative interpretations is not so unsettling. We note that the bosonization of two dimensional fermion field theories has a distinguished history in quantum field theory and statistical mechanics.[140,141] Our bosonization formulae (A.10) for superghosts have been independently obtained by Preitschopf.[142] The application of (A.10) to physical systems has occurred in the problem of constructing rotation and gauge group characters for string states and resolving them into irreducible representations[143,144].

The formulae we have discussed demonstrate an isomorphism between the

L_0 eigenspaces of two dissimilar systems. It is well known that it is possible to construct this isomorphism with an explicit operator mapping. We will present these mappings using the notation common for ghost systems, since that is the principal application of these results in this article.

For the Fermi case the fermion system is just that of the reparametrization ghosts b_n, c_n with

$$\{b_n, c_m\} = \delta_{n,-m}. \tag{A.14}$$

We shall denote the bosonized ghosts by g_n:

$$[g_n, g_m] = n\delta_{n,-m} \tag{A.15}$$

The unphysical nature of ghosts only enters in their hermiticity properties. Thus for ghosts

$$\begin{aligned} b_n^\dagger &= b_{-n} \\ c_n^\dagger &= c_{-n} \\ g_n^\dagger &= -g_{-n}. \end{aligned} \tag{A.16}$$

For example, the last of (A.16) implies that the ghost numner g_0 which has *real* eigenvalues is *antihermitian.* We note, by the way, that physical fermions would have hermiticity conditions $b_n^\dagger = c_{-n}$ so that g_0 would be hermitian. The formula relating g to b, c is simply

$$g_n = \sum_k : c_{-k} b_{n+k} : -\frac{1}{2}\delta_{n,0} \tag{A.17}$$

Here the normal ordering is standard for nonzero modes and b_0 stands on the right. A similar formula would apply for half integer moded fermions with k running over half odd integers and the $\delta_{n,0}$ deleted. With due attention paid to operator ordering, (A.17) implies the commutation relations (A.15) together with the identity

$$
\begin{aligned}
L_n^{gh} &= \sum_k (k-n) : c_{-k} b_{k+n} : -\delta_{n0} \\
&= \frac{1}{2} \sum_m : g_{-m} g_{n+m} : -\frac{3}{2} n g_n - \frac{9}{8} \delta_{n,0}
\end{aligned}
\tag{A.18}
$$

which shows the identification of L_0 for the two systems. Normal ordering is always with respect to the enclosed operators: for the g's it means the positive moded g's stand to the right.

The formulae relating b, c to g are more complicated than (A.17), but equally well-known:

$$
\begin{aligned}
c(z) &\equiv \sum_k c_{-k} z^k =: \exp(\sum_{n \neq 0} \frac{g_{-n}}{n} z^n) : U z^{g_0 + \frac{1}{2}} \\
b(z) &\equiv \sum_k b_{-k} z^k =: \exp(-\sum_{n \neq 0} \frac{g_{-n}}{n} z^n) : U^{-1} z^{-g_0 + \frac{1}{2}}
\end{aligned}
\tag{A.19}
$$

Here U is an operator which creates one unit of ghost number and satisfies

$$
[g_n, U] = U \delta_{n,0}. \tag{A.20}
$$

If we denote by $|0, g\rangle$ the ket annihilated by all the g_n with $n > 0$ which is an eigenstate of g_0 with eigenvalue g, U can be defined by its action

$$
U \, |0, g\rangle = |0, g+1\rangle \, . \tag{A.21}
$$

It is important to appreciate that all these formulae give an *exact* operator mapping rigorously preserving all algebraic relations. In proving them, one must of course pay careful attention to the ordering prescriptions. In particular when multiplying two normal ordered expressions, an important step is to normal order the product by Wick's theorem in which one only moves positive moded operators (never negative moded operators) to the right.

The expression for L_n in (A.18) differs from that for an ordinary coordinate (Eq. 2.16) by the presence of the last two terms.[145,146] The second term has the effect of altering the value of the central charge of the Virasoro algebra. It is useful to interpret these extra terms locally in the world sheet parameters σ, τ. For this purpose we replace the $\frac{3}{2}$ by an arbitrary parameter κ, so we will be able to read off corresponding results for superghost systems.

$$\begin{aligned} L_n &= \frac{1}{2}\sum_m : g_{-m}g_{n+m} : -\kappa n g_n - \frac{\kappa^2}{2}\delta_{n,0} \\ \bar{L}_n &= \frac{1}{2}\sum_m : \bar{g}_{-m}\bar{g}_{n+m} : -\bar{\kappa} n \bar{g}_n - \frac{\bar{\kappa}^2}{2}\delta_{n,0} \end{aligned} \tag{A.22}$$

Then the energy momentum tensor is given by[112]

$$\begin{aligned} T_\pm \equiv T_{00} \pm T_{01} &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \begin{pmatrix} \bar{L}_n - \bar{c}\delta_{n0}/24 \\ L_n - c\delta_{n0} \end{pmatrix} e^{-in(\tau\pm\sigma)} \\ &= \frac{1}{\pi}\left[\frac{1}{2} : (\dot{g} \pm g')^2 : -i\begin{pmatrix} \bar{\kappa} \\ \kappa \end{pmatrix}\frac{\partial}{\partial\tau}(\dot{g} \pm g') - \frac{1}{24}\right] \end{aligned} \tag{A.23}$$

where we have used

$$\dot{g} \pm g' \equiv \sum_n \begin{pmatrix} \bar{g}_n \\ g_n \end{pmatrix} e^{-i(\tau\pm\sigma)}. \tag{A.24}$$

Using the equations of motion for g, one can rewrite (A.23) to display the two dimensional tensor structure of T_{ab}

$$\begin{aligned} T_{ab} = &\frac{1}{\pi}[\partial_a g \partial_b g + G_{ab}\tfrac{1}{2}(\partial g)^2] \\ &- \frac{i}{2}\frac{\kappa + \bar{\kappa}}{\pi}(\partial_a\partial_b - G_{ab}\partial^2)g - \frac{i}{2}\frac{\bar{\kappa} - \kappa}{2\pi}(\epsilon_a{}^c\partial_b\partial_c + \epsilon_b{}^c\partial_a\partial_c)g. \end{aligned} \tag{A.25}$$

Here we define the antisymmetric symbol ϵ_{ab} with the convention $\epsilon_{01} = \sqrt{-G}$ in the presence of a nontrivial world sheet metric G_{ab}. Of course the formulas (A.23) are written for the special case of a flat Minkowski metric $G_{ab} = \mathrm{diag}(-1, 1)$.

The energy momentum tensor can be generally defined as the variation of a generally covariant action with respect to a deformation of the world sheet metric:

$$\delta S = \frac{1}{2}\int dA\delta G^{ab}T_{ab} \tag{A.26}$$

and the κ independent terms of (A.25) follow from the standard action for a scalar field in the presence of a background metric

$$S_0 = \frac{1}{2\pi}\int dAG^{ab}\partial_a g\partial_b g.$$

We can ask what term can be added to the usual action to produce the κ dependent terms in the case of a Minkowski metric. This will tell us how the bosonized ghost field should enter the Polyakov path integral. A simple calculation shows that

$$\begin{aligned}\delta\int dARg = -\int & dA\delta G^{ab}[(\partial_a\partial_b - G_{ab}\partial^2)g \\ & + \text{terms which vanish for constant metric}] \\ & + \text{boundary terms.}\end{aligned} \tag{A.27}$$

Thus adding to S_0 a term

$$i\frac{\kappa+\bar{\kappa}}{4\pi}\int dARg \tag{A.28}$$

generates the corresponding term in T_{ab}. The boundary terms in (A.27) involve both the deformation δG and its first derivatives $\partial\delta G$. To maintain a well-defined energy momentum tensor at the boundary it is desirable to add terms to the action so that when varied the boundary terms only involve δG and not its derivatives. To discover these additional terms note that the Euler character

$$\chi = \frac{1}{2\pi}\left[\frac{1}{2}\int dAR + \int_{\partial} d\ell k\right]$$

being a topological invariant is independent of all metric deformations. This means that $\delta(R\sqrt{G})$ is a total derivative of terms linear in δG and $\partial\delta G$. Further the variation $\delta(\int_{\partial m} d\ell k)$ involves terms linear in δG and $\partial\delta G$ which must cancel the boundary terms from the variation of $\frac{1}{2}\int dAR$ up to a total derivative with respect to ℓ which integrates to zero because the boundary of any manifold is closed. Since these cancellations must happen locally in σ (since δG is arbitrary), we see that the variation of

$$S_1 = i\frac{\kappa+\bar{\kappa}}{2\pi}\left[\frac{1}{2}\int dARg + \int_{\partial} d\ell kg\right] \tag{A.29}$$

can have no boundary terms involving $\partial\delta G$.

This gives the minimal modification to the action which produces the $\kappa+\bar{\kappa}$ term in the flat energy momentum tensor.[80] One could contemplate higher powers of the curvature, but these would never arise from ghost systems which have no explicit curvature dependence: curvature dependence only comes about because of an anomaly in the ghost current which is always linear in R. For nonheterotic closed strings as well as for open strings, κ is equal to $\bar{\kappa}$ so adding the term S_1 to S_0 gives the correct bosonized ghost action. In particular for the bosonic string $\kappa=\bar{\kappa}=3/2$.

For left—right asymmetric string models κ is not equal to $\bar{\kappa}$ and there must be further terms in the bosonized ghost action to account for the $\kappa-\bar{\kappa}$ terms in T_{ab}. These terms are best described using the language of zweibeins and spin connections. In two dimensions the spin connection is an abelian gauge field $\omega_\mu^{ab}=\epsilon^{ab}A_\mu$ and the topological invariant

$$\int dAR \propto \int dA(\partial_0 A_1 - \partial_1 A_0)$$

so S_1 is proportional to $\int dA\epsilon^{ab}A_a\partial_b g$. Thus we expect the $\kappa-\bar{\kappa}$ terms to be generated by something like $\int dAG^{ab}A_a\partial_b g$. Since we do not treat these chiral models in this report we do not attempt to develop the necessary formalism for "chiral bosonization" on a curved world sheet.

We next turn to the operator isomorphism implied by the identity between (A.9) and (A.10). Friedan, Martinec, and Shenker[68] introduce the scalar boson field ϕ, defined by

$$\phi_n = -\sum_m \gamma_{-m}\beta_{m+n} \qquad \text{for } n \neq 0 \tag{A.30}$$

where m runs either over integers or half odd integers for the R or NS sectors respectively. For the NS sector, ϕ_0 is given by

$$\phi_0 = -\sum_{r>0}(\gamma_{-r}\beta_r + \beta_{-r}\gamma_r) \tag{A.31}$$

and for the R sector,

$$\phi_0 = -\frac{1}{2}(\gamma_0\beta_0 + \beta_0\gamma_0) - \sum_{m=1}^{\infty}(\gamma_{-m}\beta_m + \beta_{-m}\gamma_m). \tag{A.32}$$

It follows that ϕ_n satisfies the hermiticity and commutation relations

$$\begin{aligned} \phi_n^\dagger &= -\phi_{-n} \\ [\phi_n, \phi_m] &= -n\delta_{n,-m} \end{aligned} \tag{A.33}$$

which confirms the "time-like" character of the superghost scalar field. The ϕ_n's, of course, generate only a subspace of the space generated by β and γ. To generate the rest of the space, FMS introduce a pair of fermion fields ξ and η with integer modes and anticommutation relations $\{\xi_m, \eta_n\} = \delta_{m,-n}$

$$\begin{aligned} \xi(z) &= \sum_k \xi_{-k} z^k \\ \eta(z) &= \sum_k \eta_{-k} z^k. \end{aligned} \tag{A.34}$$

Then the relation between Virasoro generators follows

$$\begin{aligned} L_n^{\beta\gamma} &\equiv \begin{cases} \sum_r (\frac{3n}{2} - r) : \gamma_{n-r}\beta_r : + \frac{1}{2}\delta_{n0} & \text{NS sector} \\ \sum_m (\frac{3n}{2} - m) : \gamma_{n-m}\beta_m : + \frac{3}{8}\delta_{n0} & \text{R sector} \end{cases} \\ &= L_n^{\phi} + L_n^{\eta\xi} \end{aligned} \tag{A.35}$$

with

$$L_n^\phi = -\frac{1}{2}\sum_k :\phi_{-k}\phi_{k+n}: -n\phi_n + \frac{1}{2}\delta_{n0} \tag{A.36}$$

and

$$L_n^{\eta\xi} = \sum_k (k-n) :\xi_{n-k}\eta_k:, \tag{A.37}$$

which implies that ξ and η have conformal weights 0 and 1 respectively.

The ξ,η system can itself be bosonized in terms of another boson field χ given by

$$\chi_n = \sum_k :\xi_{-k}\eta_{n+k}: -\frac{1}{2}\delta_{n0}$$

with

$$L_n^\chi = \frac{1}{2}\sum_k :\chi_{-k}\chi_{k+n}: -\frac{n}{2}\chi_n - \frac{1}{8}\delta_{n0}. \tag{A.38}$$

The χ_n enjoy the same hermiticity and commutation relations as the g_n. The ϕ dependence of β and γ is determined by the commutators $[\phi,\beta]$ and $[\phi,\gamma]$ to be proportional to $V^{-1} : e^{-\sum_{n\neq 0}\frac{\phi_n}{n}z^{-n}} : z^{+\phi_0-\frac{1}{2}}$ and $V : e^{+\sum_{n\neq 0}\frac{\phi_n}{n}z^{-n}} : z^{-\phi_0-\frac{1}{2}}$ respectively where $[\phi_0, V] = V$. But these expressions have conformal weight $1/2$ and $-3/2$ respectively so they must each be multiplied by a conformal weight 1 operator built from η and ξ. It is convenient to choose β and γ neutral with respect to the total change $\phi_0 + \chi_0$, so we take

$$\begin{aligned}\beta(z) &\equiv \sum_k \beta_{-k}z^k = V^{-1} : e^{-\sum_{n\neq 0}\frac{\phi_n}{n}z^{-n}} : z^{+\phi_0-\frac{1}{2}}(z\partial_z\xi(z))\\ \gamma(z) &\equiv \sum_k \gamma_{-k}z^k = V : e^{+\sum_{n\neq 0}\frac{\phi_n}{n}z^{-n}} : z^{-\phi_0-\frac{1}{2}}\eta(z)\end{aligned} \tag{A.39}$$

with $\eta(z) = \sum_k \eta_{-k}z^k$ and $\xi(z) = \sum_k \xi_{-k}z^k$. To confirm these identifications we must check the algebra $[\gamma_k, \beta_\ell] = \delta_{k,-\ell}$. This is most easily done by evaluating

$$[\gamma_k, \beta_\ell] = \frac{1}{2\pi i}\oint dz z^{k-1} \frac{1}{2\pi i} \oint_{|z'|<|z|} dz' z'^{\ell-1} \gamma(z)\beta(z')$$
$$- \frac{1}{2\pi i}\oint dz' z'^{\ell-1} \frac{1}{2\pi i} \oint_{|z|<|z'|} dz z^{k-1} \beta(z')\gamma(z)$$

normal ordering each term to extract the singularity at $z = z'$, and finally deforming the contours of the first term to agree with the second.

$$\gamma(z)\beta(z') =: e^{\sum_{n\neq 0}(z^{-n}-z'^{-n})\phi_n/n} : \left(\frac{z'}{z}\right)^{\phi_0 - 1/2} (1 - \frac{z'}{z})$$
$$[: \eta(z)(z'\partial\xi(z')) : + \frac{zz'}{(z-z')^2}]$$
$$\beta(z')\gamma(z) =: e^{\sum_{n\neq 0}(z^{-n}-z'^{-n})\phi_n/n} : \left(\frac{z'}{z}\right)^{\phi_0 + 1/2} (1 - \frac{z}{z'})$$
$$[: (z'\partial\xi(z'))\eta(z) : - \frac{zz'}{(z-z')^2}]$$

The first terms in square brackets are not singular as $z \to z'$ so they cancel in $[\gamma_k, \beta_\ell]$. The second term gives a simple pole at $z' = z$ so the contour deformation from $| z' |<| z |$ to $| z' |>| z |$ yields

$$[\gamma_k, \beta_\ell] = \oint \frac{dz}{2\pi i z} z^{k+\ell} = \delta_{k,-\ell}.$$

An important feature of (A.39) is that neither β nor γ depends on ξ_0. This means that an arbitrary function of β and γ commutes with η_0, so it is consistent to restrict the states (kets) to those annihilated by η_0. The states in the dual space (bras) are then annihilated by ξ_0. Moreover, since β and γ commute with $\phi_0 + \chi_0$, each β, γ Fock space can be taken to have a fixed eigenvalue of this operator. This operator is related to the picture number introduced earlier. To establish this we note the "vacuum" in picture P (see (2.71)) has superghost number $\phi_0 = P + 1$. Since this state has the lowest value of $L_0^{\beta\gamma}$ for this ghost number, it must have $\chi_0 = \pm\frac{1}{2}$. Then $\chi_0 = -\frac{1}{2}$ is chosen by our side condition that the kets are annihilated by η_0. Thus

$$P = \phi_0 + \chi_0 - \frac{1}{2}. \tag{A.40}$$

It is then straightforward to check that the above counting formulas ((A.9) and (A.10)) in bosonized language exactly correspond to the restrictions of fixing P and demanding $\eta_0 \, |kets\rangle = 0$.

A final feature of the superghost formalism of importance for superstring field theory is the construction of a "picture changing operator" which maps the $\beta\gamma$ Fock space in one picture to that in another. An operator like $\xi(z)$ certainly changes the value of P by + 1. But it is neither BRST invariant nor independent of the unphysical zero mode ξ_0. Both of these faults are remedied by constructing

$$X(z) = \{Q, \xi(z)\} \tag{A.41}$$

which is BRST invariant because it is a bracket with Q, and it is independent of ξ_0 because Q is independent of ξ_0

$$[\eta_0, X(z)] = -[Q, \{\eta_0, \xi_0\}] = 0.$$

Although X is a trivial operator in the enlarged space including the action of ξ_0 it is not trivial on the physical subspace annihilated by η_0. It is important that $z\partial_z\xi$, being independent of ξ_0, acts within the "small" space annihilated by η_0. This means that $\partial_z X$ is a truly trivial operator in the $BRST$ sense. Also, since ξ has zero conformal weight, so does X.

In addition to X we also make use of the "inverse picture changing operator" Y(z) with total ghost number -1. This is a BRST invariant operator which is the inverse of X(z) in the sense that

$$\lim_{z' \to z} Y(z) X(z') = 1.$$

It is given by

$$Y(z) = c(z) \big(z \partial_z \xi(z) \big) V^{-2} : e^{-2 \sum_{n \neq 0} z^{-n} \phi_n / n} : z^{2\phi_0 - 2}, \tag{A.42}$$

and it is simple to check that Y also has zero conformal weight. The two operators X and Y are used as insertions in the definition of the product and integration operation for superstring field theory.

Finally we mention that, by considering the various operator product relations, Verlinde and Verlinde,[147] making use of identities noted by Yamron[148] have made the following identifications of $\xi, \eta, e^{\pm i\phi}$ with various distribution functions of γ and β:

$$\begin{aligned} \xi(z) &= \Theta[\beta(z)] \\ \eta(z) &= z \partial_z \gamma(z) \delta[\gamma(z)] \\ e^{i\phi} &= \delta[\gamma(z)] \\ e^{-i\phi} &= \delta[\beta(z)] \end{aligned}$$

The right hand sides of these relations are what appear naturally in defining the world sheet path integral for spinning string amplitudes.[147] For a thorough explanation of these relations, see also Ref.[149].

APPENDIX B

Notation and Useful Formulae

Minkowski Spacetime Metric

$$g^{\mu\nu} = \mathrm{diag}(-1, +1, +1, \cdots, +1) \qquad \mu, \nu = 0, 1, 2, \cdots, \mathrm{D} - 1 \tag{B.1}$$

Mode Expansions

$$\begin{aligned} \frac{1}{i}\frac{\delta}{\delta x_\mu} \pm \frac{x^{\mu\prime}}{\pi} &= \frac{1}{\pi}\sum_m \alpha_m^\mu e^{\mp im\sigma}. \\ \frac{1}{i}\frac{\delta}{\delta g} \pm \frac{g'}{\pi} &= \frac{1}{\pi}\sum_m g_m e^{\mp im\sigma}. \end{aligned} \tag{B.2}$$

The mode expansions for the "bosonized" superghost fields ϕ and χ are identical to that for g.

$$\begin{aligned} c(z) &= \sum_k c_{-k} z^k & b(z) &= \sum_k b_{-k} z^k \\ \gamma(z) &= \sum_k \gamma_{-k} z^k & \beta(z) &= \sum_k \beta_{-k} z^k \\ \xi(z) &= \sum_k \xi_{-k} z^k & \eta(z) &= \sum_k \eta_{-k} z^k \end{aligned} \tag{B.3}$$

Zero Modes (Open String)

$$q^\mu = \frac{1}{\pi}\int_0^\pi d\sigma x^\mu \qquad\qquad \alpha_0^\mu = p^\mu$$

Mode Commutation Relations

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= m g^{\mu\nu}\delta_{m,-n} & [q^\mu, p^\nu] &= i g^{\mu\nu} \\ \{c_m, b_n\} &= \delta_{m,-n} & [g_n, g_m] &= n\delta_{n,-m} \\ \{\psi_m^\mu, \psi_n^\nu\} &= g^{\mu\nu}\delta_{m,-n} & & \\ [\gamma_m, \beta_n] &= \delta_{m,-n} & [\phi_n, \phi_m] &= -n\delta_{n,-m} \\ \{\xi_m, \eta_n\} &= \delta_{m,-n} & [\chi_m, \chi_n] &= m\delta_{m,-n} \end{aligned} \tag{B.4}$$

Hermiticity Conditions

$$\begin{array}{cccc}
\alpha_n^{\mu\dagger} = \alpha_{-n}^{\mu} & b_n^{\dagger} = b_{-n} & c_n^{\dagger} = c_{-n} & g_n^{\dagger} = -g_{-n} \\
\psi_n^{\mu\dagger} = \psi_{-n}^{\mu} & \gamma_n^{\dagger} = \gamma_{-n} & \beta_n^{\dagger} = -\beta_{-n} & \phi_n^{\dagger} = -\phi_{-n} \\
\xi_n^{\dagger} = \xi_{-n} & \eta_n^{\dagger} = \eta_{-n} & \chi_n^{\dagger} = -\chi_{-n} &
\end{array} \tag{B.5}$$

Picture Definition

$$P = \phi_0 + \chi_0 - \frac{1}{2}. \tag{B.6}$$

$$\begin{aligned}
\beta_n \left|0, P\right\rangle &= 0 \qquad \text{for } n > -P - \frac{3}{2} \\
\gamma_n \left|0, P\right\rangle &= 0 \qquad \text{for } n > +P + \frac{1}{2}
\end{aligned} \tag{B.7}$$

Virasoro and Super Virasoro Generators

$$\left(\frac{1}{i}\frac{\delta}{\delta x} \pm \frac{x'}{\pi}\right)^2 = \frac{2}{\pi^2}\sum_m L_m e^{\mp im\sigma} + const \tag{B.8}$$

$$\begin{aligned}
L_m &= \frac{1}{2}\sum_k : \alpha_{-k} \cdot \alpha_{k+m} : \\
L_n^{gh} &= \sum_k (k-n) : c_{-k} b_{k+n} : -\delta_{n0} \\
&= \frac{1}{2}\sum_m : g_{-m} g_{n+m} : -\frac{3}{2} n g_n - \frac{9}{8}\delta_{n,0} \\
\hat{L}_m &= \{b_m, Q\} = L_m + L_m^{gh} \\
F_n &= \sum_k \alpha_{-k} \cdot \psi_{k+n} \\
G_r &= \sum_k \alpha_{-k} \cdot \psi_{k+r} \\
L_n^R &= \frac{1}{2}\sum_k : \alpha_{-k} \cdot \alpha_{k+n} : + \frac{1}{2}\sum_k \left(k + \frac{n}{2}\right) : \psi_{-k} \cdot \psi_{k+n} : + \frac{D}{16}\delta_{n,0} \\
L_n^{NS} &= \frac{1}{2}\sum_k : \alpha_{-k} \cdot \alpha_{k+n} : + \frac{1}{2}\sum_s \left(s + \frac{n}{2}\right) : \psi_{-s} \cdot \psi_{s+n} :
\end{aligned} \tag{B.9}$$

$$
\begin{aligned}
L_n^{gh,R} &= \sum_m (m-n) : c_{-m} b_{m+n} : + \sum_m (\frac{3n}{2} - m) : \gamma_{n-m}\beta_m : -\frac{5}{8}\delta_{n,0} \\
L_n^{gh,NS} &= \sum_m (m-n) : c_{-m} b_{m+n} : + \sum_r (\frac{3n}{2} - r) : \gamma_{n-r}\beta_r : -\frac{1}{2}\delta_{n,0} \\
F_n^{gh} &= -2\sum_m \gamma_{-m} b_{m+n} + \sum_m (\frac{m}{2} - n) c_{-m}\beta_{m+n} \\
G_r^{gh} &= -2\sum_s \gamma_{-s} b_{r+s} + \sum_m (\frac{m}{2} - r) c_{-m}\beta_{m+r}
\end{aligned}
$$

$$
\begin{aligned}
L_n^{\beta\gamma} &\equiv \begin{cases} \sum_r (\frac{3n}{2} - r) : \gamma_{n-r}\beta_r : +\frac{1}{2}\delta_{n0} & \text{NS sector} \\ \sum_m (\frac{3n}{2} - m) : \gamma_{n-m}\beta_m : +\frac{3}{8}\delta_{n0} & \text{R sector} \end{cases} \\
&= L_n^{\phi} + L_n^{\eta\xi} \\
L_n^{\phi} &= -\frac{1}{2}\sum_k : \phi_{-k}\phi_{k+n} : -n\phi_n + \frac{1}{2}\delta_{n0} \\
L_n^{\chi} &= \frac{1}{2}\sum_k : \chi_{-k}\chi_{k+n} : -\frac{n}{2}\chi_n - \frac{1}{8}\delta_{n0} \\
L_n^{\eta\xi} &= \sum_k (k-n)\xi_{n-k}\eta_k
\end{aligned}
\tag{B.10}
$$

Virasoro and Super Virasoro Algebras

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m,-n}. \tag{B.11}
$$

$$
\begin{aligned}
[L_m^R, L_n^R] &= (m-n)L_{m+n}^R + \frac{D}{8}(m^3 - m)\delta_{m,-n} \\
[L_m^R, F_n] &= (\frac{m}{2} - n)F_{m+n} \\
\{F_m, F_n\} &= 2L_{m+n}^R + \frac{D}{2}(m^2 - \frac{1}{4})\delta_{m,-n}.
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
[L_m^{NS}, L_n^{NS}] &= (m-n)L_{m+n}^{NS} + \frac{D}{8}(m^3 - m)\delta_{m,-n} \\
[L_m^{NS}, G_r] &= (\frac{m}{2} - r)G_{m+r} \\
\{G_r, G_s\} &= 2L_{m+n}^{NS} + \frac{D}{2}(r^2 - \frac{1}{4})\delta_{r,-s}.
\end{aligned}
\tag{B.13}
$$

Ghost Number Operators

$$
\begin{aligned}
G^b &= \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{k=1}^{\infty}(c_{-k} b_k - b_{-k} c_k) \\
G^s &= \phi_0 = -\sum_m : \gamma_{-m}\beta_m : + \frac{1}{2} \qquad R \text{ sector} \\
G^s &= \phi_0 = -\sum_r : \gamma_{-r}\beta_r : \qquad NS \text{ sector},
\end{aligned}
\tag{B.14}
$$

BRST Operators

$$
\begin{aligned}
Q &= \sum_m c_{-m} L_m - \frac{1}{2}\sum_{m,n}(m-n) : c_{-m} c_{-n} b_{m+n} : - c_0 \\
Q_R &= \sum_m c_{-m} L_m + \sum_m \gamma_{-m} F_m - \frac{1}{2}\sum_{m,n}(m-n) : c_{-m} c_{-n} b_{m+n} : \\
&\quad - \sum_{m,n} \gamma_{-m}\gamma_{-n} b_{m+n} + \sum_{m,n}(\frac{3m}{2} - n) : c_{-m}\gamma_{m-n}\beta_n : -\frac{5}{8} c_0 \\
Q_{NS} &= \sum_m c_{-m} L_m + \sum_r \gamma_{-r} G_r - \frac{1}{2}\sum_{m,n}(m-n) : c_{-m} c_{-n} b_{m+n} : \\
&\quad - \sum_{r,s} \gamma_{-r}\gamma_{-s} b_{r+s} + \sum_{m,r}(\frac{3m}{2} - r) : c_{-m}\gamma_{m-r}\beta_r : -\frac{1}{2} c_0.
\end{aligned}
\tag{B.15}
$$

Bosonization Formulae

$$
g_n = \sum_k : c_{-k} b_{n+k} : -\frac{1}{2}\delta_{n,0} \tag{B.16}
$$

Here the normal ordering is standard for nonzero modes and b_0 stands on the right.

$$
\begin{aligned}
c(z) &\equiv \sum_k c_{-k} z^k =: \exp(\sum_{n\neq 0} \frac{g_{-n}}{n} z^n) : U z^{g_0 + \frac{1}{2}} \\
b(z) &\equiv \sum_k b_{-k} z^k =: \exp(-\sum_{n\neq 0} \frac{g_{-n}}{n} z^n) : U^{-1} z^{-g_0 + \frac{1}{2}}
\end{aligned}
\tag{B.17}
$$

Here U is an operator which creates one unit of ghost number and satisfies

$$[g_n, U] = U\delta_{n,0}. \tag{B.18}$$

$$\phi_n = -\sum_m \gamma_{-m}\beta_{m+n} \qquad \text{for } n \neq 0 \tag{B.19}$$

where m runs either over integers or half odd integers for the R or NS sectors respectively.

$$\chi_n = \sum_k :\xi_{-k}\eta_{n+k}: -\frac{1}{2}\delta_{n0}$$

$$\begin{aligned}\beta(z) &\equiv \sum_k \beta_{-k}z^k = V^{-1} : e^{-\sum_{n\neq 0}\frac{\phi_n}{n}z^{-n}} : z^{+\phi_0-\frac{1}{2}}(z\partial_z\xi(z)) \\ \gamma(z) &\equiv \sum_k \gamma_{-k}z^k = V : e^{+\sum_{n\neq 0}\frac{\phi_n}{n}z^{-n}} : z^{-\phi_0-\frac{1}{2}}\eta(z)\end{aligned} \tag{B.20}$$

where V creates one unit of superghost number and satisfies

$$[\phi_n, V] = V\delta_{n,0}.$$

Bosonized Ghost Action for Curved World Sheets

$$\begin{aligned} G_{ab} &\equiv \text{World Sheet Metric} \\ dA &\equiv \text{Element of World Sheet Area} \\ d\ell &\equiv \text{Element of Length on World Sheet Boundary} \\ \chi &\equiv \text{Euler Character} = \frac{1}{2\pi}\left[\frac{1}{2}\int dARg + \int_\partial d\ell kg\right]\end{aligned} \tag{B.21}$$

$$\begin{aligned} S_{ghost} &= \frac{1}{2\pi}\int_0^\pi dAG^{ab}\partial_a g\partial_b g + \frac{3i}{2\pi}\left[\frac{1}{2}\int dARg + \int_\partial d\ell kg\right] \\ S_{superghosts} &= \frac{1}{2\pi}\int dAG^{ab}(\partial_a\chi\partial_b\chi - \partial_a\phi\partial_b\phi) \\ &\quad + \frac{1}{2\pi}\left[\frac{1}{2}\int dA\ R(\chi+2\phi) + \int d\ell\ k(\chi+2\phi)\right]\end{aligned} \tag{B.22}$$

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FIGURE CAPTIONS

Figure 1 The parameter space for the path integral representation for the light cone open string evolution operator. Initial and final states are described by imposing the boundary conditions $\mathbf{x}(\sigma, 0) = \mathbf{x}(\sigma)$ and $\mathbf{x}(\sigma, T) = \mathbf{y}(\sigma)$ on the vertical lines.

Figure 2 The cubic term in the second quantized light cone hamiltonian glues together three rectangular strips (representing free evolution operators) into a single world sheet. In the corresponding path integral, $\mathbf{x}(\sigma, \tau)$ and its derivatives are discontinuous across the horizontal line emanating from V.

Figure 3 A pair of light cone Feynman diagrams which together form the world sheet for a single dual diagram. They are clearly smooth continuations of one another.

Figure 4 (a). The nonplanar one loop light cone diagram for the two string function involving only cubic open string vertices. The lines labelled A are identified. The integral over the modulus T terminates at $T = 0$, causing a violation of Lorentz invariance.
(b). The light cone diagram that completes the Koba Nielsen integral for the diagram in (a). The vertices at V and W involve a direct open string — closed string coupling: The part of the diagram between V and W is a cylinder since the lines labelled A are identified.

Figure 5 (a). Diagrams for the four string function with missing regions of the Koba Nielsen integrals. One of the initial open strings breaks at V_1, three open strings propagate to V_2, after which the top and bottom open strings join.
(b). The interpolating diagram that completes the Koba Nielsen integration region. Horizontal lines are identified as indicated by matching letters. There is a direct four open string vertex at V. The parameter α is integrated from 0 to πP_4^+.

Figure 6 (a). The ovelap defining the string field product, suggested by the light cone gauge, identifies two strings parametrized on intervals of length α_2 and α_3 with a single string parametrized on an interval of length $\alpha_1 = \alpha_2 + \alpha_3$.
(b). World sheet produced by attaching rectangular strips to each string parameter interval in (a). The boundary of this world sheet has a curvature singularity at V with excess angle π.

Figure 7 (a). Representation of a string field product as an overlap identifying strings parametrized on intervals of three different lengths.
(b). The corresponding world sheet which has a conical singularity with angular excess π at the interior point V.

Figure 8 (a). Overlap defining Witten's associative string field product. It identifies strings parametrized on three intervals of equal length.
(b). The world sheet obtained by attaching three equal width rectangular strips to each string parameter interval in (a).

Figure 9 The world sheet defining integration by BRST invariant path integrals. The points A on the imaginary axis above $V = i\pi/2$ are identified with their reflections A' about V. The resulting world sheet has a conical singularity at V with angular deficit π.

Figure 10 The tadpole diagram for the one loop correction to the one point function in Witten's open string field theory. T is the modulus associated with the internal propagator and is to be integrated from 0 to ∞.

Figure 11 The image of Figure 9 under the conformal map $z(\rho) = \sinh \rho$. The identified points A and A' are brought together by this map.

Figure 12 Figure 8 drawn on the complex ρ plane cut on the positive imaginary axis above $\pi/2$. The rectangular strip attached to string 3 is on the second sheet, and the points A and A' are identified.

Figure 13 (a). The one loop nonplanar two string diagram in Witten's open string field theory. The moduli T_1 and T_2 are each integrated from 0 to ∞.

(b). Closed string poles arise from the corner of moduli space T_1, $T_2 \sim 0$. In this limit a single point divides the diagram into two pieces.

Figure 14 Extraction of a closed string propagation kernel from the nonplanar diagram of Figure 13.

Figure 15 The two diagrams contributing to a given dual diagram in Witten's open string field theory may be regulated as indicated by shortening the lengths of the parameter intervals of two nonadjacent strings.

Figure 16 A string field vertex, modified to produce the regularization indicated in Figure 15 in some diagrams, would also imply the duiagrams shown here which are still singular.

Figure 17 A closed string field product might be patterned after the light cone vertex for three closed strings.

Figure 18 The world sheet used to define the product of closed string fields by Lykken and Raby. There are now two conical curvature singularities of angular excess π at V_1 and V_2.